SOME CONTRIBUTIONS TO ANOVA IN ONE OR MORE DIMENSIONS: II

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0. Introduction and notation. This paper presents certain natural extensions, to the multi-dimensional or multivariate situation, of the results contained in the first paper [10] by the authors. We shall use the same notation as before and, in addition, we shall use the following notation: $c(A)$ will denote all the characteristic roots of the matrix A, and if A is at least positive semi-definite, then $c_{\min}(A)$ and $c_{\max}(A)$ will denote, respectively, the smallest and the largest of these roots; $D_a(p \times p)$ will denote a diagonal matrix whose elements are a_1, $a_2, \cdots, a_p$; $\tilde{T}(p \times p)$ will denote a triangular matrix whose non-zero elements are along and below the diagonal; $|A|$ will denote the determinant of a square matrix A; and, $A(p \times p) \cdot \times B(q \times q)$ will denote the Kronecker product or right direct product [5] of the matrices A and B. Also min (p, q) will denote the lesser of the two real numbers p and q.

1. Resume of problems and results under the multivariate Model I of ANOVA.

1.1 *The multivariate Model I*: Let $X(p \times n) = p[\mathbf{x}_1 \mathbf{x}_2 \cdots \mathbf{x}_n]$ be a set of n observable stochastic p-vectors such that

$$
\begin{aligned}
X'(n \times p) &= A(n \times m)\xi(m \times p) + \epsilon(n \times p), \qquad m < n, \\
&= n\underset{\;r\qquad (m-r)}{[A_I \quad A_D\;]} \begin{bmatrix} \xi_I \\ \xi_D \end{bmatrix}_{p} \begin{matrix} r \\ (m-r) \end{matrix} + \epsilon(n \times p) \quad \text{(say)},
\end{aligned} \tag{1.1.1}
$$

where, as in the univariate situation, A is the *design* matrix with rank $(A) = r \leqq m \leqq n$, and A_I is a basis of A with a consequent partitioning of ξ into $\xi_I(r \times p)$ and $\xi_D(\overline{m - r} \times p)$, and where

(i) $\xi(m \times p)$ is a set of unknown parameters;

(ii) $\epsilon(n \times p)$, whose elements are physically of the nature of errors, is a random sample of size n from the non-singular p-variate normal $N[\mathbf{O}(p \times 1), \Sigma(p \times p)]$. Furthermore, we assume here that $p \leqq (n - r)$.

Under this model it is seen that $\mathbf{x}_i(p \times 1)$, for $i = 1, 2, \cdots, n$, are n independent stochasic p-vectors such that $\mathbf{x}_i$ is $N[E(\mathbf{x}_i), \Sigma]$, where the unknown dispersion matrix $\Sigma(p \times p)$ is the same for all the n vectors, and $E(\mathbf{x}_i)$, for $i = 1, 2, \cdots, n$, is given by

$$E[X'](n \times p) = A(n \times m)\xi(m \times p) = [A_I \; A_D] \begin{bmatrix} \xi_I \\ \xi_D \end{bmatrix}. \tag{1.1.2}$$

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Here again, as in the univariate case, the assumption of normality in (ii) of the above model is not necessary for purposes of linear estimation. In fact, since the linear estimation part of the present problem can be easily handled and may not be of much additional interest, we skip it and proceed directly to the solutions of the problems of testing of linear hypotheses and the associated confidence bounds.

1.2 *Testing of linear hypotheses.* The hypothesis that we seek to test, under the model of section 1.1, is

$$H_0\colon C(s \times m)\xi(m \times p)M(p \times u) = 0(s \times u),$$

(1.2.1) $$\text{or,}\quad \begin{matrix} s[C_1 & C_2\ ] \\ r & (m-r) \end{matrix} \begin{bmatrix} \xi_I(r \times p) \\ \xi_D((m-r) \times p) \end{bmatrix} M(p \times u) = 0(s \times u),$$

against

$$H\colon C\xi M = \eta(s \times u) \neq 0(s \times u),$$

where C and M are matrices given by the hypothesis, and hence called the *hypothesis* matrices, such that rank $(C) = s \leqq r \leqq m < n$ and rank $(M) = u \leqq p$ and $\eta(s \times u)$ is an arbitrary unspecified nonnull matrix. Notice that s may be greater than or equal to or less than u. One can, of course, verify that (1.2.1) is by no means the most general type of linear hypothesis imaginable, although it includes a wide variety of linear hypotheses in which we might be interested. The main results follow. [12, pp. 84a–84i]

(1.2.2) All the following results are invariant under the choice of a basis A_I of A (with a consequent determination of ξ_I and C_1).

(1.2.3) Whether we use the likelihood ratio criterion or the one used by the authors, [12], we have a similar notion of *testability* for this situation as for the univariate case, and the *testability* condition is the same as (1.3.3) of [10].

(1.2.4) The test itself is given by the following rule:

Reject H_0 against H if $c_{\max}(S_1S^{-1}) \geqq c_\alpha(u, s, n-r)$ and accept (do not reject) H_0 against H otherwise, where $c_{\max}(S_1S^{-1})$ denotes the largest characteristic root (necessarily positive except on a set of probability measure zero) of S_1S^{-1}, $c_\alpha(u, s, n-r)$ to be called c_α, for shortness, is a constant which depends on the level of significance α and the degrees of freedom u, s and $(n-r)$ and which is being tabulated from the relation

(1.2.5) $$P[c_{\max}(S_1S^{-1}) \geqq c_\alpha \mid H_0] = \alpha,$$

the distribution involved being long available [11, 12]. Here, S_1 is an $u \times u$ symmetric and at least positive semi-definite matrix of rank, almost everywhere, (i.e., except on a set of probability measure zero), $\min(u, s)$, being given by

(1.2.6) $$sS_1(u \times u) = M'XA_I(A_I'A_I)^{-1}C_1'[C_1(A_I'A_I)^{-1}C_1']^{-1}C_1(A_I'A_I)^{-1}A_I'X'M,$$

and S in an $u \times u$ symmetric and, almost everywhere, positive definite matrix of rank u (necessarily), given by

(1.2.7) $$(n-r)S(u \times u) = M'X[I(n) - A_I(A_I'A_I)^{-1}A_I']X'M.$$

We shall call the matrix on the right side of (1.2.6) the *matrix due to the hypothesis* (1.2.1) and the matrix on the right side of (1.2.7) the *matrix due to error*.

The reduction, to a canonical form, of the relevant distribution problem is one in which the characteristic roots of S_1S^{-1} are the same as those of $[((n - r) / s)Y_1Y_1'(YY')^{-1}]$, where $Y_1(u \times s)$ and $Y(u \times (n - r))$ have, in general, i.e., under H, the distribution

$$(1.2.8) \quad (2\pi)^{-[u(n-r+s)]/2} \exp\left[\frac{-1}{2}\left\{\text{tr } (Y_1 Y_1' + YY') + \sum_{i=1}^{\min(u,s)} \gamma - 2\sum_{i=1}^{\min(u,s)} (Y_1)_{ii}\sqrt{\gamma_i}\right\}\right] dY_1\, dY,$$

where γ_i's $(i = 1, 2, \cdots, \min(u, s))$ are the possibly non-zero characteristic roots of the $u \times u$ matrix $\eta'[C_1(A_1'A_I)^{-1}C_1']^{-1}\eta(M'\Sigma M)^{-1}$. It is to be noted that the u characteristic roots of this matrix are all non-negative and t of them are positive while the rest, $(u - t)$ in number, are zero, where $t(\leqq \min(u, s))$ is the rank of η. All the roots are zero if, and only if, $\eta = 0$, i.e., under H_0, and in this case we have, for Y_1 and Y, the distribution

$$(1.2.9) \quad (2\pi)(^{-[u(n-r+s)]/2} \exp\left[-\tfrac{1}{2} \text{ tr } (Y_1Y_1' + YY')\right]dY_1\, dY.$$

The distribution of $c_{\max}(S_1S^{-1})$, i.e., of $c_{\max}[(n - r / s)Y_1Y_1'(YY')^{-1}]$, on the null hypothesis H_0, was obtained earlier [11, 12] starting from (1.2.9), and this forms the basis of the tables, now being prepared, giving $c_\alpha(u, s, n - r)$ when α, u, s and $(n - r)$ are prescribed. It may be noted, from (1.2.8) and (1.2.9), that Y_1 and Y are independently distributed.

We can introduce here, just as in the univariate case, the notion of two or more different hypotheses like (1.2.1) being testable in a *quasi-independent* manner and can derive a set of necessary and sufficient conditions for this. In fact, when the hypotheses differ only in their C matrices and have the same M matrix, so that, we have, for instance

$$(1.2.10) \quad H_{0i}: \; C_i(s_i \times m)\xi(m \times p)M(p \times u) = 0(s_i \times u), \qquad \text{for } i = 1, 2, \cdots, k,$$

against respective alternatives H_i, like H defined under (1.2.1), where rank (C_i) = rank $[\underset{r}{C_{i1}}\; \underset{(m-r)}{C_{i2}}]s_i = s_i$, and $\sum_{i=1}^{k} s_i \leqq r \leqq m < n$, then, the necessary and sufficient condition for being able to test the k hypotheses (1.2.10) in a *quasi-independent* manner is that

$$(1.2.11) \quad C_{i1}(A_I'A_I)^{-1}C_{j1}' = 0(s_i \times s_j), \qquad (i \neq j = 1, 2, \cdots, k),$$

which is the same as condition (1.3.7) of [10] for the univariate case.

1.3. *The associated confidence bounds.* Going back to (1.2.1), we observe that $\eta(\neq 0)$ represents a deviation from H_0. The main results follow [9].

