BALANCED FACTORIAL EXPERIMENTS[1]

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1. Introduction and summary. Usually, in a factorial experiment, the block size of the experiment is not large enough to permit all possible treatment combinations to be included in a block. Hence we resort to the theory of confounding. With respect to symmetric factorial designs, the theory of confounding has been highly developed by Bose [1], Bose and Kishen [2] and Fisher [4], [5]. An excellent summary of the results of this research appears in Kempthorne [6]. Some examples of asymmetric factorial designs can be found in Yates [14], Cochran and Cox [3], Li [8], Kempthorne [6] and Nair and Rao [9], [10]. Nair and Rao [11] have given the statistical analysis of a class of asymmetrical two-factor designs in considerable detail. The author [13] has considered the problem of achieving "complete balance" over various interactions in factorial experiments. In the present paper a class of factorial experiments, balanced factorial experiments (BFE) (Definition 4.2) is considered. The theorems proved in Section 5 outline a detailed analysis of BFE's, including estimates of various interactions at different levels. Finally, a method of constructing BFE's is given in Section 6.

It should be noted that Theorems 5.2 to 5.5 are generalisations of the corresponding theorems by Zelen [15], and the method of construction in Section 6 is a general form of the one indicated by Yates [14], Nair and Rao [9], [10] and Kempthorne [6] (Section 18.7).

2. Notation. Let there be v treatments, each replicated r times in b blocks of k plots each. Let $\mathbf{N} = [n_{ij}](i = 1, 2, \cdots, v; j = 1, 2, \cdots, b)$ be the incidence matrix of the design, whern n_{ij} is equal to the number of times the ith treatment occurs in the jth block. The set up assumed is

$$y_{ij} = \mu + t_i + b_j + \epsilon_{ij}, \tag{2.1}$$

where y_{ij} is the yield of the plot in the jth block to which the ith treatment is applied, μ is the over all effect, t_i is the effect of the ith treatment, b_j is the effect of the jth block, and ϵ_{ij} is the experimental error. The effects μ, t_i, b_j are assumed to be fixed constants, while the errors ϵ_{ij}'s are assumed to be independent normal variates with mean zero and variance σ^2. Let T_i be the total yield of all the plots having the ith treatment, B_j be the total yield of all the plots of the jth block and $\hat{t}_i$ be a solution for t_i in the normal equations. Further denote the column vectors with elements $\{T_1, T_2, \cdots, T_v\}$, $\{B_1, B_2, \cdots, B_b\}$, $\{t_1, t_2, \cdots, t_v\}$ and

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$\{\hat{t}_1, \hat{t}_2, \cdots, \hat{t}_v\}$ by **T**, **B**, **t** and $\hat{\mathbf{t}}$ respectively. It is well known that the reduced normal equations for the intra-block estimates of treatment contrasts are

$$\mathbf{Q} = \mathbf{C}\hat{\mathbf{t}}, \tag{2.2}$$

where

$$\mathbf{Q} = \mathbf{T} - \frac{1}{k}\mathbf{NB} \tag{2.3}$$

and

$$\mathbf{C} = r\mathbf{I}(v) - \frac{1}{k}\mathbf{NN}', \tag{2.4}$$

where $\mathbf{I}(v)$ is the $v \times v$ Identity matrix. The matrix **C** defined in (2.4) will be called the **C**-matrix of the design.

3. Some useful results.

DEFINITION 3.1. If $\mathbf{l}'\mathbf{l} = 1$($\mathbf{l}$ is a $v \times 1$ matrix), the contrast $\mathbf{l}'\mathbf{t}$ will be called a normalised contrast.

DEFINITION 3.2. A normalised contrast $\mathbf{l}'\mathbf{t}$ will be called a canonical contrast of the design, if $\mathbf{l}$ is a canonical vector of the **C**-matrix of the design.

LEMMA 3.1. *A necessary and sufficient condition for a normalised contrast* $\mathbf{l}'\mathbf{t}$ *to be a canonical contrast is*

$$\mathbf{l}'\mathbf{Q} = \theta\mathbf{l}'\hat{\mathbf{t}}, \tag{3.1}$$

where

$$\theta = r - \frac{1}{k}(\mathbf{l}'\mathbf{NN}'\mathbf{l}). \tag{3.2}$$

LEMMA 3.2. *A canonical contrast* $\mathbf{l}'\mathbf{t}$ *is estimable, if the* θ *given by* (3.2) *is not equal to zero and then*

$$\mathbf{l}'\hat{\mathbf{t}} = \mathbf{l}'\mathbf{Q}/\theta \tag{3.3}$$

and

$$V(\mathbf{l}'\hat{\mathbf{t}}) = \sigma^2/\theta. \tag{3.4}$$

LEMMA 3.3. *Let* $\mathbf{l}$ *be a normalised contrast and* θ *be given by* (3.2). *Then each of the following three conditions implies the other two:*

(i) $\theta = r$.

(ii) $\mathbf{N}'\mathbf{l} = 0$.

(iii) $\mathbf{l}'\mathbf{t}$ is estimable with the minimum variance σ^2/r.

