ABSTRACTS OF PAPERS

(Abstracts of papers presented at the Stanford Annual Meeting of the Institute, August 23–26, 1960. Additional abstracts will appear in the December 1960 issue.)

1. Estimating the Infinitesimal Generator of a Finite State Continuous Time Markov Process. Arthur Albert.

Let $\{Z(t), t > 0\}$ be a separable, continuous time Markov Process with stationary transition probabilities $P_{ij}(t)$, $i, j = 1, 2, \cdots, M$. Under suitable regularity conditions, the matrix of transition probabilities, $P(t)$, can be expressed in the form $P(t) = \exp tQ$, where Q is an $M \times M$ matrix and is called the "infintesimal generator" for the process.

In this paper, a density on the space of sample functions over $[0, t)$ is constructed. This density depends upon Q. If Q is unknown, the maximum likelihood estimate

$$\hat{Q}(k, t) = \|\hat{q}_{ij}(k, t)\|,$$

based upon k independent realizations of the process over $[0, t)$ can be derived. If each state has positive probability of being occupied during $[0, t)$ and if the number of independent observations, k, grows large (t held fixed), then $\hat{q}_{ij}$ is strongly consistent and the joint distribution of the set $\{(k)^{\frac{1}{2}}(\hat{q}_{ij} - q_{ij})\}_{i\neq j}$ (suitably normalized), is asymptotically normal with zero mean and covariance equal to the identity matrix. If k is held fixed (at one, say) and if t grows large, then $\hat{q}_{ij}$ is again strongly consistent and the joint distribution of the set $\{(t)^{\frac{1}{2}}(q_{ij} - q_{ij})\}_{i\neq j}$ (suitably normalized), is asymptotically normal with zero mean and covariance equal to the identity matrix, provided that the process $\{Z(t)\ t > 0\}$ is metrically transitive (but not necessarily stationary) and has no transient states.

The asymptotic variances of the $\hat{q}_{ij}$ are computed in both cases.

2. The Sequential Design of Experiments for Infinitely Many States of Nature. Arthur Albert. (By title)

In a recent paper (*Ann. Math. Stat.* Vol. 30 (1959), pp. 755–770) Chernoff discussed a problem which he called "The Sequential Design of Experiments" as it applied to the two action (hypothesis testing) case. In that paper, a procedure was exhibited for which the risk was approximately $-c \log c/I(\theta)$, when θ is the true state of nature, $I(\theta)$ is an appropriately defined information number and c, the cost per experimental trial, is small. It was also shown that in order for some other procedure to do significantly better for some value of the parameter, it must do worse by an order of magnitude (as $c \rightarrow 0$) at some other value of the parameter. These results were obtained under the assumption that the parameter space is finite. In the present paper, the assumption of finiteness is dispensed with. The procedures proposed here are closely akin to Chernoff's procedure, and analogous (though slightly weaker) optimality properties are derived.

3. Maximal Independent Stochastic Processes. C. B. Bell, University of California, Berkeley. (By title)

R. Pyke (1958) asked: What is the maximum cardinality, M_a, of a family of independent random variables defined on an abstract space Ω of cardinality a? (1) For $a < \aleph_0$, an elementary counting process yields $M_a = [\log_2 a]$. (2) For $a = \aleph_0$, a construction and a result of E. Marczewski (*Colloq. Math.*, 1955) yield $M_{\aleph_0} = \aleph_0$. (3) $M_{\mathfrak{C}} = 2^{\mathfrak{C}}$ follows from a result of Kakutani and Oxtoby (*Ann. Math.*, 1950) for the real line. (4) For $a \geqq \mathfrak{C}^n$, one notes that a subset of Ω has the cardinality of a cartesian product of n real lines. Consequently, an elementary construction provides $M_a \geqq n \cdot 2^{\mathfrak{C}}$. (5) Following the method mentioned in (4) above and using the Generalized Continuum Hypothesis it is established that $M_{\aleph_r} \geqq \max [\aleph_2, \aleph_{r-1}]$ for ordinals r. *Open problem*: Can the Kakutani-Oxtoby construction be generalized to yield $M_a = 2^a$ for all $a > \mathfrak{C}$?

