THE SUPREMUM AND INFIMUM OF THE POISSON PROCESS[1]

By Ronald Pyke

Columbia University

1. Introduction. Let $\{X(t);\, t \geqq 0\}$ be a separable Poisson process with shift such that

$$\log E(e^{i\omega X(t)}) = -it\omega\alpha + \lambda t(e^{i\omega} - 1) \tag{1}$$

for all real ω, and $\alpha, \lambda > 0$. Set

$$\sigma(x, T) = P[\sup_{0 \leqq t \leqq T} X(t) \leqq x].$$

The task of obtaining $\sigma(x, T)$ explicitly for general stochastic processes is intrinsically difficult. However, Baxter and Donsker [1], following the methods and results of Spitzer, have obtained the double Laplace transform of $\sigma(x, T)$ for processes with stationary and independent increments. Their result as it pertains to the Poisson process is as follows.

Theorem. *Let $\{X(t);\, t \geqq 0\}$ be a separable process satisfying $X(0) = 0$ a.s and*

$$\log E(e^{i\omega X(t)}) = t\psi(\omega)$$

for all $t \geqq 0$ where exp $(\psi(\omega))$ *is the Lévy-Khintchine representation of the characteristic function of an infinitely divisible distribution. If $\psi(\omega)$ is complex and for some $\delta > 0$,*

$$\int_{-\delta}^{\delta} \left| \frac{\psi(\omega)}{\omega} \right| d\omega < \infty,$$

then for all $u, v \geqq 0$,

$$\begin{aligned} u \int_0^\infty \int_{0-}^\infty e^{-uT-vx}\, d_x\, \sigma(x, T)\, dT \\ = \exp\left\{\frac{1}{2\pi} \int_u^\infty \int_{-\infty}^\infty \frac{v}{\omega(\omega - iv)} \frac{\psi(\omega)}{s[s - \psi(\omega)]}\, d\omega\, ds\right\}. \end{aligned} \tag{2}$$

Theoretically, therefore, to obtain $\sigma(x, T)$ explicitly, one should evaluate the double integral on the right hand side of (2) and then perform a double inversion on it. For most cases this is virtually impossible except by numerical methods. Baxter and Donsker, however, have evaluated the right hand side of (2) for several important cases. Moreover, for the Gaussian process and for the process determined by coin tossing at random times, they were able to make the inversions.

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For the Poisson process, Baxter and Donsker showed that

$$(3) \qquad u \int_0^\infty \int_{0-}^\infty e^{-uT-vx}\, d_x\, \sigma(x, T)\, dT = (1 - v / s_u)[1 - \psi(iv) / u]^{-1}$$

where s_u satisfies $\mathscr{I}_m(s_u) \geqq 0$ and

$$u = \alpha s_u + \lambda(e^{-s_u} - 1).$$

From (3) they obtained $\sigma(x, +\infty)$. It is the purpose of this paper to obtain $\sigma(x, T)$ for finite T which is done in Section 3. In Section 4, the corresponding equation to (3), as well as the exact distribution, for the infimum of this process are derived. Applications of these results to queueing theory are given in Section 5. First of all, a lemma with applications to the theory of distribution-free statistics, is proven.

2. A lemma. Let $X_1, X_2, \cdots, X_n$ be independent random variables on a common measure space $(\Omega, \mathfrak{a}, P)$ such that $P[X_j \leqq x] = x$ for all $0 \leqq x \leqq 1$. Define U_j as the jth smallest component of $(X_1, X_2, \cdots, X_n)$. Therefore U_j is well defined a.s. Define for all real x, a and integral n

$$F(x:a, n) = P[\max_{1 \leqq i \leqq n} (ai - U_i) \leqq x].$$

LEMMA 1. *For* $0 \leqq a \leqq 1$, $0 \leqq na - x < 1$,

$$(4) \qquad F(x:a, n) \doteq (1 + x - na) \sum_{j=0}^{[x/a]} \binom{n}{j} (ja - x)^j (1 + x - ja)^{n-j-1}$$

where $[y]$, *the greatest integer contained in* y, *is a left continuous function. When* $na - x \geqq 1$ *or* $na - x < 0$, $F(x:a, n)$ *is equal to* 0 *or* 1 *respectively.*