Then

$$\mathbf{l}'\mathbf{Q} = \mathbf{l}'\mathbf{T}. \tag{3.5}$$

Hence $\mathbf{l}'\mathbf{t}$, *its variance and the sum of squares due to* $\mathbf{l}'\hat{\mathbf{t}}$ *are the same as those in randomised block design.*

LEMMA 3.4. *If* $\mathbf{l}_1'\mathbf{t}, \mathbf{l}_2'\mathbf{t}, \cdots, \mathbf{l}_n'\mathbf{t}$ *are n linearly independent contrasts, such that* $\mathbf{l}_i'\hat{\mathbf{t}}(i = 1, 2, \cdots, n)$ *is uncorrelated with the estimate of any contrast orthogonal to all of* $\mathbf{l}_i'\mathbf{t}$, *and i evefry normalised contrast of the form* $\mathbf{l}'\hat{\mathbf{t}} = \sum a_i\mathbf{l}_i'\hat{\mathbf{t}}$ *has the same variance, then any normalised contrast of the form* $\mathbf{l}'\mathbf{t}$ *is a canonical contrast. Further any two contrasts* $\sum a_i\mathbf{l}_i'\hat{\mathbf{t}}$ *and* $\sum b_i\mathbf{l}_i'\hat{\mathbf{t}}$ *are uncorrelated, if they are orthogonal.*

4. Factorial experiments. Let $F_1, F_2, \cdots, F_m$ be m factors at $s_1, s_2, \cdots, s_m$ levels respectively. Let $v = s_1s_2 \cdots s_m$ treatments be denoted by the levels of the factors as $(x_1x_2 \cdots x_m)$, where x_i is the level of the ith factor and takes values $0, 1, \cdots, s_i - 1$. Let $t(x_1x_2 \cdots x_m)$ be the effect of the treatment combination $(x_1x_2 \cdots x_m)$. The contrast $\sum c_{x_1x_2\cdots x_m}t(x_1x_2 \cdots x_m)$ [where summation is over all the values of $(x_1x_2 \cdots x_m)$] will be called a contrast belonging to the interaction $F_{i_1}F_{i_2} \cdots F_{i_q}$, if and only if $c_{x_1x_2\cdots x_m}$ is a function of the q levels $x_{i_1}, x_{i_2}, \cdots, x_{i_q}$ only and

$$\sum_{x_j=1}^{s_j} c_{x_1x_2\cdots x_m} = 0 \quad \text{for} \quad j = i_1, i_2, \cdots, i_q .$$

DEFINITION 4.1. "Complete balance" is achieved over an interaction, if and only if all the normalised contrasts belonging to the same interaction are estimated with the same variance.

DEFINITION 4.2. An experiment will be called a balanced factorial experiment (BFE), if the following conditions are satisfied:

(a) Each of the treatments is replicated the same number of times.

(b) Each of the blocks has the same number of plots.

(c) Estimates of contrasts belonging to different interactions are uncorrelated with each other.

(d) "Complete balance" is achieved over each of the interactions.

THEOREM 4.1. *A normalised contrast belonging to an interaction is a canonical contrast of a* BFE.

The proof of Theorem 4.1 follows from Definition 4.2 and Lemma 3.4.

In a factorial experiment, it is well known that each treatment effect can be expressed in terms of main effects and interactions as given by

$$t(x_1x_2 \cdots x_m) = \sum_{i=1}^{m} t(F_i)_{x_i} + \sum_{j=2}^{m}\sum_{i=1}^{j-1} t(F_iF_j)_{x_1x_j} + \cdots + t(F_1F_2 \cdots F_m)x_1x_2 \cdots x_m . \tag{4.1}$$

The $t(F_i)_{x_i}$ is a constant associated with the main effect of the factor F_i at the level x_i, the $t(F_iF_j)_{x_ix_j}$ is a constant associated with the interaction between the factors F_i and F_j at the levels x_i and x_j respectively, etc.

In further discussion that follows in Sections 4 and 5, we shall state and prove results for the first q factors $F_1, F_2, \cdots, F_q$ only, for the convenience of notation. However, the results are true for any q factors $F_{i_1}, F_{i_2}, \cdots, F_{i_q}$.

The parameters defined in (4.1) are not all linearly independent and satisfy the following relations:

$$\sum_{x_j=0}^{s_j-1} t(F_1F_2 \cdots F_q)_{x_1x_2\cdots x_q} = 0 \qquad \text{for } j = 1, 2, \cdots, q. \tag{4.2}$$

The estimate of $t(F_1F_2 \cdots F_q)_{x_1x_2\cdots x_q}$ will be denoted by $t(F_1F_2 \cdots F_q)_{x_1x_2\cdots x_q}$.

Following the notation used by Zelen [15] let us define S-functions as follows:

$$S(t; F_1F_2 \cdots F_q \mid x_1x_2 \cdots x_q) = \frac{1}{v} \prod_{j=1}^{q} s_j \sum{}' t(y_1y_2 \cdots y_m), \tag{4.3}$$

where $\sum'$ refers to the sum over all treatments which have the same levels $x_1, x_2, \cdots, x_q$ for the q factors $F_1, F_2, \cdots, F_q$ respectively. Then, it can be shown that

$$\begin{aligned} S(t; F_1F_2 \cdots F_q \mid x_1x_2 \cdots x_q) &= \sum_{j=1}^{q} t(F_j)_{x_j} + \sum_{j=2}^{q} \sum_{h=1}^{j=1} t(F_jF_h)_{x_jx_h} \\ &\quad + \cdots + t(F_1F_2 \cdots F_q)_{x_1x_2\cdots x_q}, \end{aligned} \tag{4.4}$$

$$t(F_1F_2 \cdots F_q)_{x_1x_2\cdots x_q} = (-1)^q \sum_{w=1}^{q} (-1)^w \{w(t)\}, \tag{4.5}$$

where $\{w(t)\}$ denotes the sum of the functions $S(t; \cdots)$ involving exactly w factors out of the q factors $F_1, F_2, \cdots, F_q$ only.