With a joint confidence coefficient $\geqq (1 - \alpha)$, for a preassigned α, we have the following simultaneous confidence bounds:

$$c_{\max}^{\frac{1}{2}}(sS_1) - [sc_\alpha]^{\frac{1}{2}}c_{\max}^{\frac{1}{2}}(S) \leqq c_{\max}^{\frac{1}{2}}[\eta'(C_1(A_I'A_I)^{-1}C_1')^{-1}\eta] \leqq c_{\max}^{\frac{1}{2}}(sS_1) + [sc_\alpha]^{\frac{1}{2}}c_{\max}^{\frac{1}{2}}(S); \tag{1.3.1}$$

and similar confidence bounds in terms of the same c_α but of truncated η, S_1 and S obtained by going back to (1.2.1) for η, to (1.2.6) for S_1 and to (1.2.7) for S, and then (i) cutting out any row of η' and the corresponding row of M', or any two rows of η' and the corresponding two rows of M' and so on till we get down to just any row of η' and the corresponding row of M', and also (ii) cutting out any column of η' and the corresponding row of C_1, any two columns of η' and the two corresponding rows of C_1 and so on; and finally (iii) combining any case of truncation under (i) with any case of truncation under (ii). Thus, with a joint probability $\geqq (1 - \alpha)$, we have $(2^u - 1) \times (2^s - 1)$ statements of which (1.3.1) is the first one.

As in the univariate case [10, Section 1.4], it is to be observed that, taken together, the $(2^u - 1) \times (2^s - 1)$ confidence statements enable us to put confidence bounds on parametric functions which are not only measures of deviation from the total null hypotheses (1.2.1) but also on all the component parts of it.

It may be noted that if, in the H_0 of (1.2.1), $M(p \times u)$ is not present, as will be seen in the hypotheses of Section 2.4, then all the above results go through if we replace u by p and $M(p \times u)$ by the identity matrix $I(p)$.

2. Multivariate Variance Components.

2.1 *The multivariate Model II*: Let $X(p \times n) = [\mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_n]$ be a set of n observable stochastic p-vectors such that

$$\begin{aligned} X'(n \times p) &= A(n \times m)\xi(m \times p) + \epsilon(n \times p), \qquad m < n, \\ &= {}_n[A_1\, A_2 \cdots A_k]_{m_1\, m_2 \cdots m_k} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_k \end{bmatrix}_p \begin{matrix} m_1 \\ m_2 \\ \vdots \\ m_k \end{matrix} + \epsilon(n \times p), \sum_{i=1}^k m_i = m, \quad \text{(say)}, \end{aligned} \tag{2.1.1}$$

where A is the *design* matrix of rank $r \leqq m < n$, and where

(i) $\xi_i(m_i \times p)$ is a random sample of size m_i from the p-variate non-singular normal population $N[\boldsymbol{\mu}_i(p \times 1), \Sigma_i(p \times p)]$ for $i = 1, 2, \cdots, k$, and $\epsilon(n \times p)$ and ξ_i's (for $i = 1, 2, \cdots, k$) are mutually independent;

(ii) $\epsilon(n \times p)$, whose elements are physically of the nature of errors, is a random sample of size n from the p-variate non-singular $N[\mathbf{O}(p \times 1), \Sigma(p \times p)]$. Furthermore, we assume that $p \leqq (n - r)$.

Writing

$$X'(n \times p) = {}_n[\mathbf{x}_1\, \mathbf{x}_2 \cdots \mathbf{x}_p] \quad \text{and} \quad \mathbf{x}(pn \times 1) = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_p \end{bmatrix},$$

we see that, under the above model, the elements of $X'(n \times p)$, i.e., of $\mathbf{x}(pn \times 1)$, have a pn-variate non-singular normal distribution $N[E(\mathbf{x}),\ \Sigma^*(pn \times pn)]$, where

$$E(\mathbf{x})(pn \times 1) = A^*(pn \times pm) \begin{bmatrix} \mu_{11}\cdot\mathbf{1} \\ \vdots \\ \mu_{k1}\cdot\mathbf{1} \\ \hline \vdots \\ \hline \mu_{1p}\cdot\mathbf{1} \\ \vdots \\ \mu_{kp}\cdot\mathbf{1} \end{bmatrix} \begin{matrix} m_1, \\ \vdots \\ m_k \\ \vdots \\ m_1 \\ \vdots \\ m_k \end{matrix} \tag{2.1.2}$$

and where

$$\boldsymbol{\mu}_i(p \times 1) = \begin{bmatrix} \mu_{i1} \\ \mu_{i2} \\ \vdots \\ \mu_{ip} \end{bmatrix} \quad \text{and} \quad \begin{aligned} A^*(pn \times pm) &= A(n \times m)\cdot\times I(p) \\ &= \begin{matrix} n \\ n \\ \cdot \\ n \end{matrix}\begin{bmatrix} A & 0 & \cdots & 0 \\ 0 & A & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & A \end{bmatrix}, \\ &\qquad\ \ m \quad\ m \quad \cdots \quad m \end{aligned}$$

and

$$\begin{aligned} \Sigma^*(pn \times pn) &= E(\mathbf{x}\mathbf{x}') - E(\mathbf{x})E(\mathbf{x}') \\ &= A_1A_1'\cdot\times\Sigma_1 + A_2A_2'\cdot\times\Sigma_2 + \cdots + A_kA_k'\cdot\times\Sigma_k \\ &\qquad + I(n)\cdot\times\Sigma \end{aligned} \tag{2.1.3}$$

if we recall and use the Kronecker product notation $A\cdot\times B$.

We shall, in this paper, consider, in detail, only the relatively more restricted model wherein

$$\Sigma_i(p \times p) = \sigma_i^2\Sigma(p \times p), \tag{2.1.4}$$

since, as will be shown in section 2.3, the more general set-up of the model defined above does not lend itself to an easy mathematical treatment. We shall call the model defined at the beginning of this section, taken together with the restriction (2.1.4), as the restricted multivariate Model II of ANOVA. Federer [3] points out that models, where dispersion matrices are proportional, have been tentatively proposed for a certain type of genetical problem so that our restricted model might still be meaningful in certain physical situations.[2]

[2] Since this paper was written up and submitted in July, 1957 further investigation showed that even without this (rather severe and unrealistic) restriction it was still possible to go ahead with (i) point estimation, (ii) testing of hypothesis and (iii) confidence interval estimation, but in terms of a different set of statistics leading up to results less sharp than those aimed at here. The mathematical tools needed are those given here plus some further tools. Thus, from a physical standpoint, this paper might be regarded as an indispensable first step toward handling the more realistic situation that does not involve the very restrictive assumption of proportionality. The justification of the present paper from a physical standpoint, in terms of a possible genetical application, is thus today entirely redundant.

Our objectives will be: (i) to estimate any *estimable* linear function of the elements of $\mathfrak{y}_1, \cdots, \mathfrak{y}_k$ and to test *testable* linear hypotheses on $\mathfrak{y}_1, \cdots, \mathfrak{y}_k$; (ii) to obtain estimates of, and test hypotheses on, the multivariate variance components, viz., the characteristic roots $c(\Sigma_1), c(\Sigma_2), \cdots, c(\Sigma_k)$ and $c(\Sigma)$; and (iii) to obtain confidence bounds (simultaneous and/or separate) on $c(\Sigma_1), \cdots, c(\Sigma_k)$ and $c(\Sigma)$. Under the restricted Model II, of course, (ii) is equivalent to obtaining estimates of, and testing hypotheses on, $\sigma_1^2, \cdots, \sigma_k^2$ and $c(\Sigma)$, while (iii) is equivalent to obtaining confidence bounds on $\sigma_1^2, \sigma_2^2, \cdots, \sigma_k^2$ and $c(\Sigma)$.

2.2 *Linear estimation and testing of linear hypotheses.* Recall that for the restricted k-way classification the design matrix $A(n \times m)$ is such that, for all $i = 1, 2, \cdots, k$, the submatrix $A_i(n \times m_i)$ has one and only one non-zero element, equal to unity, in each row, such that rank $(A) = (m - k + 1)$. When we select n individuals under this design and measure each on not one but p variates we have a multivariate restricted k-way classification analogous to the univariate restricted k-way classification discussed in [10]. Multivariate analogues of the usual complete and incomplete *connected* designs are included under this general case. Using a result given in [12] we can establish the following lemma [4, pp. 96–97]:

LEMMA 1: *For the multivariate restricted k-way classification, the necessary and sufficient condition for the estimability of* $\sum_{i=1}^{k} \mathbf{l}_i'(1 \times p)\mathfrak{y}_i(p \times 1)$ *is that*

$$\mathbf{l}_1'(1 \times p) = \mathbf{l}_2'(1 \times p) = \cdots = \mathbf{l}_k'(1 \times p),$$

so that, a linear function $\sum_{i=1}^{k} \mathbf{l}_i'(1 \times p)\mathfrak{y}_i(p \times 1)$ *of all the elements of* $\mathfrak{y}_1, \cdots, \mathfrak{y}_k$, *which is estimable, and hypotheses on which are testable, is of the form* $\mathbf{l}'(1 \times p)[\mathfrak{y}_1 + \cdots + \mathfrak{y}_k]$, *and hence, neither linear functions of the elements of each* $\mathfrak{y}_i$ *nor the elements of each* $\mathfrak{y}_i$ *are separately* estimable *and linear hypotheses on these separate functions or elements are not* testable.

2.3. Estimation of the multivariate variance components. Analogous to the univariate χ^2-distribution, we shall introduce, for the multivariate situation, the pseudo-Wishart distribution a definition of which follows.

