APPLICATION OF STORAGE THEORY TO QUEUES WITH POISSON ARRIVALS

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0. Summary. This paper is concerned with the waiting time process, $W(t)$, for the queueing system in which (1) there is only one counter, (2) the customers arrive at random and are served in the order of arrival, and (3) the service time distribution has a general form. It is observed that the Pollaczek-Khintchine formula for the transform of the limiting distribution of $W(t)$ is similar to the one occurring in the theory of continuous time storage processes, and it is inverted by the method used in that theory. Further, $W(t)$ is shown to be a special case of the storage process, and known methods and results of the storage theory are used to obtain the transition distribution function of $W(t)$.

1. Introduction. Several analogies of storage processes with those occurring in the theory of queues have been pointed out by Smith [19], Gani [4], Prabhu [16], and many others; by making use of one of these, Gani and Prabhu [5] obtained further results in storage theory. It has been remarked, however, that the analogy between the two situations is in their mathematical formalisms rather than in their physical models. This statement is essentially true if we confine ourselves to discrete time storage processes, but there exists an exact analogy if we consider continuous time models. The initial attempts to set up such a model were by using limiting methods (Moran [14], Gani [3], Downton [2]); this procedure has, however, proved cumbersome, and has obscured the essential features of the underlying stochastic process. In some recent work, Gani and Prabhu ([6], [7], [8], [9]) have given a systematic treatment of the various problems occurring in continuous time storage processes. The model they consider is the one based on Moran's [12] discrete time model for the dam, and is specified by the following assumptions.

(a) Let $X(t)$ represent the input during a time interval of length t; we assume that $X(t)$ is an additive process with stationary increments. Let $K(x, t)$ be the cumulative distribution function (c.d.f.) of $X(t)$, so that

$$K(x, t) = \Pr\{X(t) \leqq x\} \qquad (0 \leqq x < \infty, 0 \leqq t < \infty); \tag{1.1}$$

it is known that the Laplace transform of $X(t)$ is given by

$$\int_0^\infty e^{-\theta x}\, dK(x, t) = e^{-t\xi(\theta)} \qquad (R(\theta) > 0), \tag{1.2}$$

where $\xi(\theta)$ is a function of a specified type.

(b) The release is continuous and occurs at a unit rate except when the store is empty.

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(c) The store has infinite capacity.

If we denote by $Z(t)$ the storage at any time t, it follows from the above assumptions that

$$(1.3) \qquad Z(t + dt) = Z(t) + dX(t) - \min\{Z(t) + dX(t), dt\}$$

for $0 \leqq t < \infty$. Clearly $Z(t)$ is a temporally homogeneous Markov process; this process has been studied by Gani and Prabhu in the case of a Poisson input, and also in the case of a continuous infinitely divisible input of the Poisson type.

Next, consider the queueing system in which (a) the customers arrive 'at random', i.e. the inter-arrival times have the negative exponential distribution $\lambda e^{-\lambda t}\, dt (0 \leqq t < \infty)$; (b) the queue discipline is 'first come, first served', and (c) there is only one counter and the service time has the distribution $dB(t)$ $(0 \leqq t < \infty)$. Let $W(t)$ denote the waiting time of a customer who arrives at time t (i.e. the time spent by him in the queue before the commencement of his service); $W(t) > 0$ as long as the counter is occupied, but at any time that the counter becomes free, $W(t)$ becomes zero and remains zero until a customer arrives. It is easily seen that $W(t)$ is a temporally homogeneous Markov process; the Laplace transform of its distribution when the queue is in 'statistical equilibrium' is given by the Pollaczek-Khintchine formula,

$$(1.4) \qquad \varphi^*(\theta) = \frac{(1-\rho)\theta}{\theta - \lambda + \lambda\psi(\theta)}, \qquad R(\theta) \geqq 0,$$

where $\psi(\theta)$ is the Laplace transform of the service time distribution $dB(t)$, $\rho = -\lambda\psi'(0)$ is the relative traffic intensity measured in erlangs, and it is assumed that $\rho < 1$ (Pollaczek [15], Khintchine [11]). Further, let us denote the transition d.f. of $W(t)$ by $F(z_0 ; z, t)$, so that