The equations (4.3) and (4.5) give S-functions and factor-interactions in terms of treatment effects. We define similar functions in terms of the adjusted treatment totals $Q(y_1y_2\cdots y_m)$ as follows:

$$S(Q; F_1F_2 \cdots F_q \mid x_1x_2 \cdots x_q) = \frac{1}{v} \prod_{j=1}^{q} s_j \sum{}' Q(y_1y_2 \cdots y_m) \tag{4.6}$$

and

$$Q(F_1F_2\cdots F_q)_{x_1x_2\cdots x_q} = (-1)^q \sum_{w=1}^{q} (-1)^w \{w(Q)\}, \tag{4.7}$$

where $\sum'$ is as defined in (4.3) and $\{w(Q)\}$ denotes the sum of S-functions for Q involving exactly w factors from $F_1, F_2, \cdots, F_q$ only. It can be shown that the functions defined in (4.6) and (4.7) satisfy the relations exactly similar to (4.1), (4.2) and (4.4).

LEMMA 4.1. *If* $\sum c_{x_1x_2\cdots x_m} t(x_1x_2\cdots x_m)$ *is any contrast belonging to the q-factor interaction* $F_1F_2\cdots F_q$*, then it can be expressed as a contrast in terms of the factor interactions* $t(F_1F_2\cdots F_q)_{x_1x_2\cdots x_q}$ *at different levels but belonging to the same q-factor interaction. A similar result holds also for the corresponding Q-functions.*

The proof of Lemma 4.1 follows from (4.1) and (4.2) with a little algebra.

5. Analysis of BFE. The following vectors, matrices, and matrix operators will be useful in later results.

(i) $\mathbf{t}(s_i)$ = the column vector $\{t(0), t(1), \cdots, t(s_i - 1)\}$.

(ii) $Q(s_i)$ = the column vector $\{Q(0), Q(1), \cdots, Q(s_i - 1)\}$.

(iii) $\mathbf{t}(F_i)$ = the column vector $\{t(F_i)_0, t(F_i)_1, \cdots, t(F_i)_{s_i-1}\}$.

(iv) $\mathbf{Q}(F_i)$ = the column vector $\{Q(F_i)_0, Q(F_i)_1, \cdots, Q(F_i)_{s_i-1}\}$.

(v) $\boldsymbol{\lambda}(1)$ = the column vector $\{\lambda_0, \lambda_1\}$.

(vi) $\boldsymbol{\theta}(1)$ = the column vector $\{\theta_0, \theta_1\}$.

(vii) $\mathbf{I}(m)$ = the $m \times m$ Identity matrix.

(viii) $\mathbf{I}^*(m)$ = the $m \times m$ matrix obtained by replacing 0 for 1 in the last row and last column of $\mathbf{I}(m)$.

(ix) $\mathbf{E}_{mn}$ = the $m \times n$ matrix with all the elements equal to unity.

(x) $\mathbf{E}(s)$ = $s^{-\frac{1}{2}}\mathbf{E}_{s1}$.

(xi) $\mathbf{L}(s)$ = an $s \times s - 1$ matrix whose columns are mutually orthogonal normalised vectors also orthogonal to $\mathbf{E}(s)$.

(xii) $\mathbf{M}(s)$ = $[\mathbf{L}(s)|\mathbf{E}(s)]$.

(xiii) $\mathbf{N}(s)$ = $\mathbf{I}(s) - \mathbf{E}(s)\mathbf{E}'(s) = \mathbf{L}(s)\mathbf{L}'(s)$.

(xiv) $\mathbf{G}(s) = \begin{bmatrix} s-1 & 1 \\ -1 & 1 \end{bmatrix}$.

The operator "$\times$" denotes the Kronecker product of matrices defined by

$$\text{(5.1)} \qquad \mathbf{A} \times \mathbf{B} = a_{ij} \times B = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \cdots & a_{2n}\mathbf{B} \\ \vdots & \vdots & & \vdots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix}.$$

The operator "$\otimes$" denotes the symbolic Kronecker product of subscripts and suffixes defined by the following illustrations:

$$\text{(5.2)} \qquad Q(3) \otimes Q(2) = \begin{bmatrix} Q(00) \\ Q(01) \\ Q(10) \\ Q(11) \\ Q(20) \\ Q(21) \end{bmatrix}.$$

$$\text{(5.3)} \qquad \begin{bmatrix} Q(F_i)_0 \\ Q(F_i)_1 \end{bmatrix} \otimes \begin{bmatrix} Q(F_j)_0 \\ Q(F_j)_1 \end{bmatrix} = \begin{bmatrix} Q(F_iF_j)_{00} \\ Q(F_iF_j)_{01} \\ Q(F_iF_j)_{10} \\ Q(F_iF_j)_{11} \end{bmatrix}.$$

THEOREM 5.1. *A* BFE *in m factors* $F_1, F_2, \cdots, F_m$ *at* $s_1, s_2, \cdots, s_m$ *levels respectively is a* PBIB *with relevant parameters and conversely. The two treatments are the* $p_1p_2\cdots p_m$*th associates, where* $p_i = 1$, *if the ith factor occurs at the same level in both the treatments and* $p_i = 0$ *otherwise;* $\lambda_{p_1p_2\cdots p_m}$ *will denote the number of times these treatments occur together in a block. Now, if any contrast belonging to the interaction* $F_{i_1}F_{i_2}\cdots F_{i_q}$ *is estimated with the variance*

$$\sigma^2/\theta_{q_1q_2\cdots q_m}, \tag{5.4}$$

where

$$q_j = \begin{cases} 1, & \text{if } j = i_1, i_2, \cdots, i_q; \\ 0, & \text{otherwise,} \end{cases} \tag{5.5}$$

then the relation between θ*'s and* λ*'s is*

$$\begin{aligned} \boldsymbol{\theta}(1) \otimes \boldsymbol{\theta}(1) \otimes \cdots \otimes \boldsymbol{\theta}(1) = -\frac{1}{k}[\mathbf{G}(s_1) \times \mathbf{G}(s_2) \times \cdots \times \mathbf{G}(s_m)] \\ \cdot [\boldsymbol{\lambda}(1) \otimes \boldsymbol{\lambda}(1) \otimes \cdots \otimes \boldsymbol{\lambda}(1)], \end{aligned} \tag{5.6}$$

where

$$\theta_{00\cdots0} = 0 \quad \text{and} \quad \lambda_{11\cdots1} = -r(k-1). \tag{5.7}$$

The proof of Theorem 5.1 follows from Theorem 6.1 of [12], on substituting $m_1 = m_2 = \cdots = m_h = 1$ and $h = m$.