Suppose $X(p \times n)$ has the distribution

$$(2.3.1) \qquad (2\pi)^{-pn/2} |\Sigma|^{-n/2} \exp\left[\frac{-1}{2} \operatorname{tr} \Sigma^{-1}(X - \zeta)(X' - \zeta')\right] dX$$

where the elements of X, x_{ij} are such that $-\infty < x_{ij} < \infty$ $(i = 1, \cdots, p)$ $(j = 1, \cdots, n)$ so that, $E(X) = \zeta(p \times n)$ and the symmetric positive definite matrix $\Sigma(p \times p)$ is interpreted as, $n\Sigma(p \times p) = E[(X - \zeta)(X' - \zeta')]$. Then we shall call the distribution of the symmetric at least positive semi-definite matrix, $S(p \times p) = (1/n)XX'$, the pseudo-Wishart distribution with degrees of freedom n, and the distribution is central or non-central according as $\zeta = 0$ or $\zeta \neq 0$, i.e., according as $\zeta\zeta'$ is the null matrix $0(p \times p)$ or not. Conversely, we shall say that any symmetric at least positive semi-definite matrix, $S(p \times p)$, has the pseudo-Wishart distribution (in general, non-central) with degrees of freedom n, if we can write $S(p \times p) = (1/n)X(p \times n)X'(n \times p)$, where $X(p \times n)$ has the distribution (2.3.1). Further, if $E(X)E(X')$ is the null matrix $0(p \times p)$ then, and then only, will the distribution be said to be central. In particular, if in the above definition, $p \leqq n$ and rank $(S) = p$ then a pseudo-Wishart distribution for S is equivalent to the ordinary Wishart distribution.

Starting from (2.3.1), it can be shown that the distribution of the ith diagonal element, for $i = 1, 2, \cdots, p$, of XX', where of course $(1/n)XX'$ has a pseudo-Wishart distribution with degrees of freedom n, is distributed as $\sigma_{ii}\chi^2_{(n)}$ where σ_{ii} denotes the ith diagonal element of $\Sigma(p \times p)$ and where $\chi^2_{(n)}$ stands for the χ^2 variate, with degrees of freedom n, being central or non-central according as the pseudo-Wishart distribution of $(1/n)XX'$ is central or non-central.

We shall next proceed to problems of estimating and testing hypotheses on the multivariate variance components. For these purposes, by analogy with the univariate case [10, section 2.3], we shall seek $(k + 1)$ matrices, $S_i(p \times p) = (1/n_i)X(p \times n)Q_i(n \times n)X'(n \times p)$ (for $i = 0, 1, \cdots, k$), where $Q_i(n \times n)$ is symmetric and at least positive semi-definite of rank $n_i \leqq n$, $\sum_{i=0}^{k} n_i \leqq n$, such that

(2.3.2) $(1/\lambda_i)S_i$, of rank $\leqq p$, has a central pseudo-Wishart distribution with degrees of freedom n_i $(i = 0, 1, \cdots, k)$, where λ_i is a positive constant;

$$(2.3.3) \qquad \frac{1}{\lambda_0}S_0, \quad \frac{1}{\lambda_1}S_1, \cdots, \frac{1}{\lambda_k}S_k \quad \text{are mutually independent.}$$

LEMMA 2: *If $X(p \times n)$ has the distribution* (2.3.1), *then a set of necessary and sufficient conditions for $S_0, S_1, \cdots, S_k$ to satisfy the above restrictions is given by*

(a) $Q_i^2(n \times n) = \lambda_i Q_i$ *which is equivalent to the statement that* $Q_i = \lambda_i L_i' L_i$ *where* $L_i(n_i \times n)L_i'(n \times n_i) = I(n_i)(i = 0, 1, \cdots, k)$;

(b) $E(X)Q_iE(X') = 0(p \times p)$;

(c) $Q_iQ_j = 0(n \times n)$ *which, taken together with* (a), *is equivalent to the statement that* $L_i(n_i \times n)L_j'(n \times n_j) = 0(n_i \times n_j)$ $(i \neq j = 0, 1, \cdots, k)$.

PROOF: *Necessity of* (a), (b), *and* (c). Suppose that the matrices $S_0, S_1, \cdots, S_k$ satisfy (2.3.2) and (2.3.3). Then, since

$$(1 / \lambda_i)S_i = (1/n_i\lambda_i)XQ_iX' = XP_iX'$$

(where $P_i(n \times n) = (1/\lambda_i)Q_i(n \times n)$) has a central pseudo-Wishart distribution with degrees of freedom n_i, therefore, by our previous discussion, the jth $(j = 1, 2, \cdots, p)$ diagonal element of XP_iX' is *necessarily* distributed as a constant times $\chi^2_{(n_i)}$ where $\chi^2_{(n_j)}$ denotes a (central) χ^2 variate with n_j degrees of freedom. Now, if

$$X(p \times n) = \underset{n}{\begin{bmatrix} \mathbf{x}_1' \\ \mathbf{x}_2' \\ \vdots \\ \mathbf{x}_p' \end{bmatrix}}$$

(say) has the distribution (2.3.1) then $\mathbf{x}_j'(1 \times n)$ has the distribution

$$\frac{1}{(2\pi\sigma_{jj})^{n/2}} \exp\left[-\frac{1}{2\sigma_{jj}}(\mathbf{x}_j' - \zeta_j')(\mathbf{x}_j - \zeta_j)\right] d\mathbf{x}_j,$$

where

$$\zeta(p \times n) = E(X(p \times n)) = \underset{n}{\begin{bmatrix} \zeta_1' \\ \vdots \\ \zeta_p' \end{bmatrix}}$$

and σ_{jj} is the jth diagonal element of $\Sigma(p \times p)$. Using the result of [2, 6], we have that, if $\mathbf{x}_j(n \times 1)$ has the above distribution, then, in order that the jth diagonal element of XP_iX', i.e., $\mathbf{x}_j'P_i\mathbf{x}_j$, where $P_i(n \times n)$ is symmetric at least positive semi-definite of rank $n_i \leqq n$, may be distributed as $\sigma_{jj}\chi^2_{(n_i)}$, we must *necessarily* have

$$P_i^2 = P_i, \text{ i.e. } \quad Q_i^2 = \lambda_i Q_i.$$

Hence the necessity of (a).

Next, since $Q_i(n \times n)$ is symmetric positive semi-definite of rank n_i and λ_i is a positive constant, therefore, by a well-known result [12, pp. A-16 and A-17], there exists a transformation

$$\frac{1}{\lambda_i} Q_i = \begin{matrix} n_i \\ n - n_i \end{matrix}\underset{n_i}{\begin{bmatrix} \tilde{T}_1 \\ T_2 \end{bmatrix}} [\tilde{T}_1' \;\; T_2'].$$

If now

$$Y_i(p \times n_i) = X(p \times n) \begin{bmatrix} \tilde{T}_1 \\ T_2 \end{bmatrix},$$

then

$$\frac{1}{n_i\lambda_i} XQ_i X' = \frac{1}{n_i} Y_i Y_i'.$$

Thus, if $(1/n_i\lambda_i)XQ_iX' = (1/n_i)Y_iY_i'$ has a central pseudo-Wishart distribution, then, by definition, $E(Y_i)E(Y_i') = 0(p \times p)$, i.e., $E(X)Q_iE(X') = 0(p \times p)$, which proves the necessity of (b).

Finally, if $(1/\lambda_0)S_0, \cdots, (1/\lambda_k)S_k$ are distributed mutually independently in pseudo-Wishart forms with respective degrees of freedom $n_0, \cdots, n_k$, then, necessarily, their lth $(l = 1, 2, \cdots, p)$ diagonal elements, viz.,

$$\mathbf{x}_l'P_0\mathbf{x}_l, \cdots, \mathbf{x}_l'P_k\mathbf{x}_l,$$

where $P_i = (1/\lambda_i)Q_i\,(i = 0, 1, \cdots, k)$, are distributed as constant multiples of mutually independent χ^2 variates with respective degrees of freedom n_0,

$n_1, \cdots$ and n_k. Hence, from [2, 6], we necessarily have

$$P_iP_j = 0(n \times n) \quad \text{or} \quad Q_iQ_j = 0(n \times n)$$

for $i \neq j = 0, 1, \cdots, k$, which proves the necessity of (c).

Sufficiency of (a), (b), *and* (c). We shall now assume that $S_0, S_1, \cdots, S_k$ satisfy (a), (b) and (c), so that

$$\frac{1}{\lambda_i} Q_i(n \times n) = L_i'(n \times n_i)L_i(n_i \times n)$$

where $L_iL_i' = I(n_i)$ for $i = 0, 1, \cdots, k$ and $L_iL_j' = 0(n_i \times n_j)$ for $i \neq j = 0, 1, \cdots, k$, and, also, $E(X)Q_iE(X') = 0(p \times p)$ for $i = 0, 1, \cdots, k$. When $L_0, L_1, \cdots, L_k$ satisfy these conditions, it is well known that we can find a completion $L^*((n - \sum_{i=0}^k n_i) \times n)$ of the matrix

$$\begin{bmatrix} L_0 \\ L_1 \\ \vdots \\ L_k \end{bmatrix}$$

such that the completed matrix

$$L(n \times n) = \begin{bmatrix} L_0 \\ L_1 \\ \vdots \\ L_k \\ L^* \end{bmatrix}$$

is orthogonal. Let us now make the transformation

$$Y(p \times n) = p[\underset{n_0}{Y_0}\, \underset{n_1}{Y_1} \cdots \underset{n_k}{Y_k} \quad \underset{\left(n - \sum_{i=0}^{k} n_i\right)}{Y^*}] = X(p \times n)[L_0' \; L_1' \cdots L_k' \; L^{*\prime}],$$

so that, the Jacobian of the transformation is unity. Notice that

$$YY' = \sum_{i=0}^{k} Y_i Y_i' + Y^* Y^{*\prime} = \sum_{i=0}^{k} \frac{1}{\lambda_i} XQ_i X' + XL^{*\prime}L^*X'.$$