$$(1.5) \qquad F(z_0 ; z, t) = \Pr\{W(t) \leqq z \mid W(0) = z_0\},$$

where $0 \leqq F(z_0 ; z, t) \leqq 1$ for $0 \leqq z < \infty$ and $0 \leqq t < \infty$. We may sometimes simplify this notation for the d.f. to $F(z, t)$. The forward Kolmogorov equation of the process $W(t)$ is

$$(1.6) \qquad \frac{\partial}{\partial t} F(z,t) - \frac{\partial}{\partial z} F(z,t) = -\lambda F(z,t) + \lambda \int_0^z F(z-u,t)\,dB(u)$$

$$z \geqq \max(0, z_0 - t),$$

a result due to Takács [20]. The purpose of this paper is to show that $W(t)$ is a special case of the continuous time storage process described above and to apply known methods and results to obtain $F(z, t)$ explicitly from (1.6). For this purpose we first invert the formula (1.4); this is done in the next section.

2. Inversion of the Pollaczek-Khintchine formula. This formula, (1.4), implies that if $\rho < 1$, then the limiting distribution $F^*(z) = \lim_{t\to\infty} F(z, t)$ exists, and the Laplace transform

$$\varphi^*(\theta) = \int_{0-}^{\infty} e^{-\theta z}\, dF^*(z) \tag{2.1}$$

is given by (1.4). Beneš [1] has inverted this and obtained $F^*(z)$ in the form of a compound geometric distribution. An alternative method of inversion is the one used by Daniels (see the discussion in Kendall [10]) to deal with Moran's formula for the limiting distribution of the dam storage. To apply this method let us assume that an analytic extension of $\psi(\theta)$ to a full neighbourhood of the origin in the θ-plane exists. We then note that, if $\rho < 1$, there exists a real $-c < 0$ such that $\theta - \lambda + \lambda\psi(\theta) < 0$ for $-c < \theta < 0$; the formula (1.4) is then valid also for $-c < \theta < 0$. For this range of θ we can write

$$1/(\lambda - \lambda\psi(\theta) - \theta) = \int_0^{\infty} e^{-[\lambda - \lambda\psi(\theta) - \theta]t}\, dt. \tag{2.2}$$

However, we have

$$\int_0^{\infty} e^{-\theta x}\, dK(x, t) = e^{-t[\lambda - \lambda\psi(\theta)]}, \tag{2.3}$$

where $dK(x, t)$ is the compound Poisson distribution,

$$dK(x, t) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!}\, dB_n(x), \qquad (0 \leqq x < \infty), \tag{2.4}$$

where $B_n(x)$ is the n-fold convolution of $B(x)$ with itself, and $B_0(x) = 0$ if $x < 0$ and $= 1$ if $x \geqq 0$. Substituting (2.3) in (2.2) we obtain

$$\begin{aligned} 1/(\lambda - \lambda\psi(\theta) - \theta) &= \int_0^{\infty}\int_0^{\infty} e^{-\theta(z-t)}\, dK(z, t) \\ &= \int_{t=0}^{\infty}\int_{z=-t}^{\infty} e^{-\theta z}\, dK(z + t, t) \\ &= \int_{z=0-}^{\infty} e^{-\theta z}\int_{t=0}^{\infty} dK(z + t, t) \\ &\quad + \int_{z=0+}^{\infty} e^{\theta z}\int_{t=z}^{\infty} dK(t - z, t)\, dt. \end{aligned} \tag{2.5}$$

Now

$$\int_{t=z}^{\infty} dK(t - z, t)\, dt = \sum_{n=0}^{\infty}\int_{t=z}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}\, dB_n(t - z) \qquad (z > 0)$$

is the constant term in the series

$$\begin{aligned} \sum_{n=0}^{\infty}\int_{t=z}^{\infty} e^{-(\lambda-\lambda\alpha)t}\alpha^{-n}\, dB_n(t - z) &= \sum_0^{\infty} e^{-(\lambda-\lambda\alpha)z}\alpha^{-n}\{\psi(\lambda - \lambda\alpha)\}^n \\ &= e^{-(\lambda-\lambda\alpha)z}\sum_0^{\infty}\left\{\frac{G(\alpha)}{\alpha}\right\}^n, \end{aligned} \tag{2.6}$$