THEOREM 5.2. *In a* BFE, *if a normalised contrast belonging to the interaction* $F_1F_2\cdots F_q$ *is estimated with the variance* σ^2/θ, *then the estimates of the same interaction at different levels are given by*

$$\hat{t}(F_1F_2\cdots F_q)_{x_1x_2\cdots x_q} = \frac{1}{\theta} Q(F_1F_2\cdots F_q)_{x_1x_2\cdots x_q}. \tag{5.8}$$

PROOF. Using Definition 4.2, Theorem 4.1 and Lemma 3.4, it can be shown that

$$\begin{aligned} \frac{1}{\theta} \mathbf{H}_1 \times \mathbf{H}_2 \times \cdots \times \mathbf{H}_m \cdot \mathbf{Q}(s_1) \otimes \mathbf{Q}(s_2) \otimes \cdots \otimes \mathbf{Q}(s_m) \\ = \mathbf{H}_1 \times \mathbf{H}_2 \times \cdots \times \mathbf{H}_m \cdot \hat{\mathbf{t}}(s_1) \otimes \hat{\mathbf{t}}(s_2) \otimes \cdots \otimes \hat{\mathbf{t}}(s_m), \end{aligned} \tag{5.9}$$

where

$$\mathbf{H}_j = \begin{cases} \mathbf{L}'(s_j), & \text{if } j = 1, 2, \cdots, q; \\ \mathbf{E}'(s_j), & \text{otherwise.} \end{cases} \tag{5.10}$$

By the substitution of treatment effects in terms of main effects and interactions, the right hand side of the equation (5.9) can be simplified to

$$\begin{aligned} \frac{1}{v} \prod_{j=1}^{q} s_j^{-\frac{1}{2}} \mathbf{L}'(s_1) \times \mathbf{L}'(s_2) \times \cdots \times \mathbf{L}'(s_q). \\ \hat{\mathbf{t}}(F_1) \otimes \hat{\mathbf{t}}(F_2) \otimes \cdots \otimes \hat{\mathbf{t}}(F_q). \end{aligned} \tag{5.11}$$

The left hand side of the equation (5.9) can also be simplified in the same way, and we obtain

$$\frac{1}{\theta}\mathbf{L}'(s_1)\times\mathbf{L}'(s_2)\times\cdots\times\mathbf{L}'(s_q)\mathbf{Q}(F_1)\otimes\mathbf{Q}(F_2)\otimes\cdots\otimes\mathbf{Q}(F_q)$$
$$=\mathbf{L}'(s_1)\times\mathbf{L}'(s_2)\times\cdots\times\mathbf{L}'(s_q)\hat{\mathbf{t}}(F_1)\otimes\hat{\mathbf{t}}(F_2)\otimes\cdots\otimes\hat{\mathbf{t}}(F_q).$$

Then, on introducing the marginal relations (4.2),

$$(5.12)\quad \frac{1}{\theta}\mathbf{M}'(s_1)\times\mathbf{M}'(s_2)\times\cdots\times\mathbf{M}'(s_q)\mathbf{Q}(F_1)\otimes\mathbf{Q}(F_2)\otimes\cdots\otimes\mathbf{Q}(F_q)$$
$$=\mathbf{M}'(s_1)\times\mathbf{M}'(s_2)\times\cdots\times\mathbf{M}'(s_q)\hat{\mathbf{t}}(F_1)\otimes\hat{\mathbf{t}}(F_2)\otimes\cdots\otimes\hat{\mathbf{t}}(F_q).$$

Hence, on multiplying both sides by the Kronecker product of the corresponding **M** matrices, (5.12) simplifies to

$$(5.13)\quad \frac{1}{\theta}\mathbf{Q}(F_1)\otimes\mathbf{Q}(F_2)\otimes\cdots\otimes\mathbf{Q}(F_q)=\hat{\mathbf{t}}(F_1)\otimes\hat{\mathbf{t}}(F_2)\otimes\cdots\otimes\hat{\mathbf{t}}(F_q).$$

This proves Theorem 5.2.

THEOREM 5.3. *If, in a* BFE, *two factor interactions* $t(F_1F_2\cdots F_q)_{x_1x_2\cdots x_q}$ *and* $t(F_{i_1}F_{i_2}\cdots F_{i_p})_{y_1y_2\cdots y_p}$ *do not have all the factors identical, then their estimates are uncorrelated.*

PROOF. It can be seen from (5.9) and (5.13) that the estimates of the factor interactions are obtained from the contrasts belonging to the corresponding interactions. In a BFE contrasts belonging to different interactions are uncorrelated and hence the estimates of the factor interactions belonging to different interactions are uncorrelated.

THEOREM 5.4. *If, in a* BFE, *the variance of any normalised contrast belonging to the q-factor interaction* $F_1F_2\cdots F_q$ *is* σ^2/θ, *then the variance of* $\hat{t}(F_1F_2\cdots F_q)_{x_1x_2\cdots x_q}$ *is* $\prod_{j=1}^{q}(s_j-1)\sigma^2/v\theta$ *and the covariance between* $\hat{t}(F_1F_2\cdots F_q)_{x_1x_2\cdots x_q}$ *and* $\hat{t}(F_1F_2\cdots F_q)_{y_1y_2\cdots y_q}$, *provided exactly h of the* x_j *are equal to the corresponding* y_j, *is* $(-1)^{q-h}\prod'(s_j-1)\sigma^2/v\theta$, *where* $\prod'$ *represents the product for those factors for which* $x_j=y_j$.