Starting from the distribution (2.3.1) of $X(p \times n)$, we therefore have for the joint distribution of $Y_0, \cdots, Y_k$ and Y^*

$$(2\pi)^{-pn/2} |\Sigma|^{-n/2} \exp\left[\frac{-1}{2} \operatorname{tr} \Sigma^{-1}\left\{\sum_{i=0}^{k} (Y_i - \eta_i)(Y_i' - \eta_i') + (Y^* - \eta^*)(Y^{*\prime} - \eta^{*\prime})\right\}\right] \times \prod_{i=0}^{k} dY_i \cdot dY^*,$$

where

$$p[\eta_0 \; \eta_1 \; \cdots \; \eta_k \quad \underset{\left(n - \sum_{i=0}^{k} n_i\right)}{\eta^*}] = \zeta(p \times n)[L_0' \; L_1' \; \cdots \; L_k' \; L^{*\prime}],$$
$$\quad n_0 \; n_1 \; \cdots \; n_k$$

so that, $E(Y_i) = \eta_i \, (i = 0, 1, \cdots, k)$, $E(Y^*) = \eta^*$. The elements of each Y matrix, of course, vary between $-\infty$ and ∞. Integrating out over Y^*, we have for the joint distribution of $Y_0, Y_1, \cdots, Y_k$,

$$(2.3.4) \qquad (2\pi)^{\frac{-p(n_0+\cdots+n_k)}{2}} |\Sigma|^{-\frac{n_0+\cdots+n_k}{2}} \cdot \exp\left[\frac{-1}{2} \operatorname{tr} \Sigma^{-1}\left\{\sum_{i=0}^{k} (Y_i - \eta_i)(Y_i' - \eta_i')\right\}\right] dY_0 \cdots dY_k,$$

where, of course, $E(Y_i) = \eta_i$ and $E[(Y_i - \eta_i)(Y_i' - \eta_i')] = n_i\Sigma(p \times p)$ for $i = 0, 1, \cdots, k$. From (2.3.4), it follows, by definition, that

$$(2.3.5) \qquad \frac{1}{n_i} Y_i Y_i' = \frac{1}{n_i} XL_i' L_i X' = \frac{1}{n_i \lambda_i} XQ_i X' = \frac{1}{\lambda_i} S_i,$$

for $i = 0, 1, \cdots, k$, has a pseudo-Wishart distribution with n_i degrees of freedom. Also, if $E(X)Q_iE(X') = 0(p \times p)$, then, since λ_i is a positive constant' $(1/\lambda_i)E(X)Q_iE(X') = E(X)L_i'L_iE(X') = E(Y_i)E(Y_i') = \eta_i\eta_i' = 0(p \times p)$. Hence, again by definition, the pseudo-Wishart distribution of $(1/\lambda_i)S_i$ $(i = 0, 1, \cdots, k)$ is central. Finally, from (2.3.4), we observe that $Y_0, Y_1, \cdots, Y_k$ are mutually independent, and, hence it follows from (2.3.5) that

$$(1/\lambda_0)S_0, \cdots, (1/\lambda_k)S_k$$

are mutually independent.

Hence the sufficiency of (a), (b) and (c).

LEMMA 3: *If $X(p \times n)$ has the distribution*

$$(2.3.6) \quad (2\pi)^{-pn/2} |\Sigma|^{-n/2} |B|^{-p/2} \exp[-\tfrac{1}{2} \operatorname{tr} \Sigma^{-1}(X - \zeta)B^{-1}(X' - \zeta')] \, dX, \qquad -\infty < x_{ij} < \infty,$$

where $B(n \times n)$ and $\Sigma(p \times p)$ are symmetric positive definite, then, a set of necessary and sufficient conditions for $S_0, S_1, \cdots, S_k$ (defined immediately before (2.3.2)) to satisfy the conditions (2.3.2) *and* (2.3.3) *is given by*

(α) $\quad Q_iBQ_i = \lambda_iQ_i, \qquad i = 0, 1, \cdots, k;$

(β) $\quad E(X)Q_iE(X') = 0(p \times p), \qquad i = 0, 1, \cdots, k;$ and

(γ) $\quad Q_iBQ_j = 0(n \times n), \qquad i \neq j = 0, 1, \cdots, k.$

PROOF: Since $B(n \times n)$ is symmetric positive definite, therefore, there exists the transformation

$$B(n \times n) = \tilde{T}(n \times n)\tilde{T}'(n \times n),$$

so that,

$$B^{-1} = (\tilde{T}')^{-1}\tilde{T}^{-1}$$

Writing $Y(p \times n) = X(p \times n)\,(\tilde{T}')^{-1}$, or, $X = Y\tilde{T}'$, and $\theta(p \times n) = \zeta(p \times n)\,(\tilde{T}')^{-1}$, we have the Jacobian of the transformation to be $|\tilde{T}'|^p = |B|^{p/2}$. Then we notice that $(1 / n_i\lambda_i)\, XQ_i X' = (1 / n_i\lambda_i) Y\tilde{T}'Q_i \tilde{T}Y'$, and, from (2.3.6), the distribution of $Y(p \times n)$ is

$$(2\pi)^{-pn/2}\, |\Sigma|^{-n/2} \exp\left[-\tfrac{1}{2} \operatorname{tr} \Sigma^{-1}(Y - \theta)(Y' - \theta')\right] dY, \qquad -\infty < y_{ij} < \infty,$$

which is of the same form as (2.3.1). Now applying Lemma 2 to the matrices $(1 / n_i \lambda_i) Y\tilde{T}'Q_i \tilde{T}Y'$ (notice that rank $(\tilde{T}'Q_i\tilde{T})$ = rank $(Q_i) = n_i$), we obtain that a set of necessary and sufficient conditions for these matrices to satisfy (2.3.2) and (2.3.3) is

(α) $\quad \tilde{T}'Q_i\tilde{T}\tilde{T}'Q_i\tilde{T} = \lambda_i\tilde{T}'Q_i\tilde{T}$, or, $Q_iBQ_i = \lambda_iQ_i \quad (i = 0, 1, \cdots, k)$;

(β) $\quad E(Y)\tilde{T}'Q_i\tilde{T}E(Y') = 0(p \times p)$, or $E(X)Q_iE(X') = 0(p \times p)$, $(i = 0, 1, \cdots, k)$; and, finally,

(γ) $\quad \tilde{T}'Q_i\tilde{T}\tilde{T}'Q_j\tilde{T} = 0(n \times n)$, or, $Q_iBQ_j = 0(n \times n)$,

$$(i \neq j = 0, 1, \cdots, k).$$

Hence the lemma is proved.

Next, under the general multivariate Model II, we have noted that

$$\mathbf{x}(pn \times 1) = \begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_p \end{bmatrix}$$

has the distribution

$$(2.3.7) \qquad (2\pi)^{-pn/2}\, |\Sigma^*|^{-1/2} \cdot \exp\left[-\frac{1}{2}\left\{[(\mathbf{x}_1' \cdots \mathbf{x}_p') - E(\mathbf{x}_1' \cdots \mathbf{x}_p')]\, \Sigma^{*-1} \left[\begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_p \end{bmatrix} - E\begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_p \end{bmatrix}\right]\right\}\right] d\mathbf{x},$$

where

$$E(\mathbf{x}) = E\begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_p \end{bmatrix}$$

and Σ^* are defined respectively in (2.1.2) and (2.1.3). In order that this distribution of $X(p \times n)$ be, essentially, of the same form as (2.3.6), we should be able to express the exponent in (2.3.7), except for a constant factor $(-1) / 2$, in the form

$$\operatorname{tr} M_2^{-1}\,(X - \zeta)\, M_1^{-1}\,(X' - \zeta'),$$

where $M_1(n \times n)$ and $M_2(p \times p)$ are symmetric positive definite matrices and where $E(X) = \zeta$.

LEMMA 4: *A necessary and sufficient condition for this is that*

$$\Sigma^*(pn \times pn) = M_1(n \times n) \cdot \times M_2(p \times p). \tag{2.3.8}$$

PROOF: *Sufficiency of the condition.* If $\Sigma^* = M_1 \cdot \times M_2$ then it is known [5] that $\Sigma^{*-1} = M_1^{-1} \cdot x\, M_2^{-1}$. Now, let

$$M_1^{-1}(n \times n) = \begin{bmatrix} m_{11}^{(1)} & m_{12}^{(1)} & \cdots & m_{1n}^{(1)} \\ m_{21}^{(1)} & m_{22}^{(1)} & \cdots & m_{1n}^{(1)} \\ \vdots & \vdots & \cdots & \vdots \\ m_{n1}^{(1)} & m_{n2}^{(1)} & \cdots & m_{nn}^{(1)} \end{bmatrix}, \qquad m_{ij}^{(1)} = m_{ij}^{(1)},$$

and

$$M_2^{-1}\,(p \times p) = \begin{bmatrix} m_{11}^{(2)} & m_{12}^{(2)} & \cdots & m_{1p}^{(2)} \\ m_{21}^{(2)} & m_{22}^{(2)} & \cdots & m_{2p}^{(2)} \\ \vdots & \vdots & \cdots & \vdots \\ m_{p1}^{(2)} & m_{p2}^{(2)} & \cdots & m_{pp}^{(2)} \end{bmatrix}, \qquad m_{ij}^{(2)} = m_{ji}^{(2)}.$$

Then,

$$[\mathbf{x}_1' - E(\mathbf{x}_1'), \quad \cdots, \quad \mathbf{x}_p' - E(\mathbf{x}_p')]\, \Sigma^{*-1} \begin{bmatrix} \mathbf{x}_1 & - & E(\mathbf{x}_1) \\ & \vdots & \\ \mathbf{x}_p & - & E(\mathbf{x}_p) \end{bmatrix}$$

$$= \sum_{i,j=1}^{p} [\mathbf{x}_i' - E(\mathbf{x}_i')]\, M_1^{-1}\,(n \times n) \cdot m_{ij}^{(2)}\, [\mathbf{x}_j - E(\mathbf{x}_j)]$$

$$= \operatorname{tr} M_2^{-1} \begin{bmatrix} \mathbf{x}_1' & - & E(\mathbf{x}_1') \\ & \vdots & \\ \mathbf{x}_p' & - & E(\mathbf{x}_p') \end{bmatrix} \qquad M_1^{-1}\,[\mathbf{x}_1 - E(\mathbf{x}_1), \cdots, \mathbf{x}_p - E(\mathbf{x}_p)]$$

$$= \operatorname{tr} M_2^{-1}\,[X - E(X)]\, M_1^{-1}\,[X' - E(X')].$$

Hence the sufficiency of the condition (2.3.8).