PROOF: It may be shown that whenever $0 \leqq na - x < 1$, $F(x:a, n)$ has the representation

$$F(x:a, n) = n! \int_{na-x}^1 dz_n \int_{(n-1)a-x}^{z_n} dz_{n-1} \cdots \int_{ka-x}^{z_{k+1}} dz_k \int_0^{z_k} dz_{k-1} \cdots \int_0^{z_2} dz_1$$

where $k = [x/a] + 1$. That this multiple integral reduces to the expression (4) may be shown by a generalization of the method of Birnbaum and Tingey [7] along with an application of Lemma 1 of [2]. The evaluation of an essentially equivalent multiple integral has already been given by Chapman in equation (3) of [9].[2] Lemma 1 has also been obtained using probabilistic methods by Dempster [10].

3. The derivation of $\sigma(x, T)$. Let $\{Y(t), t \geqq 0\}$ be a Poisson process with parameter $\lambda > 0$; that is

$$E\{e^{i\omega Y(t)}\} = e^{\lambda t(e^{i\omega}-1)}.$$

[2] I am grateful to the referee for this reference.

Write $X(t) = Y(t) - \alpha t$. Then

$$\sigma(x, T) = P[\sup_{0 \leqq t \leqq T} X(t) \leqq x] = P[Y(t) \leqq \alpha t + x; 0 \leqq t \leqq T].$$

It is well known (cf. [3] Chap. VIII) that the conditional distribution of the first n discontinuity points of the Poisson process given that there were n such points in $(0, T)$, is the distribution of $(TU_1, TU_2, \cdots, TU_n)$ where the U_i's are as defined above. Using this fact, one may write

$$\sigma(x, T) = \sum_{n=0}^{[\alpha T + x]} P[\max_{1 \leqq i \leqq n} (i / \alpha T - U_i) \leqq x / \alpha T] e^{-\lambda T} \frac{(\lambda T)^n}{n!}.$$

Evaluating the summands by means of Lemma 1 gives

$$\sigma(x, T) = \sum_{n=0}^{[x]} \frac{e^{-\lambda T}(\lambda T)^n}{n!} + \sum_{n=[x]+1}^{[\alpha T + x]} e^{-\lambda T} \frac{(\lambda T)^n}{n!} (\alpha T + x - n)(\alpha T)^{-n} \sum_{j=0}^{[x]} \binom{n}{j} (j - x)^j (\alpha T + x - j)^{n-j-1}. \tag{5}$$

By Lemma 2 of [2], (5) may be rewritten to give

THEOREM 1. *Let $\{Y(t), t \geqq 0\}$ be a Poisson process with parameter $\lambda > 0$. Then for all $\alpha, x > 0$*

$$\sigma(x, T) = e^{-\lambda T} \sum_{n=0}^{[\alpha T + x]} \frac{(\alpha T + x - n)(\lambda/\alpha)^n}{n!} \cdot \sum_{j=0}^{[x]} \binom{n}{j} (j - x)^j (\alpha T + x - j)^{n-j-1} \tag{6}$$

where $\binom{n}{j} = 0$ for $j > n$.

COROLLARY: *For all $T > 0$, $\alpha > 0$,*

$$\sigma(0, T) = e^{-\lambda T} \sum_{n=0}^{[\alpha T]} \frac{(\lambda T)^n}{n!} (1 - n / \alpha T).$$

The limiting case of $\sigma(x, +\infty)$ may be obtained from Theorem 1 by an application of the Central Limit Theorem for Poisson variables. More specifically, by the Central Limit Theorem, for $j \geqq [x]$,

$$\lim_{T \to \infty} \sum_{n=j}^{[\alpha T + x]} e^{-\lambda T} \frac{\{\lambda T + (x - j)\lambda / \alpha\}^{n-j-1}}{(n - j)!} \{\lambda T + (x - n)\lambda / \alpha\} = \begin{cases} (1 - \lambda / \alpha) e^{\lambda(x-j)/\alpha}, & \text{if } \lambda < \alpha, \\ 0, & \text{if } \lambda \geqq \alpha. \end{cases}$$