PROOF. The right hand side of equation (5.9) represents a set of normalised orthogonal contrasts belonging to the interaction $F_1F_2\cdots F_q$. Hence by Lemma 3.4, its variance-covariance matrix is $(\sigma^2/\theta)\mathbf{I}$. Consequently, the variance-covariance matrix of (5.11) can be written as

$$(5.14)\quad \frac{\sigma^2}{\theta}\mathbf{I}(s_1-1)\times\mathbf{I}(s_2-1)\times\cdots\times\mathbf{I}(s_q-1).$$

Hence it can be deduced that the variance-covariance matrix of the right hand side of (5.12) is

$$(5.15)\quad \frac{\sigma^2}{v\theta}\mathbf{I}^*(s_1)\times\mathbf{I}^*(s_2)\times\cdots\times\mathbf{I}^*(s_q).$$

Now, on applying

$$\mathbf{M}(s)\mathbf{I}^*(s)\mathbf{M}'(s) = \mathbf{L}(s)\mathbf{L}'(s) = \mathbf{N}(s), \tag{5.16}$$

it follows that the variance-covariance matrix of the right hand side of the equation (5.13) is

$$\frac{\sigma^2}{v\theta}\prod_{j=1}^{q} s_j \cdot \mathbf{N}(s_1) \times \mathbf{N}(s_2) \times \cdots \times \mathbf{N}(s_q). \tag{5.17}$$

The required expressions for variances and covariances can be obtained from (5.17).

THEOREM 5.5. *If, in a* BFE, *the variance of any normalised contrast belonging to the interaction* $F_1F_2\cdots F_q$ *is* σ^2/θ, *then the sum of squares due to the same interaction is given by*

$$\left(\prod_{j=1}^{q} s_j\right)^{-1} v\theta \sum \hat{t}(F_1F_2\cdots F_q)^2_{x_1x_2\cdots x_q} = \left(\prod_{j=1}^{q} s_j\right)^{-1} \frac{v}{\theta}\sum Q(F_1F_2\cdots F_q)^2_{x_1x_2\cdots x_q}, \tag{5.18}$$

where the summation is over all possible values of $(x_1x_2\cdots x_q)$. *Its expected value is*

$$\left(\prod_{j=1}^{q} s_j\right)^{-1} v\theta \sum t(F_1F_2 \cdots F_q)^2_{x_1x_2\cdots x_q} + \prod_{j=1}^{q} (s_j - 1)\sigma^2, \tag{5.19}$$

and it is distributed as $\sigma^2\chi^2$ *variate with* $\prod_{j=1}^{q}(s_j - 1)$ *degrees of freedom under the null hypothesis that the interaction* $F_1F_2\cdots F_q$ *is zero at all the levels.*

Theorems 5.1 to 5.5 indicate a method of analysis for BFE. This method is useful only when estimates of interactions at different levels are required. For obtaining the analysis of variance table, a simple course would be to employ the method outlined in [13].

6. A method of construction. In this section we shall derive a method of constructing a BFE in $(m + n)$ factors from two known BFE's in $(n + 1)$ factors and m factors respectively.

The method employs replacement of different levels of a factor in one design, by the distinct sets of treatment combinations forming the blocks of another design. By the statement, that the level x_0 of the first factor in the treatment $(x_0x_1\cdots x_n)$ is replaced by the block (of another design) containing treatments $(y_{11}y_{12}\cdots y_{1m})$, $(y_{21}y_{22}\cdots y_{2m})$, $\cdots$, $(y_{k1}y_{k2}\cdots y_{km})$; we shall mean that the treatment $(x_0x_1\cdots x_n)$ is replaced by a set of k treatments $(y_{i1}y_{i2}\cdots y_{im}x_1x_2\cdots x_n)$, $i = 1, 2, \cdots, k$ respectively; these treatments belong to a new factorial design in $(m + n)$ factors. As an illustration, if the block A contains treatments (120), (203), (111) and (112), then the statement that the level 0 of the first factor (0120) is replaced by the block A will mean that the treatment (0120) is replaced by a set of 4 treatments (120210), (203210), (111210) and (112210).

Further, for employing this method, we need a known BFE with some specific properties. We shall assume that there exists a BFE in m factors $F_1, F_2, \cdots, F_m$ at $s_1, s_2, \cdots, s_m$ levels respectively with $s_1 s_2 \cdots s_m = v^*$ treatments, each replicated r^* times in b^* blocks of k^* plots each, with the incidence matrix

$$\text{(6.1)} \qquad \mathbf{N}^* = [n_{ij}^*] = [A_1 | A_2 | \cdots | A_{b^*}].$$

Further assume that $b^* = pq$, and it is possible to put pq blocks in p groups, each containing q blocks, in such a way that the design consisting of p blocks formed by adding together all the blocks of a group is a BFE. Without loss of generality it can be assumed that the incidence matrix of this BFE is

$$\text{(6.2)} \qquad \mathbf{N}_{pq}^* = \left[\sum_{j=1}^{q} \mathbf{A}_j \middle| \sum_{j=1}^{q} \mathbf{A}_{q+j} \middle| \cdots \middle| \sum_{j=1}^{q} A_{pq-q+j} \right].$$

It can be seen that for a resolvable design $\mathbf{N}^*$, the corresponding design $\mathbf{N}_{pq}^*$ exists with $p = r^*$. Another simple example is that the design $\mathbf{N}^*$ is a 2^3 factorial design in 3 factors A, B and C, in 4 blocks of two plots each, obtained by confounding the interactions AB, BC and AC; and the design $\mathbf{N}_{22}^*$ is the design in 2 blocks of 4 plots each, formed by confounding the interaction AB only.