4. **The Covariance Function of a Simple Trunk Group, with Applications to Traffic Measurement.** V. E. BENEŠ, Bell Telephone Laboratories and Dartmouth College. (By title)

Erlang's classical model for telephone traffic is considered: N trunks, calls arriving in a Poisson process, and negative exponential holding-times. Let $N(t)$ be the number of trunks in use at t. An explicit formula for the covariance $R(\cdot)$ of $N(\cdot)$ in terms of the characteristic values of the transition matrix of the Markov process $N(\cdot)$ is obtained. Also, $R(\cdot)$ is expressed purely in terms of constants and the "recovery function," i.e., the transition probability $\Pr\{N(t) = N \mid N(0) = N\}$. $R(\cdot)$ is accurately approximated by $R(0) \exp \{r_1 t\}$, where r_1 is the largest negative characteristic value, itself well approximated (underestimated) by $-E\{N(\cdot)\}/R(0)$. Exact and approximate formulas for sampling error in traffic measurement are deduced from these results.

5. **Limiting Distribution of the Maximum in an Infinite Sequence of Exchangeable Random Variables.** SIMEON M. BERMAN, Columbia University. (By title)

Let $\{X_n : n = 1, 2, \cdots\}$ be an infinite sequence of exchangeable random variables (r.v.'s), i.e., the joint distribution function (d.f.) of any m of these r.v.'s does not depend on their subscripts but only on their number m. The limiting d.f. of $Z_n = \max (X_1, X_2, \cdots, X_n)$ is characterized; a necessary and sufficient condition for the convergence of the d.f. of Z_n is given under an assumption on the d.f. of X_1. Let $\Phi(x)$ be one of the three limiting d.f.'s of maxima of independent r.v.'s with a common d.f.; let $A(y)$ be any d.f. such that $\lim_{y \to 0+} A(y) = 0$ and $\lim_{y \to \infty} A(y) = 1$. Then a d.f. is a limiting d.f. of Z_n if and only if is of the form $\int_0^\infty [\Phi(x)]^y \, dA(y)$. Let $\{Y_n : n = 1, 2, \cdots\}$ be independent r.v.'s with the same marginal d.f.'s as X_1; suppose that $W_n = \max (Y_1, Y_2, \cdots, Y_n)$ has the limiting d.f. $\Phi(x)$, that is, there exist sequences $\{a_n\}$ and $\{b_n\}$ such that for all t,

$$\lim_{n \to \infty} P\{a_n^{-1}(W_n - b_n) \leqq t\} = \Phi(t).$$

Under this assumption a necessary and sufficient condition is given for

$$\lim_{n \to \infty} P\{a_n^{-1}(Z_n - b_n) \leqq t\} = \int_0^\infty [\Phi(t)]^y \, dA(y).$$

For each integer k and real number u, let

$$\mu_k(u) = P\{X_1 > u, X_2 > u, \cdots, X_k > u\}[P\{X_1 > u\}]^{-k};$$

then $\{\mu_k(u) : k = 1, 2, \cdots\}$ is a moment sequence which uniquely determines a d.f. $A_u(y)$. The condition is that the d.f.'s $A_u(y)$ converge completely to $A(y)$ as $u \to \infty$.

6. **Elements of the Sequential Design of Experiments.** STUART A. BESSLER, Sylvania Electronic Defense Laboratories.

An experimenter observes a physical phenomenon, the outcome of which depends upon some unknown parameter θ belonging to a finite parameter space Θ. The experimenter wishes to choose which of k-alternative hypothesis best describes the parameter θ. To aid him in his decision he may perform experiments, e, selected from an infinite experiment space $\mathfrak{F}$. At each stage of the experimental process the experimenter must either stop experimenting and chôose a terminal action or continue experimenting in which case he must choose the next experiment. The "rule" which the experimenter uses in making these decisions will be called a sequential decision procedure. A sequential decision procedure is proposed and its optimal character is described. The procedure is demonstrated by applying it to the problem of choosing which of three normal populations with common variances has the largest mean. Several other examples are discussed. A measure of efficiency is defined, and for each example the efficiency of a common alternative decision procedure is computed.

