PROPER SPACES RELATED TO TRIANGULAR PARTIALLY BALANCED INCOMPLETE BLOCK DESIGNS[1]

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1. Summary. The proper spaces of the matrix NN', where N is the incidence matrix of a triangular partially balanced incomplete block design, are exhibited explicitly; this provides a convenient form for the Gramian of a basis of the join of two of these spaces.

2. Introduction. The matrix N, the incidence matrix for an incomplete block design, is the matrix with v rows (v is the number of treatments) and b columns (b is the number of blocks) where the typical element, n_{ij}, is unity if the ith treatment occurs in the jth block, and is zero otherwise. The non-negative symmetric matrix $Q = NN'$ of order v has elements q_{ij}, where q_{ii} is equal to the number of replicates of the ith treatment, and q_{ij} $(i \neq j)$ is equal to the number of blocks in which the ith and the jth treatment occur together.

In the case of partially balanced incomplete block designs with two associate classes, of which the class of triangular designs as developed by R. C. Bose and T. Shimamoto [1], is a subclass, the numbers q_{ii} are all equal (to r, the common number of replications), while the q_{ij} $(i \neq j)$ are either λ_1 or λ_2, depending on whether the pair of treatments i and j are first or second associates respectively.

The knowledge of proper values and spaces of Q is of interest in finding conditions of existence of designs with given sets of parameters; in addition it can contribute to a better understanding of the analysis of actually constructed designs and lead to the attachment of a physical meaning to their association schemes. We will pay attention to the last-mentioned points in a later paper.

Knowledge of the proper *values* of Q for several cases, including the triangular designs, as given by Connor and Clatworthy [2] has already been utilized for the derivation of necessary conditions for the parameters of such designs. Knowledge of the proper *spaces* of Q for triangular designs, as will be shown in this note, provided Ogawa [3] other conditions for the existence of such designs.

3. The proper spaces of Q. In order to consider the proper values and spaces of $Q = NN'$ we conceive Q as the matrix of the linear transformation Q of a vector space A consisting of vectors $x = (x_1, x_2, \cdots, x_v)$ into itself, where the coordinate x_i corresponds to the ith treatment.

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Since triangular designs are special cases of partially balanced incomplete block designs with two associate classes, the number of first and second associates of a fixed treatment is independent of the chosen fixed treatment. Hence the ith cordinate y_i in $y = Qx$ is equal to $rx_i + \lambda_1 S_1 + \lambda_2 S_2$, where S_j $(j = 1, 2)$ represents the sum of the coordinates in x corresponding to the jth associates of treatment i.

We see immediately that if $x = (1, 1, \cdots, 1)$ then y_i is equal to $r + \lambda_1 n_1 + \lambda_2 n_2$ where n_j $(j = 1, 2)$ is the number of jth associates of treatment i. As the n_j are independent of i in all partially balanced incomplete block designs it follows that $s = (1, 1, \cdots, 1)$ is a proper vector of Q with proper value $r + \lambda_1 n_1 + \lambda_2 n_2 = rk$ where k is the block size.

For further investigation of the proper spaces of Q we shall consider the $(v - 1)$-dimensional subspace A^* of A orthogonal to s. For every vector x in A^*, $x_i + S_1 + S_2 = 0$. Hence the coordinate y_i in Qx, if x is restricted to A^*, is

$$(r - \lambda_2)x_i + (\lambda_1 - \lambda_2)S_1 \tag{1}$$

For triangular designs with parameter n (n is an integer greater than 3), the first and second associates of each treatment can be read from an association scheme which is constructed as follows. Consider an n by n square array in which the diagonal positions are denoted by $*$ and where the remaining $n(n - 1)$ positions each contain one of the $\frac{1}{2}n(n - 1)$ treatment indices such that each index occurs twice and symmetrically with respect to the diagonal. For $n = 5$ e.g. this might be

*	1	2	3	4
1	*	5	6	7
2	5	*	8	9
3	6	8	*	10
4	7	9	10	*

The first associates of any treatment are all those treatments which occur in the same row or the same column as this treatment, while the second associates are those which do not occur in the same row or the same column as this treatment. We note that $n_1 = 2(n - 2)$ in this case.