Necessity of the condition. Supposing now that

$$[\mathbf{x}_1' - E(\mathbf{x}_1'), \quad \cdots, \quad \mathbf{x}_p' - E(\mathbf{x}_p')\,]\Sigma^{*-1} \begin{bmatrix} \mathbf{x}_1 & - & E(\mathbf{x}_1) \\ & \vdots & \\ \mathbf{x}_p & - & E(\mathbf{x}_p) \end{bmatrix}$$

$$= \operatorname{tr} M_2^{-1}\,[X - E(X)]\, M_1^{-1}\,[X' - E(X')],$$

and writing M_1^{-1} and M_2^{-1} as before, we can argue backwards in the proof of the sufficiency of the condition and obtain that $\Sigma^{*-1} = M_1^{-1} \cdot \times M_2^{-1}$, so that, $\Sigma^* = M_1 \cdot \times M_2$.

Hence the lemma is proved.

Further, we can establish that for $\Sigma^*(pn \times pn)$, defined in (2.1.3), under the general Multivariate Model II with a perfectly general design matrix, to be expressible as $M_1 \cdot\times M_2$ we have the necessary and sufficient conditions

$$\Sigma_i(p \times p) = \sigma_i^2\, \Sigma(p \times p), \qquad (i = 1, 2, \cdots, k),$$

which yield the restricted Model II. That these conditions on $\Sigma_i(p \times p)$ are sufficient is easily verified. That they are necessary can be demonstrated as follows where, for simplicity of argument, we assume that $p = 2$.

Suppose $\Sigma^*(pn \times pn) = \Sigma^*(2n \times 2n)$, since $p = 2$, here,

$$= M_1(n \times n) \cdot\times M_2(2 \times 2),$$

where M_1 and M_2 are symmetric positive definite matrices. Then, from (2.1.3), we have

$$A_1A_1' \cdot\times \Sigma_1 + \cdots + A_kA_k' \cdot\times \Sigma_k + I(n) \cdot\times \Sigma = M_1 \cdot\times M_2 .$$

From this we obtain the equations

$$\begin{aligned} (2.3.9)\qquad & A_1A_1' [\sigma_{11}^{(1)} - c_1\sigma_{22}^{(1)}] + A_2A_2' [\sigma_{11}^{(2)} - c_1\sigma_{22}^{(2)}] + \cdots \\ & \qquad + A_kA_k' [\sigma_{11}^{(k)} - c_1\sigma_{22}^{(k)}] + I(n)\, [\sigma_{11} - c_1\sigma_{22}] = 0(n \times n), \end{aligned}$$

and

$$\begin{aligned} & A_1A_1' [\sigma_{11}^{(1)} - c_2\sigma_{12}^{(1)}] + A_2A_2' [\sigma_{11}^{(2)} - c_2\sigma_{12}^{(2)}] + \cdots \\ & \qquad + A_kA_k' [\sigma_{11}^{(k)} - c_2\sigma_{12}^{(k)}] + I(n)\, [\sigma_{11} - c_2\sigma_{12}] = 0(n \times n), \end{aligned}$$

where $c_1 = (M_2)_{11} / (M_2)_{22}$, $c_2 = (M_2)_{11} / (M_2)_{12}$, $(M_2)_{ij}$ is the ijth $(i, j = 1, 2)$ element of $M_2(2 \times 2)$, and where

$$\Sigma_i(2 \times 2) = \begin{bmatrix} \sigma_{11}^{(i)} & \sigma_{12}^{(i)} \\ \sigma_{12}^{(i)} & \sigma_{22}^{(i)} \end{bmatrix}$$

for $i = 1, 2, \cdots, k$ and

$$\Sigma(2 \times 2) = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}.$$

For the equations (2.3.9) to hold we must have either,

$$\sigma_{11}^{(1)} / \sigma_{22}^{(1)} = \sigma_{11}^{(2)} / \sigma_{22}^{(2)} = \cdots = \sigma_{11}^{(k)} / \sigma_{22}^{(k)} = \sigma_{11} / \sigma_{22} = c_1 ,$$

and

$$\sigma_{11}^{(1)} / \sigma_{12}^{(1)} = \sigma_{11}^{(2)} / \sigma_{12}^{(2)} = \cdots = \sigma_{11}^{(k)} / \sigma_{12}^{(k)} = \sigma_{11} / \sigma_{12} = c_2 ,$$

or,

$$A_iA_i' = a_iI(n), \qquad (i = 1, 2, \cdots, k)$$

where a_i is a scalar constant. These latter conditions on the submatrices of the design matrix are too restrictive and unrealistic, so that, for a perfectly general design matrix, the former conditions hold necessarily if $\Sigma^* = M_1 \cdot\times M_2$, and they are verified to be equivalent to the conditions, $\Sigma_i = \sigma_i^2\Sigma$, for $i = 1, 2, \cdots, k$, where σ_i^2 are certain positive constants. The proof of the necessity of the conditions for general p follows exactly along the same lines.

We have thus set up, for reasons of easier mathematical treatment, the restricted multivariate Model II mentioned in section 2.1, and under this restricted set-up we have, for $X(p \times n)$, the distribution

$$(2.3.10) \quad (2\pi)^{(-pn/2)} |\Sigma|^{(-n/2)} \left|\sum_{i=1}^{k} \sigma_i^2 A_i A_i' + I(n)\right|^{(-p/2)}$$

$$\cdot \exp\left[-\frac{1}{2} \operatorname{tr} \Sigma^{-1}(X - E(X))\left(\sum_{i=1}^{k} \sigma_i^2 A_i A_i' + I(n)\right)^{-1}(X' - E(X'))\right] dX,$$

since $|\Sigma^*| = |M_1(n \times n) \cdot\times M_2(p \times p)| = |M_1|^p \, |M_2|^n$ by [5].

Next, suppose that Q_i $(n \times n)$ $(i = 0, 1, \cdots, k)$ is a symmetric at least positive semi-definite matrix of rank n_i $(\leqq n)$ such that $E(X)QE(X') = 0(n \times n)$; then, we have, under the multivariate Model II,

$$\Lambda_i\,(p \times p) = \frac{1}{n_i} E(XQ_iX')$$

$$(2.3.11) \qquad = \frac{1}{n_i} E \begin{bmatrix} \mathbf{x}_1' Q_i \mathbf{x}_1, & \cdots, & \mathbf{x}_1' Q_i \mathbf{x}_p \\ \mathbf{x}_2' Q_i \mathbf{x}_1, & \cdots, & \mathbf{x}_2' Q_i \mathbf{x}_p \\ \vdots & \cdots & \vdots \\ \mathbf{x}_p' Q_i \mathbf{x}_1, & \cdots, & \mathbf{x}_p' Q_i \mathbf{x}_p \end{bmatrix}$$

$$= \frac{1}{n_i}\left[\sum_{j=1}^{k} \operatorname{tr}(A_j A_j' Q_i)\, \Sigma_j + (\operatorname{tr} Q_i)\, \Sigma\right],$$

by using Lemma 3 of [10] and simplifying,

$$= \frac{1}{n_i}\left[\sum_{j=i}^{k} \sigma_j^2 \operatorname{tr}(A_j A_j' Q_i) + \operatorname{tr} Q_i\right] \Sigma(p \times p),$$

for the restricted multivariate Model II.

Also, if $(1/\lambda_i)S_i\,(p \times p) = (1\,/\,n_i\lambda_i)\,XQ_iX'$ has a central pseudo-Wishart distribution with degrees of freedom n_i (= rank of Q_i) then, under the restricted Multivariate Model II where $X(p \times n)$ has the distribution (2.3.10), we have

$$E(S_i) = \lambda_i\Sigma = \Lambda_i\,(p \times p),$$

so that, from (2.3.11),

$$(2.3.12) \qquad \lambda_i = \frac{1}{n_i}\left[\sum_{j=1}^{k} \sigma_j^2 \operatorname{tr}(A_j A_j' Q_i) + \operatorname{tr} Q_i\right].$$

Again, under the restricted multivariate Model II, if we apply the conditions (α), (β), and (γ) of Lemma 3 (remembering that $B(n \times n)$ of (2.3.6) is now replaced by $\sum_{i=1}^{k} \sigma_i^2 A_iA_i' + I(n))$ to the $(k+1)$ matrices $(1/\lambda_0)S_0, \cdots (1/\lambda_k) S_k$, and then require, as in the univariate case discussed in [10, Section 2.3], that these matrices satisfy the conditions (α) and (γ) for all $\sigma_1^2, \cdots, \sigma_k^2$, we have, after some simplification,

$$(2.3.13) \quad \left.\begin{aligned} Q_i A_l A_l' Q_i &= \left[\frac{1}{n_i} \operatorname{tr} A_l A_l' Q_i\right] Q_i, \quad l = 1, 2, \cdots, k; \\ Q_i^2 &= \left[\frac{1}{n_i} \operatorname{tr} Q_i\right] Q_i \end{aligned}\right\} (i = 0, 1, \cdots, k),$$

and

$$(2.3.14) \quad Q_iA_lA_l'Q_j = 0(n \times n), \quad l = 1, 2, \cdots, k; \quad Q_iQ_j = 0(n \times n) \quad \text{for } i \neq j = 0, 1, \cdots, k.$$

It is seen that these conditions, (2.3.13) and (2.3.14), are exactly the same as those obtained for the univariate problem [Cf. (2.3.4) and (2.3.5) of [10]]. Thus, for a given design matrix $A(n \times m)$, the same $Q_0, Q_1, \cdots, Q_k$ which satisfy (2.3.4) and (2.3.5) of [10] for the univariate case, also satisfy (2.3.13) and (2.3.14) under the restricted multivariate Model II set-up. Also, for given $Q_0, Q_1, \cdots, Q_k$, the same design matrix $A(n \times m)$, which satisfies (2.3.4) and (2.3.5) of [10] under the univariate Model II set-up, also satisfies (2.3.13) and (2.3.14) under the restricted multivariate Model II set-up.