Therefore, from Theorem 1, for $x \geqq 0$

$$\sigma(x, +\infty) = (1 - \lambda / \alpha) \sum_{j=0}^{[x]} \left(\frac{\lambda}{\alpha}\right)^j \frac{(j - x)^j}{j!} e^{-\lambda(j-x)/\alpha} \tag{7}$$

when $\lambda < \alpha$ and is equal to zero otherwise. This formula disagrees with (4.15) of [1]. For $x = 0$, (7) becomes

$$\sigma(0, +\infty) = \begin{cases} 1 - \lambda / \alpha, & \text{if } \lambda/\alpha < 1, \\ 0, & \text{otherwise.} \end{cases}$$

The expression (7) has also been obtained by Breakwell [4] who has computed a constant multiple of (7) for several values of the parameters.

4. The infimum of $\{X(t), t \geqq 0\}$. It is also of interest to study the infimum of the deviations of the Poisson process about the line αt. Since

$$\inf_{0 \leqq t \leqq T} X(t) = -\sup_{0 \leqq t \leqq T} \{-X(t)\},$$

a double generating function, $\Phi(u, v)$ say, may be obtained for the infimum by the same methods as used for the supremum by Baxter and Donsker [1]. It follows from (2) that for the infimum

$$u\Phi(u, v) = \exp\left\{\frac{1}{2\pi}\int_u^\infty \int_{-\infty}^\infty \frac{v}{\omega(\omega + iv)} \frac{\psi(\omega)}{s[s - \psi(\omega)]}\, d\omega\, ds\right\}.$$

By an application of Rouché's theorem, it may be shown that for all $s > 0$, $\psi(z) - s$ has as many roots in the upper half plane, $|z| \geqq 0$, as $h(z) = i\alpha z + \lambda + s$, namely one. Denote this root by iy_s; that is

$$\psi(iy_s) = s = \alpha y_s + \lambda(e^{-y_s} - 1).$$

Since $\overline{iy_s}$ is also a root, its uniqueness implies that y_s is real. Moreover the root is simple. Straightforward integration in the upper half plane yields

$$\frac{1}{2\pi}\int_{-\infty}^\infty \frac{v}{\omega(\omega + iv)} \frac{\psi(\omega)}{s[s - \psi(\omega)]}\, d\omega = -\frac{v}{y_s(y_s + v)} \frac{dy_s}{ds}.$$

Consequently,

$$u\Phi(u, v) = (1 + v / y_u)^{-1}.$$

Although unable to make a double inversion of φ, one may show that

$$\lim_{u \to 0} u\varphi(u, v) = (1 + v / y_0)^{-1}.$$

Under the assumption $\alpha < \lambda$, y_0 is the unique positive root of the real function $\alpha y + \lambda(e^{-y} - 1)$. When $\alpha \geqq \lambda$, $y_0 = 0$ and the above limit is defined to be zero. It follows, in particular, that for $\alpha < \lambda$, the infimum over $[0, \infty)$ of $X(t)$ has an exponential distribution with parameter y_0.

In the following, the explicit distribution function of the infimum over finite intervals of the Poisson process is derived. Moreover, an expression for the distribution function of the infimum over $[0, \infty)$ is obtained. It is shown by a second method that this latter distribution is exponential. The advantage of the second method is that the parameter y_0 as defined above is obtained explicitly.