For convenience, we write the coordinates of any vector x in A in the same arrangement as the corresponding indices in the upper diagonal part of the association scheme. We now construct a set of n vectors, $c_1, c_2, \cdots, c_n$, in A in the following way. Write unity in all the positions of which the corresponding indices occur in the pth row of the association scheme; write zero everywhere else. The resulting vector is called c_p.

In our example with $n = 5$ we obtain

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ & 1 & 1 & 1 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & 1 \\ & & & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ & 0 & 1 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ & 0 & 0 & 1 \\ & & 0 & 1 \\ & & & 1 \end{pmatrix},$$

for $c_1, \cdots, c_5$ respectively.

Let the n-dimensional subspace of A spanned by these n linearly independent vectors be called A_1. We note that A_1 contains the one-dimensional space spanned by s. Now consider the inner product of any vector in A_1, $\gamma_1 c_1 + \cdots + \gamma_n c_n$, and s; this is equal to $\gamma_1(c_1, s) + \cdots + \gamma_n(c_n, s) = (n - 1)(\gamma_1 + \cdots + \gamma_n)$. Hence, for any vector in A_1^*, the $(n - 1)$-dimensional subspace of A_1 orthogonal to s, we have $\gamma_1 + \cdots + \gamma_n = 0$.

We further note that, if treatment i occurs in the pth row and the qth column of the association scheme, the coordinate x_i of any vector $\gamma_1 c_1 + \cdots + \gamma_n c_n$ in A_1 corresponding to the pth row and the qth column of the association scheme is equal to $\gamma_p + \gamma_q$.

Let x be any vector in A_1^*. Then the sum of the coordinates of x corresponding to all the treatments in the row and the column of the association scheme in which treatment i occurs (treatment i is counted twice in this sum) is equal to $2x_i + S_1$ on the one hand; on the other hand, according to the last two paragraphs, it is equal to

$$(n - 1)\gamma_p + (\gamma_1 + \cdots + \gamma_n) - \gamma_p\} + \{(n - 1)\gamma_q + (\gamma_1 + \cdots + \gamma_n) - \gamma_q\}$$
$$= (n - 2)(\gamma_p + \gamma_q) = (n - 2)x_i .$$

Hence $S_1 = (n - 4)x_i$ for all vectors in A_1^*.

Now it follows from (1) that the coordinate y_i of Qx, where x is restricted to A_1^*, is $\{r + (n - 4)\lambda_1 - (n - 3)\lambda_2\}x_i$. Therefore A_1^* is a proper space of Q with proper value $r + (n - 4)\lambda_1 - (n - 3)\lambda_2$.

Finally we consider the (orthogonal) complement of A_1 with respect to A (which of course is the same as the complement of A_1^* with respect to A^*) and call this A_2. The dimension of A_2 is $\frac{1}{2}n(n - 1) - n = \frac{1}{2}n(n - 3)$.

Since every vector in A_2 is orthogonal to the given n basis vectors of A_1, the sum of its coordinates corresponding to a row (or a column) of the association scheme must be zero. Taking the sum of its coordinates corresponding to the row and the column of the association scheme in which treatment i occurs (treatment i is counted twice again), we find that $2x_i + S_1 = 0$.

Again, from (1) it follows that the coordinate y_i of Qx, where x is now restricted to A_2, is $(r - 2\lambda_1 + \lambda_2)x_i$. Therefore A_2 is also a proper space of Q and the corresponding proper value is $r - 2\lambda_1 + \lambda_2$.

4. The Gramian of A_1. In connection with conditions for constructibility one is interested (see Ogawa [3]) in the Gramian, the symmetric matrix of inner products of a set of basis vectors of proper spaces of Q or of joins of such spaces. In the present case it is quite easy to find the Gramian of the given basis of A_1, the join of the proper space A_1^* and the proper space spanned by s. We simply need the inner products of the vectors $c_1, \cdots, c_n$.

The diagonal elements of this Gramian are all equal to the number of unities in these basis vectors, i.e. $n - 1$, while the off-diagonal elements are equal to the number of treatments which two different rows of the association scheme have in common, namely 1.

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