7. Alias Sets of Error Vectors in the Theory of Error Correcting Group Codes. R. C. BOSE, University of North Carolina and Case Institute of Technology.

Consider an $n \times r$ parity check matrix A, of rank r whose elements belong to the Galois field GF(s), $s = p^m$. The letters of the code consist of all n-place row vectors γ for which $\gamma A = 0$. Suppose γ is transmitted over an s-ary channel, and the output is $\gamma + \epsilon$, $\epsilon = (e_1, e_2, \cdots, e_n)$. Then ϵ is the error vector. Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be the row vectors of A. The 2^r vectors for which $e_1\alpha_1 + e_2\alpha_2 + \cdots + e_n\alpha_n$ has a constant value may be said to form an alias set. Let Ω be the set of error vectors which we wish to correct with certainty. Then no alias set should contain more than one member from Ω. Subject to this condition one would like to maximize n for a given r. This principle is of very wide application. For example, let $s = 2$, and let Ω consist of all vectors with one non-zero or two adjacent non-zero coordinates. Then we get Abramson's (*IRE Trans.* Vol. IT5, 1959, pp. 150–157) single error and double adjacent error correcting (SEC-DAEC) code by choosing A such that the vectors $\alpha_1, \alpha_2, \cdots, \alpha_n, \alpha_1 + \alpha_2, \cdots, \alpha_{n-1} + \alpha_n$ constituting the set Ω^* are all distinct. For decoding we calculate $(\gamma + \epsilon)A = \epsilon A$. If ϵ belongs to Ω then ϵA belongs to Ω^*, and uniquely determines ϵ. The required condition is satisfied if $\alpha_i = (\beta_i, 1)$, $i = 1, 2, \cdots, 2^r - 1$, where β_i is the coefficient vector of the $(r - 2)$th degree polynomial which represents the element x^i of GF(2^{r-1}), x being a primitive element.

8. On Methods of Constructing Sets of Mutually Orthogonal Latin Squares Using a Computer. R. C. BOSE, I. M. CHAKRAVARTI, D. E. KNUTH, Case Institute of Technology and University of North Carolina. (Invited Paper)

This is in continuation of the work presented under the same title at the Midwestern Regional Meeting of IMS this year. The method is to start with module $G(2, 2t)$ whose elements are vectors $x = (a, b)$ where a is a residue class (mod 2) and b is a residue class (mod $2t$), the addition being defined by $(a_1, b_1) + (a_2, b_2) = (c, d)$ where $a_1 + a_2 = c$ (mod 2), $b_1 + b_2 = d$ (mod $2t$) and where $P_1[x_j] = a_j$ and $P_2[x_j] = b_j$ and $(0, 0)$, $(0, 1)$, $\cdots$ $(0, 2t - 1)$, $(1, 0)$ $\cdots$ $(1, 2t - 1)$ is the standard order. The existence of a set of m mutually orthogonal Latin squares based on a module G is known (Mann 1942) to be equivalent to the existence of a matrix $X_{m,4t} = ((x_{ij}))$ whose rows are elements of G and amongst the $4t$ differences of any two rows every element of G occurs once. The existence of $X_{m,4t}$ implies the existence of $A_{m,4t} = ((a_{ij})) = ((P_1[x_{ij}]))$ where $a_{ij} = 0$ or 1 and in every two-rowed submatrix of A the four possible pairs $\binom{0}{0}$, $\binom{0}{1}$, $\binom{1}{0}$ and $\binom{1}{1}$ occur as columns with equal frequency t. Starting with such a matrix $A_{m,4t}$ which exists whenever a Hadamard matrix of order $4t$ exists, a programme was written for adjoining a second coordinate b_{ij} to every a_{ij}, where b_{ij} belongs to the ring of residue classes (mod $2t$), so that a matrix $X_{m,4t} = (a_{ij}, b_{ij})$ could be obtained. For $t = 3$, this method yielded $m = 5$

mutually orthogonal Latin squares of order 12. These results have also been generalized in other directions for different orders.