For $x \leqq 0$ set

$$\mu(x, T) = P[\inf_{0 \leqq t \leqq T} X(t) \leqq x] = 1 - P[Y(t) > \alpha t + x, 0 \leqq t \leqq T]$$

for all $T \geqq 0$. It is clear that whenever $\alpha T + x < 0$, $\mu(x, T) = 0$. Suppose $\alpha T + x \geqq 0$. Then by a similar argument to that used in Section 3,

$$1 - \mu(x, T) = \sum_{n=K+1}^{\infty} e^{-\lambda T} \frac{(\lambda T)^n}{n!} P\left[TU_i < \frac{i - 1 - x}{\alpha}, i = 1, 2, \cdots, n\right]$$
$$= \sum_{n=K+1}^{\infty} e^{-\lambda T} \frac{(\lambda T)^n}{n!} P\left[\max_{1 \leqq i \leqq n} (i/\alpha T - U_i) \leqq \frac{n - x}{\alpha T} - 1\right]$$

where $K = [\alpha T + x]$, since the distribution functions of $(U_1, U_2, \cdots, U_n)$ and $(1 - U_n, 1 - U_{n-1}, \cdots, 1 - U_1)$ are the same. Therefore, by Lemma 1, one obtains

$$\mu(x, T) = 1 + x \sum_{n=K+1}^{\infty} e^{-\lambda T} \frac{(\lambda/\alpha)^n}{n!} \cdot \sum_{j=0}^{n-1-K} \binom{n}{j} (j - n + x + \alpha T)^j (n - j - x)^{n-j-1}$$

(8)

$$= \sum_{n=0}^{K} e^{-\lambda T} \frac{(\lambda T)^n}{n!} - x \sum_{r=0}^{K} e^{-\lambda(r-x)/\alpha} \frac{(\lambda/\alpha)^r (r - x)^{r-1}}{r!} + x e^{-\lambda T} \sum_{n=0}^{K} \sum_{r=0}^{K-n} (\lambda/\alpha)^{n+r} \frac{(x + \alpha T - n)^r}{r!} \frac{(n - x)^{n-1}}{n!}.$$

It may be shown, by rearranging the summations of the third term in (8) and by applying Lemma 2 of [2], that the sum of the first and third terms is zero. Therefore one obtains[3]

THEOREM 2: *For all $x \leqq 0$, α, $T \geqq 0$, the distribution function of $\inf_{0 \leqq t \leqq T} X(t)$, $\mu(x, T)$, is given by*

$$\mu(x, T) = -x \sum_{r=0}^{K} e^{-\lambda(r-x)/\alpha} \frac{(\lambda/\alpha)^r (r - x)^{r-1}}{r!}$$

As a consequence of this Theorem, it follows that

$$\mu(x, +\infty) = -x \sum_{r=0}^{\infty} e^{\lambda(x-r)/\alpha} (r - x)^{r-1} \frac{(\lambda/\alpha)^r}{r!}, \tag{9}$$

since all the summands are positive. This distribution shall be shown to be exponential over $(-\infty, 0]$. Specifically, we shall prove

[3] From discussions with Joseph M. Gani, it was learned that a result equivalent to Theorem 2, namely the first-passage-time distribution, has been obtained in Dam Storage theory (cf. [11]). The author is indebted to Gani for giving him access to the proofs of [11], without which the above simplification of (8) would not have been attempted.

THEOREM 3: *For all* $x \leqq 0, \lambda > \alpha > 0$,

$$\log \mu(x, +\infty) = x\lambda\alpha^{-1}\{1 - \mu(-1, +\infty)\}. \tag{10}$$

It will be convenient to define for all real x and $\beta \leqq e^{-1}$

$$f(x, \beta) = x \sum_{j=0}^{\infty} \frac{\beta^j}{j!} (j + x)^{j-1}. \tag{11}$$

Since $f(x, \beta)$ is a power series in β we have

$$f(x, \beta) f(y, \beta) = xy \sum_{n=0}^{\infty} \beta^j \sum_{j=0}^{n} \frac{(j + x)^{j-1}}{j!} \frac{(n - j + y)^{n-j-1}}{(n - j)!}. \tag{12}$$