THEOREM 6.1. *Let there be a* BFE $\mathbf{N}$ *in* $(n + 1)$ *factors* $F_0, F_{m+1}, F_{m+2}, \cdots, F_{m+n}$ *at* $s_0, s_{m+1}, s_{m+2}, \cdots, s_{m+n}$ *levels respectively* $(s_0 = q)$, *in* b *blocks of* k *plots each* (*with* r *replications*). *Also let there be two* BFE's $\mathbf{N}^*$ *and* $\mathbf{N}_{pq}^*$ *as given by* (6.1) *and* (6.2). *Now, if the level* $j - 1$ *of the factor* F_0 *is replaced by the block* A_{iq+j} $(j = 1, 2, \cdots, q)$ *in each of the treatments of* $\mathbf{N}$, *then the design obtained by adjoining the* p *designs so formed* (*for* $i = 0, 1, \cdots, p - 1$) *is a* BFE *in* $m + n$ *factors with* rr^* *replications in* bp *blocks of* kk^* *plots each.*

PROOF: Let the incidence matrix of the BFE in $n + 1$ factors $F_0, F_{m+1}, F_{m+2}, \cdots, F_{m+n}$ be

$$\text{(6.3)} \qquad \mathbf{N} = \begin{bmatrix} \mathbf{N}_1 \\ \mathbf{N}_2 \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{N}_q \end{bmatrix},$$

where $\mathbf{N}_j$ *is a matrix of* $s_{m+1}s_{m+2}\cdots s_{m+n} = v$ *rows corresponding to* v *treatments in each of which the factor* F_0 *occurs at the same level* $j - 1$, *and further that the order of these* v *combinations in each of the sub-matrices is the same. Then the incidence matrix of the constructed design is*

$$\mathbf{H} = \left[\sum_{j=1}^{q} \mathbf{A}_j \times \mathbf{N}_j \middle| \sum_{j=1}^{q} \mathbf{A}_{q+j} \times \mathbf{N}_j \middle| \cdots \middle| \sum_{i=1}^{q} \mathbf{A}_{pq-q+j} \times \mathbf{N}_j \right].$$

Now from Theorem 4.1, it can be shown that

$$\text{(6.4)} \qquad \begin{aligned} \mathbf{N}_i\mathbf{N}_i' &= \mathbf{N}_j\mathbf{N}_j' = \mathbf{U}, \quad \text{say;} \\ \mathbf{N}_i\mathbf{N}_j' &= \mathbf{N}_1\mathbf{N}_k' = \mathbf{W}, \text{ say,} \qquad \text{if } i \neq j \text{ and } 1 \neq k. \end{aligned}$$

This equivalent to the fact that **C**-matrix of a BFE is invariant under renaming of the levels of a factor or is symmetric with respect to all levels of any one of the factors. From the equations (6.3) and (6.4), we have

$$\mathbf{NN}' = \mathbf{I}(q) \times (\mathbf{U} - \mathbf{W}) + \mathbf{E}_{qq} \times \mathbf{W}. \tag{6.5}$$

Let $\mathbf{l}'(v)\mathbf{t}(v)$ (where $\mathbf{t}(v)$ is a column vector representing v combination of the n factors $F_{m+1}, F_{m+2}, \cdots, F_{m+n}$ and $\mathbf{t}(v)$ is a $(v \times 1)$ vector) be a normalised contrast belonging to the interaction $F_{i_1}F_{i_2}\cdots F_{i_\alpha}$ for a design in n factors. Similarly let $\mathbf{l}'(q)\mathbf{t}(q)$ be a normalised contrast in q levels of the factor F_0 only. In the design **N**, let the variances of the estimates of the contrasts $\mathbf{l}'(q) \times \mathbf{l}'(v)\mathbf{t}(q) \otimes \mathbf{t}(v)$ and $\mathbf{E}'(q) \times \mathbf{l}'(v)\mathbf{t}(q) \otimes \mathbf{t}(v)$ be σ^2/θ_{1d} and σ^2/θ_{0d} respectively; θ_{1d} and θ_{0d} are canonical roots of the **C**-matrix of **N** corresponding the normalised contrasts belonging to the interactions $F_0F_{i_1}F_{i_2}\cdots F_{i_\alpha}$ and $F_{i_1}F_{i_2}\cdots F_{i_\alpha}$ respectively. Then from Theroem 4.1 and Lemma 3.1, we have

$$\begin{aligned} \mathbf{NN}'\mathbf{l}(q) \times \mathbf{l}(v) &= k(r - \theta_{1d})\mathbf{l}(q) \times \mathbf{l}(v), \\ \mathbf{NN}'\mathbf{E}(q) \times \mathbf{l}(v) &= k(r - \theta_{0d})\mathbf{E}(q) \times \mathbf{l}(v). \end{aligned} \tag{6.6}$$

Writing $k(r - \theta_{1d}) = \psi_{1d}$ and $k(r - \theta_{0d}) = \psi_{0d}$, say, and substituting $\mathbf{NN}'$ from (6.5), we can deduce that

$$\begin{aligned} \mathbf{Ul}(v) - \mathbf{Wl}(v) &= \psi_{1d}\mathbf{l}(v), \\ \mathbf{Ul}(v) + (q-1)\mathbf{Wl}(v) &= \psi_{0d}\mathbf{l}(v). \end{aligned} \tag{6.7}$$