9. Best Fit to a Random Variable by a Random Variable Measurable with Respect to a σ-Lattice. H. D. Brunk, University of Missouri.

Let $(\Omega, \mathcal{S}, \mu)$ be a probability space and f a random variable. Let $\mathcal{L}$ be a sub-σ-lattice of $\mathcal{S}$. Then there is an $\mathcal{L}$-measurable random variable g (for real t, $\{\omega \varepsilon \Omega : g(\omega) < t\} \varepsilon \mathcal{L}$) minimizing $\int (f - g)^2 \, d\mu$ (if appropriate integrals exist) in the class of $\mathcal{L}$-measurable random variables. More generally, the squared difference may be replaced by the W. H. Young form $\Delta_\Phi(\cdot, \cdot)$ determined by an arbitrary convex function Φ: the $\mathcal{L}$-measurable random variable g minimizing $\int \Delta_\Phi(f, g) \, d\mu$ in the class of $\mathcal{L}$-measurable random variables is independent of Φ (assuming appropriate integrals exist). When $\mathcal{L}$ is a sub-σ-field of $\mathcal{S}$, then g is the conditional expectation $E(f \mid \mathcal{L})$. In special cases treated by van Eeden (*Indag. Math.*, Vol. 19 (1957), pp. 128–136, 201–211) and by Ayer, Brunk, Ewing, Reid, Silverman, and Utz (*Ann. Math. Stat.*, Vol. 26 (1955), pp. 641–647, 607–616, *Pac. J. Math.*, Vol. 7 (1957), pp. 833–846) g is the solution of a problem in maximum likelihood estimation of ordered parameters; in these cases the σ-lattice $\mathcal{L}$ is not a σ-field.

10. On the Non-null Distribution of the Studentized Difference between the Two Largest Sample Values (Preliminary Report). André Croteau and Jacques St-Pierre, University of Montreal.

The non-null distribution of the difference between the two largest sample values has already been obtained by A. Zinger and J. St-Pierre (*Biometrika*, Vol. 45, Parts 3 and 4, December 1958, pp. 436–447) in the case of normal populations with known variances. In the case of unknown variances, the distribution of the studentized difference between the two largest sample values is obtained for three normal populations. The distribution takes the form of an iterated integral involving recurrence relations leading rather easily to numerical evaluations. A generalisation in the case of "n" populations is presently studied by the authors.

11. Random Noise in Relay Control Systems. R. C. Davis, Convair Division of General Dynamics Corporation.

A general method is developed to obtain the probability distribution of the error in a single closed-loop relay control system in which one controls a linear time-invariant dynamic element in the presence of a time-varying signal perturbed additively by Gaussian noise. The noise is allowed to be of a particular nonstationary type and specifically is the output of a perfect amplifier with time variable gain in cascade with a linear time-invariant filter with a rational amplitude versus frequency response—the input being the derivative of a Wiener process. The method used is the development of the theory of a particular type of discontinuous Markoff process for which the corresponding analogy in heat conduction is the conduction of heat in a moving medium in which there is a surface of discontinuity in medium velocity. In this way both the transient and steady state probability distributions of error are obtained. The probability distribution of error is obtained explicitly and involves line integrals of the Gaussian probability density function in the phase space of the error and certain of its time derivatives.