where $G(\alpha) = \psi(\lambda - \lambda\alpha)$ is the probability generating function of the number of arrivals in a service period of arbitrary duration. Now choose a real α_1 such that $1 < \alpha_1 < \zeta$ and a real α_2 such that $G(\alpha_1) < G(\alpha_2) < \alpha_1$ (which is possible since $G(\alpha)$ is continuous in α). Here ζ is the real positive root (other than unity) of the equation $G(\alpha) = \alpha$, and $\zeta > 1$ since $G'(1) = -\lambda\psi'(0) = \rho < 1$. In the annulus $\alpha_1 < |\alpha| < \alpha_2$ we have $|G(\alpha)| < G(\alpha_2) < \alpha_1 < |\alpha|$, so that $|G(\alpha)/\alpha| < 1$ and the right hand side of (2.6) becomes

$$\alpha e^{-(\lambda-\lambda\alpha)z}/(\alpha - G(\alpha)).$$

The constant term in this is given by the formula

$$\frac{1}{2\pi i}\int_{|\alpha| \leq \alpha_1} \frac{e^{-(\lambda-\lambda\alpha)z}}{\alpha - G(\alpha)}\, d\alpha = \lim_{\alpha\to 1}\, (\alpha - 1)\, \frac{e^{-(\lambda-\lambda\alpha)z}}{\alpha - G(\alpha)} = (1-\rho)^{-1},$$

since $\alpha = 1$ is the only pole of the integrand within the circle $|\alpha| < \alpha_1$. Hence we have the result

$$\int_{t=z}^{\infty} dK(t - z, t)\, dt = (1 - \rho)^{-1} \qquad (\rho < 1, z > 0). \tag{2.7}$$

Using (2.7), we can simplify (2.5) as

$$\frac{1}{\lambda - \lambda\psi(\theta) - \theta} = \int_{z=0-}^{\infty} e^{-\theta z} \int_{t=0}^{\infty} dK(z + t, t)\, dt - \frac{1}{\theta(1 - \rho)}. \tag{2.8}$$

We now have

$$\begin{aligned}\int_{0-}^{\infty} e^{-\theta z}[1 - F^*(z)]\, dz &= \frac{1}{\theta} - \frac{1}{\theta}\int_{0-}^{\infty} e^{-\theta z}\, dF^*(z)\\ &= \frac{1}{\theta} + \frac{1-\rho}{\lambda - \lambda\psi(\theta) - \theta} \quad \text{from (1.4)}\\ &= (1-\rho)\int_{z=0-}^{\infty} e^{-\theta z}\int_{t=0}^{\infty} dK(z+t,t)\, dt\end{aligned}$$

from (2.8). This is true for $-c < \theta < 0$, and hence for all θ. Thus we obtain

$$F^*(z) = 1 - (1 - \rho)\int_0^{\infty} dK(t + z, t) \tag{2.9}$$

as the limiting distribution of the waiting time.

3. The waiting time $W(t)$ as a storage process. It is possible to demonstrate the equivalence of (2.9) and Beneš' result for the limiting distribution $F^*(z)$, but the connection between the Pollaczek-Khintchine formula (1.4) and the compound Poisson distribution does not seem to have been noticed so far. This distribution, however, is of fundamental importance in the theory of queues with Poisson arrivals. In fact, if N is the number of customers who arrive during the interval $(0, t)$, then their total service time has the distribution $dB_N(x)$, and, since N is a random variable having the Poisson distribution with mean λt, it follows that the total service time of customers arriving during $(0, t)$ has the compound Poisson distribution (2.4). This distribution can be considered as the 'service potential', which is steadily exhausted by the server at unit rate per unit time except when it is zero. Viewed in this manner, the waiting time for a queue with Poisson arrivals reduces to a special case of the storage process described in section 1, where the input $X(t)$ has the distribution (2.4). Clearly, $X(t)$ is an additive process with stationary increments, and from (2.3) it is seen that its Laplace transform is given by