We shall next present a tie-up, for the multivariate restricted k-way classification, between the analysis under Model I and the analysis under Model II.

2.4. *Tie-up between the analyses under the multivariate Models I and II for the restricted k-way classification.* We recall from section 1.2 that, under the multivariate Model I, we can obtain k matrices due to the k *testable* hypotheses of equality of the row vectors of $\xi_i(m_i \times p)$ $(i = 1, 2, \cdots, k)$, which can, by analogy with (1.2.1), be written as

$$(2.4.1) \quad \begin{aligned} H_{0i} : C_i((m_i - 1) \times m)\xi(m \times p) &= \underset{m-r}{[C_{i1} \quad C_{i2}]}\xi, \qquad \text{where } r = rank(A) = m - k + 1 \text{ and } m = \sum_{i=1}^{k} m_i, \\ &= (m_i - 1)\left[\begin{array}{c|c|c|cccc|c|c|c} \underset{m_1}{0} & \underset{m_2}{0} & \cdots & 1 & 0 & \cdots & 0 \; -1 & 0 & \cdots & \underset{m_k}{0} \\ & & & 0 & 1 & \cdots & 0 \; -1 & & & \\ & & & \cdot & \cdot & \cdots & \cdot \quad \cdot & & & \\ & & & 0 & 0 & \cdots & \underset{m_i}{1 \; -1} & & & \end{array}\right] \cdot \xi \\ &= 0((m_i - 1) \times p), \end{aligned}$$

so that rank $(C_i) = (m_i - 1)$ $(i = 1, 2, \cdots, k)$. As in section 1.2, we can obtain *k matrices due to the k hypotheses* $H_{01}, H_{02}, \cdots, H_{0k}$, viz.,

$$XA_I(A_I'A_I)^{-1}C_{i1}' [C_{i1}(A_I'A_I)^{-1}C_{i1}']^{-1} C_{i1}(A_I'A_I)^{-1}A_I'X' \tag{2.4.2}$$

for $i = 1, 2, \cdots, k$, and these matrices are symmetric and at least positive semi-definite of rank, almost everywhere, $\min(p, (m_i - 1))$. We have, also, the *matrix due to error*

$$X[I(n) - A_I(A_I'A_I)^{-1}A_I']X' \tag{2.4.3}$$

which is symmetric positive definite (almost everywhere) since we assume in the model that $p \leqq (n - r)$, where for the restricted k-way classification $r = (m - k + 1)$.

Now, under the multivariate Model II, in the notation of section 2.3, suppose we take $n_0S_0 = XQ_0X'$ as the *matrix due to error* given by (2.4.3) with $n_0 = (n - r) = (n - m + k - 1)$, and $n_iS_i = XQ_iX'$, for $i = 1, 2, \cdots, k$, as the *matrices due to hypothesis* given by (2.4.2) with $n_i = \text{rank}\ (Q_i) = (m_i - 1)$. Notice that $\sum_{k}^{i=0} n_i = (n - 1) < n$. It is seen that these Q_i, $i = 0, 1, \cdots, k$, are the same as those for the univariate case [Cf. section 2.4 of [10]]. We may verify that all these Q_i's are such that $E(X)Q_iE(X') = 0(p \times p)$ under the multivariate Model II and hence we can obtain that

$$\begin{aligned} \Lambda_i(p \times p) &= \frac{1}{n_i} E(XQ_i X') \\ &= \nu_i \Sigma_i + \Sigma, \end{aligned} \tag{2.4.4}$$

for the general multivariate Model II, where

$$\begin{aligned} \nu_i &= \frac{2}{(m_i - 1)} \left\{ \begin{matrix} \text{sum of the elements along and below the diagonal of} \\ [C_{i1}(A_I' A_I)^{-1}C_{i1}']^{-1} \end{matrix} \right\} \\ &= (\nu_i \sigma_i^2 + 1)\Sigma, \end{aligned}$$

for the restricted multivariate Model II, $(i = 1, 2, \cdots, k)$, and

$$\Lambda_0 = \frac{1}{n_0} E(XQ_0 X') = \Sigma.$$

Therefore, under the restricted multivariate Model II, we have, for the above $Q_i(n \times n)$, that

$$\lambda_i = \nu_i\sigma_i^2 + 1, \qquad i = 1, 2, \cdots, k \text{ and } \lambda_0 = 1. \tag{2.4.5}$$

If now, under the restricted multivariate Model II, we apply the conditions (α), (β), and (γ) of Lemma 3 to the set of matrices $(1/\lambda_0)S_0$, $(1/\lambda_1)S_1, \cdots, (1/\lambda_k)S_k$, taken as above, we notice that they all satisfy (β) so that their distributions are all central (if they are pseudo-Wishart at all). Furthermore, $(1/\lambda_0)S_0 = S_0$ (by 2.4.5)), where n_0S_0, the *matrix due to error*, can be seen to have the central pseudo-Wishart distribution (in fact, the ordinary Wishart distribution, since S_0 is positive definite here) with degrees of freedom n_0, and to be also distributed independently of $(1/\lambda_i)S_i$ for $i = 1, 2, \cdots, k$. Also, by applying (α) and (γ) of Lemma 3 to $(1/\lambda_i)S_i$ $(i = 1, 2, \cdots, k)$, we observe that they are distributed mutually independently in central pseudo-Wishart forms with respective degrees of freedom $n_1, n_2, \cdots, n_k$, if and only if,

$$C_{i1}(A_I' A_I)^{-1}C_{i1}' = \frac{1}{\nu_i}[I(m_i - 1) + J((m_i - 1) \times (m_i - 1))], \quad (i = 1, \cdots, k), \tag{2.4.6}$$

and

$$C_{i1}(A_I'A_I)^{-1}C_{j1}' = 0((m_i - 1) \times (m_j - 1)), \quad (i \neq j = 1, 2, \cdots, k), \tag{2.4.7}$$

where we recall that $I(p)$ denotes the identity matrix of order p and $J(p \times q)$ denotes a matrix of p rows and q columns all of whose elements are equal to unity. The conditions (2.4.6) and (2.4.7), which are independent of the unknown variance components, are the same as (2.4.5) and (2.4.6) of [10] for the univariate case.

Recalling the remarks toward the end of section 2.3, we observe that these conditions, (2.4.6) and (2.4.7), are both satisfied by the multivariate analogues of the usual univariate complete block designs.

Finally, it may be seen from (2.4.4) that we can take $(1/\nu_i)$ $(S_i - S_0)$ as an unbiased estimate of Σ_i $(p \times p)$, for $i = 1, \cdots, k$, and S_0 as an unbiased estimate of $\mathbf{\Sigma}(p \times p)$. We may, therefore, use $c[(1/\nu_i)$ $(S_i - S_0)]$ as estimates of $c(\Sigma_i)$ and $c(S_0)$ as estimates of $c(\Sigma)$.

2.5 *Tests of hypotheses on the multivariate variance components.* The usual null hypotheses may be stated as

$$H_{0i}\text{:} \quad \Sigma_i(p \times p) = 0(p \times p), \text{ or, equivalently, } c(\Sigma_i) = 0, \quad \text{for } i = 1, 2, \cdots, k. \tag{2.5.1}$$

It is easily seen, from (2.4.4), that for the restricted k-way classification H_{0i} is equivalent to $\Lambda_i(p \times p) = \Lambda_0(p \times p)$ $(i = 1, 2, \cdots, k)$, or, for the restricted multivariate Model II, to the hypotheses $\lambda_i = \lambda_0$ $(i = 1, 2, \cdots, k)$. The alternative to this last form is taken to be H_{1i}: $\lambda_i > \lambda_0$. Assuming that the restricted k-way classification has matrices like (2.4.2) and (2.4.3) which satisfy both (2.4.6) and (2.4.7), we have, by definition of the pseudo-Wishart distribution, that $XQ_0X' = Y_0(p \times n_0)Y_0'(n_0 \times p)$ and $XQ_iX' = Y_i(p \times n_i)Y'(n_i \times p)$ $(i = 1, 2, \cdots, k)$, where Y_0 and Y_i have the joint distribution

$$\text{const. exp}\left[-\tfrac{1}{2}\text{ tr } \Sigma^{-1}\left\{\frac{1}{\lambda_0} Y_0 Y_0' + \frac{1}{\lambda_i} Y_i Y_i'\right\}\right] dY_0\, dY_i\,,$$

and where $E(Y_0Y_0') = n_0\lambda_0\Sigma$ and $E(Y_iY_i') = n_i\lambda_i\Sigma$.