12. Sample Size for a Specified Width Confidence Interval on the Variance of a Normal Distribution. Franklin A. Graybill and Robert D. Morrison, Oklahoma State University. (By title)

If an experimenter decides to use a confidence interval to locate a parameter, he is concerned with at least two things: (1) Does the interval contain the parameter? (2) How wide is the interval? In general, the answer to these questions cannot be given with absolute certainty, but must be given with a probability statement. The problem the experimenter then faces is the determination of the sample size n such that (A) the probability will be equal to $1 - \alpha$ that the confidence interval contains the parameter, and (B) the probability will be equal to β^2 that the width of the confidence interval will be less than d units (where α, β^2, and d are specified). $1 - \alpha$ will be called the confidence coefficient, and β^2 will be called the width coefficient. To solve this problem will generally require two things: (1) The form of the frequency function; (2) Some previous information on the unknown parameters. This suggests that the sample be taken in two steps; the first sample will be used to determine the number of observations to be taken in the second sample so that (A) and (B) will be satisfied. For a confidence interval on the mean of a normal population with unknown variance this problem has been solved by Stein for $\beta^2 = 1$. The purpose of this paper is to illustrate a method for determining n to satisfy (A) and (B) for the variance of a normal distribution. A set of tables is presented to which will be needed for the solution of this problem.

13. On the Unbiasedness of Yates' Method of Estimation Using Interblock Information. Franklin A. Graybill and V. Seshadri, Oklahoma State University. (By title)

In a balanced incomplete block model with blocks and errors random normal variables, Yates has shown that there are two independent unbiased estimates for any treatment contrast. These are referred to as intrablock and interblock estimators. Yates has also given a method for combining these two estimators which depend on the variance (unknown) and has shown how to estimate the variances from an analysis of variance. Since this combined estimator is used quite extensively, it seems desirable to study its properties. Graybill and Weeks have shown that Yates' combined estimator is based on a set of minimal sufficient statistics and have presented an estimator which is unbiased. *The purpose of this note is to show that Yates' estimator, which is based on intrablock and interblock information, is unbiased.*

14. On the Distribution of the Ratio of the Largest of Several Chi-Squares to an Independent Chi-Square with Application to Ranking Problems. S. S. Gupta and M. Sobel. Bell Telephone Laboratories.

The distribution of $\chi^2_{\max}/\chi^2_0$ and its upper percentage points are considered where $\chi^2_{\max}$ is the maximum of p independent chi-squares and χ^2_0 is a chi-square independent of the p others. A common number ν of degrees of freedom is the principal case considered and tables of percentage points (25%, 10%, 5%, 1%) are given for $\nu = 2(2)50$ and $p = 1(1)10$; the case $p = 1$ which reduces to an F-distribution being used as a check. The computed tables have an application in the selection of a subset containing the "best" of several Gamma or Type III populations, i.e., the one with the largest scale parameter. In particular, if several exponential populations are individually observed until exactly r failures are obtained from each then the above tables can be used for selecting a subset containing the one with the largest mean life.

15. Expected Values of Normal Order Statistics. H. Leon Harter, Wright-Patterson Air Force Base.

A brief history is given of the development of the theory of order statistics and of past efforts to tabulate their expected values for samples from a normal population. A fuller account is given of the method of computation of a five-decimal-place table of the expected values of all order statistics for samples of size n from a normal population. Included is such a table for $n = 2(1)100$ and for values of n, none of whose prime factors exceeds seven, up through $n = 400$. Also included is a discussion of an approximation proposed by Blom, and a table of values of the constant α required for this approximation for selected values of n, together with interpolation formulas for estimating α for other values of n. A discussion is given of actual and potential uses of the tables.

16. **Circular Error Probabilities.** H. LEON HARTER, Wright-Patterson Air Force Base. (By title)

A problem which often arises in connection with the determination of probabilities of various miss distances of bombs and missiles is the following: Let x and y be two normally and independently distributed orthogonal components of the miss distance, each with mean zero and with standard deviations $\sigma_x \geqq \sigma_y$. Now for various values of $c = \sigma_y/\sigma_x$, it is required to determine (1) the probability P that the point of impact lies inside a circle with center at the target and radius $K\sigma_x$, and (2) the value of K such that the probability is P that the point of impact lies inside such a circle. Solutions of (1), for $c = 0.0(0.1)1.0$ and $K = 0.1(0.1)5.8$, and (2), for the same values of c and $P = 0.5, 0.75, 0.9, 0.95, 0.975, 0.99, 0.995, 0.9975$, and 0.999, are given, along with some hypothetical examples of the application of the tables.