$$\int_0^\infty e^{-\theta x} dK(x, t) = e^{-t\xi(\theta)} \tag{3.1}$$

as in (1.2), with

$$\xi(\theta) = \lambda \int_0^\infty (1 - e^{-\theta u})\, dB(u). \tag{3.2}$$

The particular case of 'regular' service time has been considered by Gani [3] and Moran [13] in their treatment of certain finite dam models, and the time-dependent solution of Takács's equation (1.6) in this case has been obtained by Gani and Prabhu [8]. In the case of an arbitrary service time distribution, this integro-differential equation is similar to the one obtained by Gani and Prabhu [9] for the storage process with continuous infinitely divisible inputs of the Poisson type, and can be solved by the method used there. This is done in section 4. As a preliminary result however, we require the probability $F(z_0\,; 0, t)$ of not having to wait (i.e. the probability of finding the counter free) at time t. To obtain this, we first find the probability $dG(z, t)$ that the counter becomes free for the first time at t, given that the waiting time of the initial customer was $W(0) = z > 0$. This is analogous to the probability distribution of the 'wet period' in a dam. Following Kendall [10] it can be proved quite generally that, for an input of the general additive input with the distribution $dK(x, t)$, this distribution is given by $dG(z, t) = (z/t)\, dK(t - z, t)$, and its Laplace transform by $e^{-z\eta(\theta)}$, where $\eta(\theta)$ satisfies the functional equation $\eta(\theta) = \theta + \xi\{\eta(\theta)\}$, $\theta > 0$. Applying these results to the waiting time process $W(t)$, we find that

$$dG(z, t) = \sum_0^\infty e^{-\lambda t} \lambda z \frac{(\lambda t)^{n-1}}{n!}\, dB_n(t - z) \tag{3.3}$$

This result has been directly proved by Prabhu [17]. Further, we obtain

$$\int_{t=z}^\infty e^{-\theta t}\, dG(z, t) = e^{-z\eta(\theta)}, \tag{3.4}$$

where $\eta(\theta)$ satisfies the functional equation

$$\eta(\theta) = \theta + \lambda - \lambda\psi\{\eta(\theta)\}, \qquad \theta > 0. \tag{3.5}$$

This equation is essentially the same as one considered by Takács [20], who proved that it has a unique solution.

Now $F(0, t)$ is the probability of finding the counter free at time t, not necessarily for the first time; by a direct enumeration of the ways in which this can happen we obtain

$$F(z_0; 0,t) = \int_{z_0}^{t} dG(z_0, \tau)F(0;0,t - \tau). \tag{3.6}$$

We assert that the solution of the integral equation (3.6) is given by

$$F(z_0; 0,t)\, dt = \begin{cases} 0 & \text{if } t < z_0 \\ \int_{z_0}^{t} dG(z, t)\, dz & \text{if } t \geqq z_0. \end{cases} \tag{3.7}$$

To prove this statement, let us multiply the right hand side of (3.6) by dt, and substitute (3.7); we obtain

$$\int_0^{t-z_0} d\zeta \int_{z_0}^{t-\zeta} dG(z_0, \tau)dG(\zeta, t - \tau) = \int_0^{t-z_0} dG(z_0 + \zeta, t)\, d\zeta$$
$$= \int_{z_0}^{t} dG(z, t)\, dz = F(z_0; 0, t)\, dt,$$

where we have used the fact that the distribution $dG(z, t)$ is additive in the parameter z, which is evident from its Laplace transform. We have thus proved our assertion, and (3.7) gives the probability of finding the counter free at time t.

The Laplace transform of (3.7) is given by

$$\begin{aligned} \int_{z_0}^{t} e^{-\theta t}F(z_0; 0, t)\, dt &= \int_{z_0}^{\infty} dz \int_{z_0}^{\infty} e^{-\theta t}\, dG(z, t) \\ &= \int_{z_0}^{\infty} e^{-z\eta(\theta)}\, dz = \frac{e^{-z_0\eta(\theta)}}{\eta(\theta)} \quad \text{from (3.4),} \end{aligned} \tag{3.8}$$

where $\eta(\theta)$ is given by (3.5).

The results (3.4) and (3.8) are due to Beneš [1]; however, the explicit expressions for $dG(z, t)$ and $F(0, t)$ have not been given previously.