As a consequence of Lemma 2 of [2], the inner summation of (12) is equal to $(n + x)^{n-1} / n!$, and so

$$f(x, \beta) f(y, \beta) = f(x + y; \beta)$$

Since $f(x, \beta)$ is a continuous function of x when $f(0, \beta)$ is defined to equal 1, it is known that

$$f(x, \beta) = e^{xg(\beta)} \tag{13}$$

for some function $g(\beta)$ independent of x. For a fixed $\beta < e^{-1}$, $g(\beta)$ may be obtained by differentiation $w \cdot r \cdot t \cdot x$ and taking limits as $x \to 0$. In this way one obtains

$$g(\beta) = \lim_{x \to 0} \frac{\partial}{\partial x} f(x, \beta) = \beta f(1, \beta). \tag{14}$$

Setting $v = \lambda/\alpha$ and $\beta = ve^{-v}$ in (9) and (11) gives

$$\mu(x, +\infty) = e^{vx} f(-x, ve^{-v}) = \exp\{xv - xve^{-v} f(1, ve^{-v})\}$$

$$= \exp\{vx - vx\,\mu(-1, +\infty)\},$$

which is the desired result.

Upon setting $x = -1$ in (10), one obtains

$$\log \mu(-1, +\infty) = -v\{1 - \mu(-1, +\infty)\},$$

or equivalently, one has shown that $v\{1 - \mu(-1, +\infty)\}$ is a non-negative root of the equation

$$\lambda(e^{-z} - 1) + \alpha z = 0. \tag{15}$$

That is, in the notation of (3), $v\{1 - \mu(-1, +\infty)\} = s_0$.

The function $\mu(x, +\infty)$ is a special case of the ruin function studied in the theory of collective risk which has already been shown to have an exponential form e^{Rx} where R is the unique solution of (15), (cf. Cramér [5]). The result for $\alpha < \lambda$ contained in Theorem 3 is new in that it gives an explicit expression for R.

Upon observing that the only non-negative root of (15) when $\lambda < \alpha$ is zero itself, we may in summary state that

$$\mu(x, +\infty) = \begin{cases} 1 & \text{if } \alpha \geqq \lambda \\ \exp\{x\lambda\alpha^{-1}[1 - \mu(-1, +\infty)]\} & \text{if } \alpha < \lambda. \end{cases}$$

5. Applications to queueing theory. Suppose that customers are arriving at times n/α, $n = 0, 1, 2, \cdots$ and that the service time for the jth customer, S_j say, is exponentially distributed with expectation $1/\lambda$. It is of importance in queueing theory to determine the distribution of the busy period of the server under the initial condition that there were k people in the queue. To this end one must compute

$B(T \mid k) \equiv P$[server is busy throughout $(0, T] \mid k$ people in the queue at $t = 0$].

Since additional customers are arriving at times n/α, $n = 1, 2, \cdots$ we have

$$\begin{aligned} B(T \mid k) &= P[S_1 + S_2 + \cdots + S_{i+k} \geqq (i + 1) / \alpha, 0 \leqq i \leqq \alpha T] \\ &= P[Y(t) \leqq \alpha t + k - 1, 0 \leqq t \leqq T], \end{aligned}$$

where $Y(t)$ is a Poisson process with parameter λ. Therefore, $B(T \mid k) = \sigma(k - 1, T)$. Define T_k as the time until the server is free under the condition that there are $k \geqq 1$ in line at time $t = 0$. Let G_k be the distribution function of T_k. Then clearly $T_k > 0$ a.s. and for all $t \geqq 0$,

$$G_k(t) = 1 - B(t \mid k) = 1 - \sigma(k - 1, t)$$

which may then be evaluated by Theorem 1. In particular T_1 represents the total busy period of the server and its distribution function is $G_1(t) = 1 - \sigma(0, t)$ which is given by the Corollary to Theorem 1.

A second application is of Theorem 2 to the queueing model in which the service times are constant and equal to $1/\alpha$ and the times between arrivals are independent random variables distributed exponentially with expectation $1/\lambda$. Let t_i denote the arrival time of the ith person after $t = 0$. As in the above let T_k denote the time until the server is free measured from $t = 0$ when it is assumed that at $t = 0$, there are k people in the queue and the server is just beginning service. Thus if G_k denotes the distribution function of T_k, $G_k(t) = 0$ for $t < 0$, and for $t \geqq 0$,

$$G_k(t) = 1 - P\left[t_i \leqq \frac{i + k - 1}{\alpha}, i = 1, 2, \cdots, N_t\right]$$

where N_t is the number of customers arriving in $(0, t]$. Therefore,

$$G_k(t) = 1 - P[Y(u) \geqq \alpha u - k, 0 \leqq u \leqq t] = \mu(-k, t),$$

which may be evaluated by Theorem 2. In particular T_1 represents the total busy period of the server and its distribution function is given by $\mu(-1, t)$.