17. **Comparison of Normal Scores and Wilcoxon Tests.** J. L. HODGES, JR. AND E. L. LEHMANN, University of California, Berkeley. (By title)

The normal scores test (i.e. the Fisher-Yates-c_1-test or the van der Waerden X-test) and the Wilcoxon test have been proposed for testing the equality of two distributions against the "shift" alternative that the populations have distributions $F(x)$ and $F(x - \theta)$. From the known limiting behavior of the test statistics one obtains an expression for the asymptotic relative efficiency $e(F)$ of Wilcoxon to normal scores. It is shown that $0 \leqq e(F) \leqq 6/\pi$ for all F, and that all values including the endpoints may be attained.

18. **Minimal Sufficient Statistics for the Two-Way Classification Mixed Model Design.** ROBERT A. HULTQUIST AND FRANKLIN A. GRAYBILL, Oklahoma State University. (By title)

A theorem proved by Rao and Blackwell reveals the importance of minimal sufficient statistics in point estimation problems. This theorem states: If Y is a vector of observations, S is a minimal sufficient statistic for a vector of parameters θ and $f(Y)$ is an unbiased estimate of $g(\theta)$, then $f(S) = E[f(Y) \mid S]$ is also an unbiased estimate of $g(\theta)$ based on S and such that variance $f(Y) >$ variance $h(S)$. We thus see that if we are interested in determining minimum variance unbiased estimators of variance components these estimators must be based on a minimal sufficient statistic. The objective of this paper is to exhibit minimal sufficient statistics for the *two-way classification mixed model* design with *unequal* numbers in the subcells.

19. **Three-Quarter Replicates of 2^3 and 2^4 Designs.** PETER W. M. JOHN, California Research Corporation.

Half replicates of 2^3 and 2^4 designs do not enable all the main effects and two-factor interactions to be estimated clear of two-factor interactions. Three-quarter replicates are obtained which give all main effects and two-factor interactions clear for the 2^4 design; for the 2^3 design main effects are clear and, if any one of the two-factor interactions is negligible, the other two are clear. In each case, the effects are estimated by extracting half replicates from the design.

20. On the Generalization of Sverdrup's Lemma and Its Applications to Multivariate Distribution Theory. D. G. KABE, Karnatak University. (By title). (Introduced by B. D. Tikkiwal)

Tikkiwal and Kabe (*Karnatak Univ. J.*, 1958) have given analytic-cum-geometric proof of the Sverdrup's lemma (*Skand. Aktur.* 1947). This lemma is now generalized for a p-variate population. Let the vectors $X_i' = (x_{i1}, x_{i2}, \cdots, x_{in})$ for $i = 1, 2, \cdots, p$ have the density $f(X_1'X_1, X_1'X_2, \cdots, X_p'X_p, BX_1, BX_2, \cdots, BX_p)$, then the density of $X_1'X_j = b_{ij}$, $BX_i = v_i (i, j = 1, 2, \cdots, p)$, B being $q \times n$ matrix of rank q, is given by

$$2^{-p} \prod_{i=1}^{p} C(n - q - p + i) | BB' |^{-\frac{1}{2}p} f(b_{11}, b_{12}, \cdots, b_{pp}, v_1, \cdots, v_p) | b_{ij} - v_1'(BB')^{-1} v_j |^{-\frac{1}{2}(n-q-p-1)}$$

$C(n)$ being the surface area of a unit n-dimensional sphere. Almost all the distributions in multivariate theory have been derived by the help of this lemma.

21. Approximations to Neyman Type A and Negative Binomial Distributions in Practical Problems (Preliminary Report). S. K. KATTI, Florida State University.