4. The transition d.f. **of** $W(t)$. We are now in a position to solve the integro-differential equation (1.6) and to obtain the transition d.f. of $F(z, t)$ for the waiting time process $W(t)$. Let us denote the Laplace transform of $dF(z, t)$ by $\varphi(\theta, t)$, so that

$$\varphi(\theta, t) = \int_0^{\infty} e^{-\theta z}\, dF(z, t) \qquad (R(\theta) > 0). \tag{4.1}$$

Taking Laplace transforms of both sides of (1.6) with respect to z, we obtain a differential equation in $\varphi(\theta, t)$, which readily yields the solution $\varphi(\theta, t) = e^{-t\xi(\theta)+\theta(t-z_0)} - \theta\int_0^t F(0, t - \tau)e^{-\tau\xi(\theta)+\theta\tau}\, d\tau$ as obtained by Takács [19]. Writing this in a slightly different way, we have

$$(4.2) \qquad \theta^{-1}\varphi(\theta, t) = \theta^{-1}e^{-t\xi(\theta)+\theta(t-z_0)} - \int_0^t F(0, t-\tau)e^{-\tau\xi(\theta)+\theta\tau}\, d\tau.$$

Now from (3.1) we have

$$(4.3) \qquad \int_0^\infty e^{-\theta z}K(z, t)\, dz = \theta^{-1}e^{-t\xi(\theta)}$$

and

$$(4.4) \qquad \int_{-z'}^\infty e^{-\theta z}K(z + z', t)\, dz = \theta^{-1}e^{-t\xi(\theta)+\theta z'},$$

so that (4.2) yields the relation

$$(4.5) \qquad F(z, t) = K(t + z - z_0, t) - \int_0^t F(0, t-\tau)\, dK(\tau + z, \tau).$$

However, the validity of this inversion rests on proving that the right hand side of (4.5) vanishes for $z < \max(0, z_0 - t)$ or $-z > \min(0, t - z_0)$. Thus we have to show that

$$(4.6) \qquad K(t - z - z_0, t) = \int_z^{t-z_0} F(0, t-\tau)\, dK(\tau - z, \tau), \qquad (0 < z < t - z_0).$$

In order to do this, consider the process $Z(t) = W(0) + X(t) - t$, which is a temporally homogeneous Markov process with the transition d.f.

$$(4.7) \qquad P(z_0; z, t) = K(t + z - z_0, t).$$

By a direct enumeration of the paths $\zeta \to -z$ in this process we obtain

$$(4.8) \qquad dP(\zeta, -z, t) = \int_0^t dG(\zeta; 0, \tau)\, dP(0; -z, t-\tau) \qquad (\zeta > 0),$$

where $dG(\zeta; 0, \tau)$ denotes the probability of the first transition $\zeta \to 0$. Clearly, this is the same as the corresponding probability for our process $W(t)$, since the first transition $\zeta \to 0$ can be made only through positive values. Thus (4.8) can be written as

$$(4.9) \qquad k(t - z - \zeta, t) = -\int_z^{t-\zeta} dG(\zeta, t-\tau)k(\tau - z, \tau) \qquad (0 < \zeta \leqq t - z),$$

where we have written $dK(x, t) = k(x, t)\, dx$ for convenience. Integrating (4.9) over $z_0 \leqq \zeta \leqq t - z$ we obtain

$$K(t - z - z_0, t) = -\int_z^{t-z_0} k(\tau - z, \tau) \int_{z_0}^{t-\tau} dG(\zeta, t-\tau)\, d\zeta$$
$$= \int_z^{t-z_0} F(0, t-\tau)k(\tau - z, \tau)\, d\tau$$

using (3.7). Thus we have proved (4.6); our inversion is therefore valid, and (4.5) gives the transition d.f. of the waiting time process $W(t)$.

By an argument similar to the one used by Gani and Prabhu [9], it can be proved that the limiting distribution $F^*(z) = \lim_{t\to\infty} F(z, t)$ exists independently of z_0 if $\rho < 1$ and is given by (2.9). This confirms the result obtained in section 2.

In conclusion it may be noted that the integro-differential equation of Takács in the general case where the Poisson parameter λ is a function of time has been studied by Reich [18], who reduces it to a Volterra equation of the first kind. However, for the case λ = constant, we believe that our solution is much more straightforward, and that our method is applicable to more general distributions of the service potential; this is being investigated.

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