Of special interest is $1 - G_1(+\infty)$, which is the probability of the server being busy for an infinite length of time, or equivalently, of there always being a waiting line. Since $1 - G_1(+\infty) = 1 - \mu(-1, +\infty)$ we have by Section 4, that $1 - G_1(+\infty) = 0$ whenever $\alpha \geqq \lambda$, and

$$1 - G_1(+\infty) = 1 - e^{-\lambda/\alpha} \sum_{j=0}^{\infty} e^{-j\lambda/\alpha}(j+1)^{j-1} \frac{(\lambda/\alpha)^j}{j!}$$

whenever $\alpha < \lambda$. That $G_1(\cdot)$ is a proper distribution function only when $\alpha \geqq \lambda$ is in keeping with the known result that the recurrent event, "the server is not busy" is ergodic whenever $\alpha > \lambda$, null recurrent whenever $\alpha = \lambda$ and transient whenever $\alpha < \lambda$ (cf. Lindley [6]).

6. Applications of Lemma 1. The result given by Lemma 1 is of use in the theory of distribution-free statistics. The special case of (4) with $a = 1/n$ is the distribution function of the D_n^+-statistic. This special case is known, having been obtained by several authors (see e.g. [7]).

A slightly modified version of the D_n^+-statistic is

$$\max_{1 \leqq i \leqq n} \left(\frac{i}{n+1} - U_i\right) = C_n^+.$$

This statistic has the same asymtotic properties as D_n^+ as well as some desirable small sample properties. For example $E(i/(n+1) - U_i) = 0$ for all i. Moreover, setting $U_{n+1} = 1$, $U_0 = 0$ one may write

$$W_i = \frac{1}{n+1} - U_i + U_{i-1}, \qquad i = 1, 2, \cdots, n+1,$$

and

$$S_j = \sum_{i=1}^{j} W_i, \qquad j = 1, 2, \cdots, n+1.$$

It is known that the random variables $(W_1, \cdots, W_n)$ are symmetrically dependent. Therefore by a result of Andersen [8]

$$P\left[C_n^+ = \frac{j}{n+1} - U_j\right] = \frac{1}{n+1}, \qquad j = 0, 1, \cdots, n.$$

That is to say, the probability that the maximum should occur at the jth observation is independent of j. This is not true for D_n^+ as was shown in [2]. The distribution function of C_n^+, i.e., $P[C_n^+ \leqq x]$, is given by $F(x:1/(n+1), n)$ for $x \geqq -1/(n+1)$ and is equal to zero for $x < -1/(n+1)$.

Lemma 1 may also be used to obtain the power of the D_n^+ or C_n^+ tests against alternatives of the form $G_c(x) = cx$ for all $x \varepsilon [0, 1/c]$. That is to say one may obtain, for example, the power of D_n^+ against G_c, namely

$$(16) \qquad P[\max_{1 \leqq i \leqq n} (i/n - Z_i) \leqq x] \equiv P[\max_{1 \leqq i \leqq n} (i/cn - U_i) \leqq x/c].$$

The latter probability may be evaluated by Lemma 1 and is equal to the first probability, where Z_i is the ith smallest component of $(V_1, \cdots, V_n)$ in which the V_i's are mutually independent random variables with the common distribution function G_c. Similarly the power of D_n^+ or C_n^+ against alternatives of the form $G_{b,c}(x) = b + cx$ for all $x \varepsilon [0, 1/c - b/c]$ and $= 0$ for $x < 0$, may be expressed as a sum of a finite number of terms of the form (16). This generalization of (16) has recently been studied by Chapman [9] for the case $b + c = 1$.

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