The Neyman Type A and the Negative Binomial distributions have been used for fitting data arising from biological phenomena with varying degrees of success, e.g. G. Beall, "The Fit and Significance of Contagious Distributions when Applied to Observations on Larval Insects", *Ecology*, Vol. 21 (1940), pp. 460–474 and C. I. Bliss and R. A. Fisher, "Fitting of Negative Binomial Distribution to Biological Data", *Biometrics*, Vol. 9, pp. 176–200. In the present work, it is shown that these distributions approximate to elementary distributions such as Poisson, Poisson with zeros added and Logarithmic in various regions of the parameter space. Preliminary fitting indicates that the elementary—and hence simple—distributions can be used with advantage as alternatives to these relatively complex distributions in many a practical situation. It is found that a reasonable judgment about the elementary distribution to be used can be made on the basis of the mean and the first frequency.

22. Two Sample Nonparametric Tests for Scale Parameter (Preliminary Report). JEROME KLOTZ, University of California, Berkeley.

Let $X_1, \cdots, X_m$ and $Y_1, \cdots, Y_n$ be samples from populations with continuous distritions F and G. We are interested in tests of the hypothesis $F = G$ that will be powerful against differences in scale when the populations are equivalent in location. Siegel and Tukey have recently devised a way to use the Wilcoxon statistic for this problem. The Pitman efficiency of their test for the normal case relative to the F-test is $6/\pi^2$. Pitman efficiency one relative to the F can be obtained for normality with the use of the following rank order statistic S. Assign weight $[\Phi^{-1}(i/N + 1)]^2$ to the ith smallest observation in the pooled sample where $N = m + n$, and let S be the sum of these weights over the observations from F. (Our weights are the squares of those used in Van der Waerden's X-test for the corresponding location problem.) As a rank order statistic S has an exact null distribution; we give a small table for the null distribution and approximations for m, n large. The Pitman efficiency of the test of Siegel and Tukey relative to the S-test can take on any value between 0 and ∞ for different F.

23. Zero Correlation and Independence. H. O. LANCASTER, University of Sydney. (By title)

Let $\{x_j\}$ be a set of n random variables. Let orthonormal functions be defined on each marginal distribution such that $x_j^{(0)} = 1$ for $j = 1, 2, \cdots, n$ and so that $\{x_j^{(i_j)}\}$ forms a basis if x_j has only a finite number (n_j) of points of increase, $i_j = 1, 2, \cdots, n_j$ and $\{x_j^{(i_j)}\}$ is a complete orthonormal set if x_j has a general type of distribution. Let generalised coefficients of correlation be defined, $\rho^{\{i_j\}} = E\{\prod_j x_j^{(i_j)}\}$. These coefficients are ordinary coefficients of correlation if precisely two of the i_j are non-zero. If more than two of the i_j are non-zero, the generalized coefficients may not be less than unity in absolute value. *Theorem*: A necessary and sufficient condition for independence of the set $\{x_j\}$ is that the generalized coefficients of correlation should all vanish. The bivariate case has been treated by Sarmanov, O. V., *Doklady Akad. Nauk SSSR*, Vol. 121 (1958), pp. 52–55 and Lancaster, H. O., *Aust. J. Stat.*, Vol. 1 (1959), pp. 53–56 and *J. Aust. Math. Soc.* (in the press), in which the multivariate case of the theorem also is proved without restriction.

24. Sequential Model Building for Prediction in Regression Analysis, I (Preliminary Report). HAROLD J. LARSON AND T. A. BANCROFT, Iowa State University.

Two different sequential procedures for deciding on the "length" of the linear regression model to use for predictions are evaluated, both assuming the population variance σ^2 to be known. In the first procedure the experimenter fits all the independent variables available, sequentially tests the coefficients of the "doubtful" ones to be zero, and deletes from the model the terms whose coefficients do not differ significantly from zero. In the second procedure the experimenter fits a set of "basic" variables he knows to be necessary, sequentially test the coefficients of the "nonbasic" variables to be zero and adds to his model those nonbasic variables whose coefficients differ significantly from zero. The expected value and the variance of the estimator are discussed for each procedure and limited tables for certain specific values of the parameters are presented to allow explicit evaluation of the bias of the estimators.