Consider $\mathbf{a}'$ $(1 \times p)Y_0(p \times n_0)$ and $\mathbf{a}'$ $(1 \times p)Y_i(p \times n_i)$ for all nonnull

$\mathbf{a}(p \times 1)$. Then, $(1/n_0)E(\mathbf{a}'Y_0Y_0'\mathbf{a}) = \lambda_0\mathbf{a}'\Sigma\mathbf{a}$ and $(1/n_i)E(\mathbf{a}'Y_iY_i'\mathbf{a}) = \lambda_i\mathbf{a}'\Sigma\mathbf{a}$. For testing $\lambda_i = \lambda_0$ against $\lambda_i > \lambda_0$ we may take as critical region

$$w_{i\mathbf{a}} : F_{\mathbf{a}}(n_i, n_0) = \frac{\mathbf{a}'Y_i Y_i' \mathbf{a} / n_i}{\mathbf{a}'Y_0 Y_0' \mathbf{a} / n_0} \geqq F_\alpha(n_i, n_0),$$

where $F_\alpha(n_i, n_0)$ is the upper 100α % point of the central F-distribution with n_i and n_0 degrees of freedom. Taking, $w_i = \bigcap_{\mathbf{a}} w_{i\mathbf{a}}$, as a critical region for the hypothesis (2.5.1) we obtain that

$$w_i\colon \quad c_{\max}(S_iS_0^{-1}) \geqq c_\alpha^*(p, n_i, n_0)$$

which is seen, from (1.2.4), to be the critical region of the test, at a level α^*, for the hypothesis (2.4.1) under the multivariate Model I discussed under Section 1 of this paper.

It must be noted that the above arguments for deriving a test were made solely to obtain, for the customary null hypotheses under the restricted multivariate Model II, if possible, a critical region which is the same as the one for the customary null hypotheses under the multivariate Model I. The use of the union-intersection principle to obtain w_i from $w_{i\mathbf{a}}$ is rather artificial since we do not have H_{0i} itself as an intersection of hypotheses $H_{0i\mathbf{a}}$.

2.6. *Confidence statements.* We shall first assume that we are dealing with restricted k-way classifications that have matrices like (2.4.2) and (2.4.3) satisfying both (2.4.6) and (2.4.7). As observed before, the multivariate analogues of the usual univariate complete block designs satisfy these requirements. Under the restricted multivariate Model II, we shall then obtain simultaneous confidence bounds on $\sigma_1^2, \cdots, \sigma_k^2$ and $c(\Sigma)$. Next, we shall relax the condition (2.4.7), i.e., we shall not require that the pseudo-Wishart distributions of the matrices like (2.4.2) in our analysis be independent. Under this relaxation, we shall obtain an alternate set of confidence bounds for the individual σ_i^2's $(i = 1, 2, \cdots, k)$.

If $(1/\lambda_i)S_i\,(p \times p) = (1/n_i\lambda_i)XQ_iX'$, for $i = 0, 1, \cdots, k$, have independent central pseudo-Wishart distributions with respective degrees of freedom n_0, n_1, $\cdots$ and n_k, then, by definition, we have $(1/n_i\lambda_i)XQ_iX' \doteq$

$$(1/n_i)Y_i\,(p \times n_i)Y_i'(n_i \times p),$$

for $i = 0, 1, \cdots, k$, where the joint distribution of $Y_0, Y_1, \cdots, Y_k$ is

$$(2\pi)^{-[p(n-1)]/2}\,|\Sigma|^{-(n-1)/2} \exp\left[-\tfrac{1}{2}\operatorname{tr}\Sigma^{-1}\left\{\sum_{i=0}^{k} Y_i Y_i'\right\}\right] dY_0 \cdots dY_k, \tag{2.6.1}$$

$$-\infty < \text{all elements of } Y_i < \infty,$$

and where $E(Y_iY_i') = n_i\Sigma(p \times p)$ and $\sum_{k}^{i=0} n_i = (n-1)$. It is well known that, for the symmetric positive definite matrix Σ, there exists an orthogonal matrix, $\Gamma(p \times p)$, such that $\Sigma(p \times p) = \Gamma' D_\gamma \Gamma$, where the p (non-zero) elements of the diagonal matrix D_γ are the p (positive) characteristic roots of Σ. Now making the transformation

$$(2.6.2) \qquad D_{1/\sqrt{\gamma}}\,\Gamma Y_i(p \times n_i) = Z_i\,(p \times n_i), \qquad (i = 0, 1, \cdots, k),$$

we can verify that the Jacobian is $|\Sigma|^{(n-1)/2}$, so that, the joint distribution of $Z_0, Z_1, \cdots, Z_k$ is

$$(2.6.3) \qquad (2\pi)^{-[p(n-1)]/2} \exp\left[-\tfrac{1}{2}\,\mathrm{tr}\left\{\sum_{i=0}^{k} Z_i Z_i'\right\}\right] dZ_0 \cdots dZ_k,$$

$$-\infty < \text{all elements of } Z_i < \infty.$$

From (2.6.3), it can be seen, by analogy with the methods used in [8, 9, 12], that we can obtain constants, $\mu_{i1}(p, n_i, \alpha_i) = \mu_{i1}$ (say) and $\mu_{i2}(p, n_i, \alpha_i) = \mu_{i2}$ (say), for $i = 0, 1, \cdots, k$, such that the statement

$$(2.6.4) \qquad \mu_{i1} \leqq c_{\min}(Z_iZ_i') \leqq c_{\max}(Z_iZ_i') \leqq \mu_{i2},$$

has probability $(1 - \alpha_i)$ and the probability of statements like (2.6.4) holding simultaneously for $i = 0, 1, \cdots, k$ is $(1 - \alpha) = \prod_{i=0}^{k}(1 - \alpha_i)$. We note that $Z_0Z_0' = (n_0/\lambda_0)D_{1/\sqrt{\gamma}}\,\Gamma S_0\Gamma' D_{1/\sqrt{\gamma}}$, where n_0S_0 $(p \times p)$ is the *matrix due to error* given by (2.4.3) and is symmetric positive definite. Therefore, starting from (2.6.4), with $i = 0$, and reasoning exactly as in section 1 of [9], we obtain the confidence statement

$$(2.6.5) \qquad \frac{n_0}{\mu_{01}}\,c_{\max}(S_0) \geqq c_{\max}(\Sigma) \geqq c_{\min}(\Sigma) \geqq \frac{n_0}{\mu_{02}}\,c_{\min}(S_0)$$

with confidence coefficient $\geqq (1 - \alpha_0)$.

Next, for any $i = 1, 2, \cdots, k$, we note that (2.6.4) is equivalent to

$$(2.6.6) \qquad \mu_{i1} \leqq \frac{n_i}{\lambda_i}\,c_{\min}(D_{1/\gamma}\,\Gamma S_i\,\Gamma') \leqq \frac{n_i}{\lambda_i}\,c_{\max}(D_{1/\gamma}\Gamma S_i\Gamma') \leqq \mu_{i2}.$$

However, it is known that the non-zero characteristic roots of $A(p \times q)$ $B\,(q \times p)$ are the same as those of $B(q \times p)A(p \times q)$ and that $c_{\min}(A_1)$ $c_{\min}(A_2) \leqq$ all $c(A_1A_2) \leqq c_{\max}(A_1)\,c_{\max}(A_2)$ where A_1 $(p \times p)$ is symmetric positive definite and $A_2(p \times p)$ is symmetric at least positive semi-definite. [Cf. pp. A-5 and A-7 of [12] for proofs.] Using these two results we have

$$c_{\min}(\Gamma S_i\,\Gamma' D_{1/\gamma}) \leqq \frac{c_{\min}(S_i)}{c_{\min}(\Sigma)} \quad \text{and} \quad c_{\max}(\Gamma S_i\,\Gamma' D_{1/\gamma}) \geqq \frac{c_{\max}(S_i)}{c_{\max}(\Sigma)},$$

so that, (2.6.6) implies the statement

$$(2.6.7) \qquad \frac{n_i}{\mu_{i2}}\,\frac{c_{\max}(S_i)}{c_{\max}(\Sigma)} \leqq \lambda_i \leqq \frac{n_i}{\mu_{i1}}\,\frac{c_{\min}(S_i)}{c_{\min}(\Sigma)},$$

which, therefore, has a probability $\geqq (1 - \alpha_i)$.

Taking the statement (2.6.5) together with all statements like (2.6.7) for

$i = 1, 2, \cdots, k$, we obtain the simultaneous statements

$$(2.6.8)\quad \begin{gathered} \frac{n_0}{\mu_{02}} c_{\min}(S_0) \leqq c_{\min}(\Sigma) \leqq c_{\max}(\Sigma) \leqq \frac{n_0}{\mu_{01}} c_{\max}(S_0) \\ \frac{n_1}{\mu_{12}} \frac{c_{\max}(S_1)}{c_{\max}(\Sigma)} \leqq \lambda_1 \leqq \frac{n_1}{\mu_{11}} \frac{c_{\min}(S_1)}{c_{\min}(\Sigma)} \\ \cdots \qquad \cdots \qquad \cdots \\ \frac{n_k}{\mu_{k2}} \frac{c_{\max}(S_k)}{c_{\max}(\Sigma)} \leqq \lambda_k \leqq \frac{n_k}{\mu_{k1}} \frac{c_{\min}(S_k)}{c_{\min}(\Sigma)} \end{gathered}$$

with a joint probability $\geqq (1 - \alpha) = \prod_{j=0}^{k} (1 - \alpha_j)$.