Now let $\mathbf{l}'(v^*)\mathbf{t}(v^*)$ be a normalised contrast belonging to an interaction $F_{j_1}F_{j_2}\cdots F_{j_\beta}$ in an m factor-design in $F_1, F_2, \cdots, F_m$ only. Let the variance of its estimate in the two designs $\mathbf{N}^*$ and $\mathbf{N}^*_{pq}$ be σ^2/θ_1 and σ^2/θ_2 respectively; also let $k^*(r^* - \theta_1) = \psi_1$ and $qk^*(r^* - \theta_2) = \psi_2$. Then by Theorem 4.1 and Lemma 3.1, $\mathbf{l}(v^*)$ is a canonical vector of $\mathbf{N}^*\mathbf{N}^{*\prime}$ and $\mathbf{N}^*_{pq}\mathbf{N}^{*\prime}_{pq}$; and

$$\begin{aligned} \left(\sum_{i=1}^{pq} \mathbf{A}_i\mathbf{A}_i'\right) \mathbf{l}(v^*) &= \psi_1\mathbf{l}(v^*), \\ \sum_{i=0}^{p-1} \left(\sum_{j=1}^{q} \mathbf{A}_{iq+j}\right)\left(\sum_{j=1}^{q} \mathbf{A}_{iq+j}'\right) \mathbf{l}(v^*) &= \psi_2\mathbf{l}(v^*). \end{aligned} \tag{6.8}$$

Now, we have

$$\mathbf{HH}' = \sum_{i=0}^{p-1} \left\{\sum_{j=1}^{q}\sum_{l=1}^{q} \mathbf{A}_{iq+j}\mathbf{A}_{iq+l}' \times \mathbf{N}_j\mathbf{N}_l'\right\}.$$

Hence, from (6.4) and (6.5),

$$\mathbf{HH}' = \left\{\sum_{i=0}^{p-1} \left(\sum_{j=1}^{q} \mathbf{A}_{iq+j}\right)\left(\sum_{j=1}^{q} \mathbf{A}_{iq+j}'\right)\right\} \times \mathbf{W} + \sum_{i=1}^{pq} \mathbf{A}_i\mathbf{A}_i' \times (\mathbf{U} - \mathbf{W}). \tag{6.9}$$

Therefore

$$\mathbf{HH}'\mathbf{1}(v^*) \times \mathbf{1}(v) = \left\{\sum_{i=0}^{p-1}\left(\sum_{j=1}^{q}\mathbf{A}_{iq+j}\right)\left(\sum_{j=1}^{q}\mathbf{A}'_{iq+j}\right)\right\}\mathbf{1}(v^*) \times \mathbf{W}\mathbf{1}(v) + \left(\sum_{i=1}^{pq}\mathbf{A}_i\mathbf{A}'_i\right) l(v^*) \times (\mathbf{U} - \mathbf{W})\mathbf{1}(v).$$

Applying the results in (6.7) and (6.8), we obtain

$$(6.10) \qquad \mathbf{HH}'\mathbf{1}(v^*) \times \mathbf{1}(v) = \frac{1}{q}\{\psi_2(\psi_{0d} - \psi_{1d}) + \psi_1\psi_{1d}\}\ \mathbf{1}(v^*)X\mathbf{1}(v).$$

From (6.10), it follows that $\mathbf{1}(v^*) \times \mathbf{1}(v)$ is a canonical vector of the matrix $\mathbf{HH}'$, hence $\mathbf{1}'(v^*) \times \mathbf{1}(v)\mathbf{t}(v^*) \otimes t(v)$ is a canonical contrast of the design $\mathbf{H}$ and its variance is σ^2/θ, where $kk^*(rr^* - \theta) = \psi_2/2(\psi_{0d} - \psi_{1d}) + \psi_1\psi_{1d}$. Therefore

$$(6.11) \qquad rr^* - \theta = (r^* - \theta_2)(\theta_{1d} - \theta_{0d}) + (r^* - \theta_1)(r - \theta_{1d}).$$

If the symbol L with the corresponding suffixes denotes the loss of information (as compared with a randomised block design) in each case, then

$$(6.12) \qquad L = L_2(L_{0d} - L_{1d}) + L_1L_{1d}.$$

The contrasts belonging to an interaction of $(m + n)$ factors can be formed by the Kronecker product of the contrasts in the m factors and n factors separately. Hence, from equations (6.10) and (6.11), it follows that the every contrast belonging to the same interaction $F_{j_1}F_{j_2}\cdots F_{j_\beta}F_{i_1}F_{i_2}\cdots F_{i_\alpha}$ is estimated with the same variance σ^2/θ, in the design $\mathbf{H}$; therefore it is a BFE.

Thus Theorem 6.1 is proved.

The variance of the estimate of the contrast $\mathbf{1}'(v^*) \times \mathbf{E}'(v)\mathbf{t}(v^*) \otimes \mathbf{t}(v)$ can be obtained from 6.11 by putting $\theta_{0d} = 0$ and taking σ^2/θ_{1d} as the variance of the estimate of a normalised contrast belonging to the main effect of F_0 in the design $\mathbf{N}$. Similarly, the variance of the estimate of the contrast $\mathbf{E}'(v^*) \times \mathbf{1}'(v)\mathbf{t}(v^*) \otimes \mathbf{t}(v)$ can be obtained from 6.11 by putting $\theta_1 = \theta_2 = 0$.