Recalling now, from (2.4.5), that $\lambda_i = \nu_i \sigma_i^2 + 1$ ($i = 1, 2, \cdots, k$), and using the leading statement in (2.6.8), we obtain the further statements implied by (2.6.8)

$$(2.6.9)\quad \begin{gathered} \frac{n_0}{\mu_{02}} c_{\min}(S_0) \leqq c_{\min}(\Sigma) \leqq c_{\max}(\Sigma) \leqq \frac{n_0}{\mu_{01}} c_{\max}(S_0) \\ \frac{1}{\nu_1}\left[\frac{n_1}{n_0}\frac{\mu_{01}}{\mu_{12}}\frac{c_{\max}(S_1)}{c_{\max}(S_0)} - 1\right] \leqq \sigma_1^2 \leqq \frac{1}{\nu_1}\left[\frac{n_1}{n_0}\frac{\mu_{02}}{\mu_{11}}\frac{c_{\min}(S_1)}{c_{\min}(S_0)} - 1\right] \\ \cdots \qquad \cdots \qquad \cdots \\ \frac{1}{\nu_k}\left[\frac{n_k}{n_0}\frac{\mu_{01}}{\mu_{k2}}\frac{c_{\max}(S_k)}{c_{\max}(S_0)} - 1\right] \leqq \sigma_k^2 \leqq \frac{1}{\nu_k}\left[\frac{n_k}{n_0}\frac{\mu_{02}}{\mu_{k1}}\frac{c_{\min}(S_k)}{c_{\min}(S_0)} - 1\right] \end{gathered}$$

which, therefore, are a set of simultaneous confidence bounds on $\sigma_1^2, \sigma_2^2, \cdots, \sigma_k^2$ and all $c(\Sigma)$ with a joint confidence coefficient $\geqq (1 - \alpha)$, for a preassigned α. These simultaneous confidence bounds on the set of $c(\Sigma)$ and $\sigma_1^2, \cdots, \sigma_k^2$ are obtained on the assumption of independence between $(1/\lambda_i)S_i$ (for $i = 1, 2, \cdots, k$). If this assumption were relaxed we would still be able to obtain individual confidence bounds on $\sigma_1^2, \cdots, \sigma_k^2$ and the set of all $c(\Sigma)$, although the simultaneous confidence bounds in this situation would be far more difficult to obtain.

We shall next obtain the alternate set of separate confidence bounds for $\sigma_1^2, \sigma_2^2, \cdots$ and σ_k^2.

If $Y_0(p \times n_0)$ and $Y_i(p \times n_i)$, where $p \leqq n_0$ but may be $\gtreqless n_i$, are such that rank $(Y_0Y_0') = p$ and rank $(Y_iY_i') = \min(p, n_i)$, and further if Y_0 and Y_i have the joint distribution

$$(2.6.10)\quad (2\pi)^{-[p(n_0+n_i)]/2} \mid \Sigma \mid^{-(n_0+n_i)/2} \exp[-\tfrac{1}{2} \operatorname{tr}\Sigma^{-1}\{Y_0Y_0' + Y_iY_i'\}]dY_0\,dY_i,$$

where $\Sigma(p \times p)$ is symmetric positive definite and $E(Y_0Y_0') = n_0\Sigma$, $E(Y_iY_i') = n_i\Sigma$, then, Rao [7], in continuation of the work of Bartlett and Wald, has shown that, for large m_i, $-m_i \log_e \Lambda_i$ has the central χ^2-distribution with pn_i degrees of freedom, where

$$(2.6.11)\quad \Lambda_i = \frac{|Y_0\,Y_0'|}{|Y_0\,Y_0' + Y_i\,Y_i'|} \qquad \text{and} \qquad m_i = \left[n_0 + n_i - \frac{p + n_i + 1}{2}\right].$$

Hence, we can find $\chi^2_{1\alpha_i}$ and $\chi^2_{2\alpha_i}$ such that the statement

$$\chi^2_{1\alpha_i} \leqq -m_i \log_e \Lambda_i \leqq \chi^2_{2\alpha_i}, \tag{2.6.12}$$

or, equivalently

$$\mu_{1\alpha_i} \leqq \Lambda_i \leqq \mu_{2\alpha_i}, \qquad [\mu_{1\alpha_i} = \exp(-\tfrac{1}{2}\chi^2_{2\alpha_i}), \qquad \mu_{2\alpha_i} = \exp(-\tfrac{1}{2}\chi^2_{1\alpha_i})],$$

has a probability $(1 - \alpha_i)$ for a preassigned α_i.

Under the restricted multivariate Model II, for a restricted k-way classification, if we take the matrices $(1/\lambda_0)S_0, (1/\lambda_1)S_1, \cdots, (1/\lambda_k)S_k$ as in Section 2.4, then we have seen that $(1/\lambda_0)S_0$ is distributed in the central Wishart form with n_0 degrees of freedom and it is distributed independently of $(1/\lambda_1)S_1, \cdots, (1/\lambda_k)S_k$. We have, also, by (2.4.5), that $\lambda_i = \nu_i\sigma_i^2 + 1$ $(i = 1, \cdots, k)$ and $\lambda_0 = 1$. Further, if $(1/\lambda_1)S_1, \cdots, (1/\lambda_k)S_k$ satisfy the condition (2.4.6), so that they have central pseudo-Wishart distributions with degrees of freedom $n_1, \cdots, n_k$ (even though these may not be independent), then, by definition, we can write $n_0S_0 = Z_0(p \times n_0)Z_0'(n_0 \times p)$ and

$$(n_i/\lambda_i)S_i = Z_i(p \times n_i)Z_i'(n_i \times p).$$

The joint distribution of Z_0 and Z_i is then of the same form as (2.6.10), and, by analogy with the statements (2.6.10) – (2.6.12), we can find, for large m_i, constants (depending on a central χ^2-distribution with pn_i degrees of freedom), $\mu_{1\alpha_i}$ and $\mu_{2\alpha_i}$, such that the statement

$$\mu_{1\alpha_i} \leqq \frac{|n_0 S_0|}{\left|n_0 S_0 + \frac{n_i}{\lambda_i} S_i\right|} \leqq \mu_{2\alpha_i}, \tag{2.6.13}$$

or equivalently,

$$\frac{1}{\mu_{1\alpha_i}} \geqq |\zeta_i S_i S_0^{-1} + I(p)| \geqq \frac{1}{\mu_{2\alpha_i}}, \tag{2.6.14}$$

where $\zeta_i = (n_i/n_0\lambda_i) > 0$, will have a probability $(1 - \alpha_i)$. If rank $(S_i) = \min(p, n_i) = s_i$ (say), then (2.6.14) is seen to be equivalent to

$$\begin{aligned}\frac{1}{\mu_{1\alpha_i}} \geqq (\zeta_i)^{s_i} \operatorname{tr}_{s_i}(S_i S_0^{-1}) + (\zeta_i)^{s_i-1} \operatorname{tr}_{s_i-1}(S_i S_0^{-1}) + \cdots \\ \cdots + \zeta_i \operatorname{tr}(S_i S_0^{-1}) + 1 \geqq \frac{1}{\mu_{2\alpha_i}},\end{aligned} \tag{2.6.15}$$

where $\operatorname{tr}_s(A)$ denotes the sum of all sth order principal minors of A. Using certain matrix factorization theorems given in [12, pp. A-15–A-17], we can prove that $\operatorname{tr}_s(S_iS_0^{-1}) = \operatorname{tr}_s$ [a symmetric at least positive semi-definite matrix] > 0 for $s \leqq s_i$. Hence, all the coefficients of powers of ζ_i in the middle part of (2.6.15) are real and positive. Next, since $|\zeta_iS_iS_0^{-1} + I(p)| > 1$, in order that the bounds in (2.6.15) may be non-trivial, we should have $1/\mu_{2\alpha_i} > 1$.

Considering now the equality signs in (2.6.15), we obtain the equations

$$(2.6.16)\qquad \begin{aligned} (\zeta_i)^{s_i} \operatorname{tr}_{s_i}(S_i S_0^{-1}) + \cdots + \zeta_i \operatorname{tr}(S_i S_0^{-1}) - \left(\frac{1}{\mu_{2\alpha_i}} - 1\right) &= 0 \\ (\zeta_i)^{s_i} \operatorname{tr}_{s_i}(S_i S_0^{-1}) + \cdots + \zeta_i \operatorname{tr}(S_i S_0^{-1}) - \left(\frac{1}{\mu_{1\alpha_i}} - 1\right) &= 0. \end{aligned}$$

From well-known results in the theory of equations, it now follows that the equations (2.6.16) each have one and only one positive real root. Let these positive real roots be denoted by $\theta_{2\alpha_i}$ and $\theta_{1\alpha_i}$. Then it is seen that (2.6.14) or (2.6.15) is equivalent to

$$(2.6.17)\qquad \theta_{1\alpha_i} \geqq \zeta_i \geqq \theta_{2\alpha_i}$$

with a probability $(1 - \alpha_i)$. Recalling that $\zeta_i = (n_i / n_0\lambda_i)$ and $\lambda_i = \nu_i\sigma_i^2 + 1$, we see that (2.6.17) is equivalent to the confidence interval statement

$$(2.6.18)\qquad \frac{1}{\nu_i}\left[\frac{n_i}{n_0\,\theta_{1\alpha_i}} - 1\right] \leqq \sigma_i^2 \leqq \frac{1}{\nu_i}\left[\frac{n_i}{n_0\,\theta_{2\alpha_i}} - 1\right]$$

with a confidence coefficient $(1 - \alpha_i)$, for a preassigned α_i. We thus have, for $i = 1, 2, \cdots, k$, separate confidence interval statements for each of $\sigma_1^2, \sigma_2^2, \cdots$ and σ_k^2, but, due to the complexity of the distribution problem involved, it would be far more difficult to obtain simultaneous confidence bounds on $\sigma_1^2, \sigma_2^2, \cdots$, and σ_k^2 by this method. Nor would the difficulty be appreciably reduced, under this approach, even if we assumed that $(1/\lambda_i)S_i$'s (for $i = 1, 2, \cdots, k$) were independent as we did under the first approach.

2.7 *Concluding remarks*: After the work presented in this paper and in [10] had been completed, it was brought to the attention of the authors that Bose [1] has, for the univariate case, given a general treatment, using slightly different methods, of a mixed model with one set of random components. A very recent paper by Zelen [13] also has some results, for the univariate case on a mixed model with one set of random components as applied to Incomplete Block Designs, which are contained in [10].

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