THEOREM 6.2. *Let there be a* BFE N_α *in* $(n + 1)$ *factors* $F_0, F_{m+1}, F_{m+2}, \cdots, F_{m+n}$ *at* $s_0, s_{m+1}, s_{m+2}, \cdots, s_{m+n}$ *levels respectively, in b blocks of k plots each. Also let there be another* BFE $\mathbf{N}_\beta$ *in m factors* $F_1, F_2, \cdots, F_m$ *at* $s_1, s_2, \cdots, s_m$ *levels respectively, in* b^* *blocks of* k^* *plots each. If* $k^* = s_0$, *then on substituting* s_0 *levels of the factor* F_0 *in* $\mathbf{N}_\alpha$ *by* $s_0 = k^*$ *distinct treatments of a block of* $\mathbf{N}_\beta$, *we obtain b new blocks corresponding to each of the blocks of* $\mathbf{N}_\beta$. *Then the design obtained by taking all the* bb^* *blocks so formed is a* BFE *in* $(m + n)$ *factors.*

Theorem 6.2 appears to be different from Theorem 6.1. However, on a close examination, Theorem 6.2 is seen to be a particular case of Theorem 6.1, on taking $\mathbf{N} = \mathbf{N}_\alpha$, $\mathbf{N}^*$ to be a BFE in b^*k^* blocks of 1 plot each and $\mathbf{N}^*_{pq} = \mathbf{N}_\beta$ with $p = b^*$ and $q = k^*$. ($\mathbf{N}^*$ is a BFE in the sense that information on every contrast is zero.) From this analogy, the proof of Theorem 6.2 follows exactly on the same lines as Theorem 6.1.

In Theorems 6.1 and 6.2 we have replaced the levels of the first factor F_0. It is known that by permuting factors and correspondingly rewriting each of the treatments the design remains the same; it only means that the treatments are given new names. Hence, in practice the replacement as in Theorems 6.1 and 6.2 can be carried out for any intermediate factor. The proper rearrangement of the factors and the renaming of the treatments can be made where necessary.

There are many BFE's known for $3^m \times 2^n$ type, but no design is available for $3^2 \times 2^2$ in blocks of 6 plots each. We shall construct two such designs by the above method.

EXAMPLE 6.1. If we take $\mathbf{N}_\alpha$ equal to the 3×2^2 design given in Cochran and Cox ([3], plan 6.9, p. 240), and $\mathbf{N}_\beta$ as the 3^2 BFE in 6 blocks of 3 plots each, obtained by confounding the first order interaction between the two facrors, then, on applying Theorem 6.2, we obtain a $3^2 \times 2^2$ design in 36 blocks of 6 plots each.

EXAMPLE 6.2. Similarly, if we take $\mathbf{N}_\alpha$ equal to the $3^2 \times 2$ design given in Cochran and Cox ([3], plan 6.11, p. 241), and $\mathbf{N}_\beta$ as the 2^2 BFE in 2 blocks of 2 plots each, obtained by confounding the first order interaction, then, on applying Theorem 6.2, we obtained a $3^2 \times 2^2$ design in 24 blocks of 6 plots each.

EXAMPLE 6.3. Take the design $\mathbf{N}^*$ to be the following design in 2×3 in 6 blocks of 2 plots each.

The Plan of the Design.

Block Number	1	2	3	4	5	6
Treatments	00	01	02	00	02	01
	11	12	10	12	11	10

The blocks 1, 2, 3 and 4, 5, 6 form two complete replications, so we can take $\mathbf{N}_{23}^*$ as the randomised block with two replications. Now, let us take a 5×3 design in 20 blocks of 3 plots each, given by Rao ([12], p. 169). Then on applying Theorem 6.1, we obtain a $2 \times 3 \times 5$ BFE in 40 blocks of 6 plots each ($r = 8$).

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REFERENCES

[1] R. C. BOSE, "Mathematical theory of the symmetric factorial designs," *Sankhyā*, Vol. 8 (1947), pp. 107–166.

[2] R. C. BOSE AND K. KISHEN, "On the problem of confounding in general symmetric factorial designs," *Sankhyā*, Vol. 5 (1940), pp. 21–26.

[3] WILLIAM G. COCHRAN AND GERTRUDE M. COX, *Experimental Designs*, 2nd ed., John Wiley and Sons, New York, 1957.

[4] R. A. FISHER, *The Design of Experiments*, 4th ed., Oliver and Boyd, Edinburgh, 1947.

[5] R. A. FISHER, "The theory of confounding in factorial experiments in relation to the theory of groups," *Ann. Eugenics*, Vol. 11 (1942), pp. 341–353.

[6] OSCAR KEMPTHORNE, *The design and Analysis of Experiments*, John Wiley and Sons, New York, 1952.

[7] TOSIO KITAGAWA AND MICHIWO MITOME, *Tables for the Designs of Factorial Experiments*, Baifukan Co. Ltd., Tokyo, 1953.

[8] JEROME C. R. LI, "Design and Statistical analysis of some confounded factorial experiments," *Iowa Agricultural Experimental Station Research Bulletin*, 333 (1934).

[9] K. R. NAIR AND C. R. RAO, "Confounded designs for asymmetrical factorial experiments," *Science and Culture*, Vol. 7 (1942), pp. 313–314.

[10] K. R. NAIR AND C. R. RAO, "Confounded designs for the $k \times p^m \times q^n \cdots$ type of factorial experiments," *Science and Culture*, Vol. 7 (1942), pp. 361–362.

[11] K. R. NAIR AND C. R. RAO, "Confounding in asymmetrical factorial experiments," *J. Roy. Stat. Soc.*, Ser. B, Vol. 10 (1948), pp. 109–131.

[12] C. R. RAO, "A general class of quasifactorial and related designs," *Sankhyā* Vol. 17 (1956), pp. 165–174.

[13] B. V. SHAH, "On balancing in factorial experiments," *Ann. Math. Stat.*, Vol. 29 (1958), pp. 766–779.

[14] F. YATES, The Design and Analysis of Factorial Experiments, Imp. Bur. Soil Sci. Harpenden, England (1937).

[15] MARVIN ZELEN, "The use of group divisible designs for confounded asymmetrical factorial arrangements," *Ann. Math. Stat.*, Vol. 29 (1958) pp. 22–40.