25. The Use of Sample Quasi-Ranges in Setting Confidence Intervals for the Population Standard Deviation. F. C. LEONE, Y. H. RUTENBERG, AND C. W. TOPP,* Case Institute of Technology and* Fenn College.

The problem is the choice of an optimal selection method of quasi-ranges for setting one sided confidence bounds and confidence intervals for the standard deviation from a given distribution. The proposed methods of optimal selection are applied to random ordered samples from the normal, exponential and rectangular distributions. Tables of confidence bounds for the standard deviation of these distributions are given for confidence levels commonly used in statistical work. These are compared with the results of standard procedures.

26. On a Property of a Test for the Equality of Two Normal Dispersion Matrices Against One-sided Alternatives. WADIE F. MIKHAIL, University of North Carolina.

The monotonic character, with respect to the variation of each noncentrality parameter, of the power function of the largest root test of normal multivariate analysis of variance or of independence between two sets of variates was proved in an earlier paper by S. N. Roy and the author. This paper obtains, using the same technique used before, similar results for four tests derived by S. N. Roy and R. Gnanadesikan for the equality of two dispersion matrices, in the normal multivariate set-up, against one-sided alternatives.

27. **A Note on Simple Sampling Plans** (Preliminary Report). T. V. NARAYANA AND S. G. MOHANTY, Queen's University.

From previous work done by one of the authors, it is known that a simple sampling plan of size n can be characterised by a unique vector of n non-negative integers satisfying certain conditions. A simple symmetric sampling plan of size n is defined as one in which the boundary points are symmetric about the line $y = x$. The following theorem is proved: The number of simple symmetric sampling plans of size n is $\binom{[3n/2]}{[(n-1)/2]} \Big/ [(n+1)/2]$ where $[x]$ is the largest integer contained in x. This theorem follows from known results on the number of compositions of an integer dominated by a given composition of this integer (*Canad. Math. Bull.*, Vol. 1, No. 3). A recursive method is suggested to obtain the number of simple sampling plans of size n and the authors hope to establish the general result that the number of such sampling plans is $\frac{1}{n}\binom{3n}{n-1}$.

28. **On Sampling with Varying Probabilities and With Replacement in Subsampling Designs.** J. N. K. RAO, Iowa State University. (Introduced by T. A. Bancroft)

In sub-sampling, it is usual practice to select the primaries with replacement and with varying probabilities, due to difficulties in the theory of sampling with varying probabilities and without replacement. This leads to three different methods of selecting the secondaries. In method 1, if the ith primary is selected λ_i times, $m_i\lambda_i$ secondaries are selected without replacement and with equal probabilities from the ith primary. In method 2, if the ith primary is selected λ_i times, λ_i sub-samples of size m_i are independently drawn of each other from the ith primary with equal probability and without replacement, each subsample being replaced after it is drawn. In method 3, when the ith primary is selected λ_i times, a fixed size of m_i is drawn from the ith primary with equal probability and without replacement and the estimate from the ith primary is weighted by λ_i. It is known that method 1 has smaller variance than method 2, and method 2 has smaller variance than method 3. But, the three methods have different expected costs, assuming that expected cost in a primary as proportional to expected sample size from the primary. Therefore it would appear more reasonable to compare the efficiency of the three methods for the same expected sample size. Here a comparison of the variances has been made for the same expected sample size but the conclusions remain the same regarding efficiency.

29. **Some Results on Transformations in the Analysis of Variance.** M. M. RAO, Carnegie Institute of Technology.

The square-root and the logarithmic transformations are considered when the mean is large in each case. In the former the variance is assumed known, and in the latter the corresponding assumption is that the coefficient of variation is small but the variance is unknown. In these cases, it is shown that the usual normal theory is applicable to test the hypotheses on means of the untransformed variables. These results extend those of E. G. Olds and N. C. Severo (These *Annals*, 1956). Sufficient conditions for the applicability of the normal theory are presented for a class of distributions depending on a finite set of parameters with one parameter large, while the others, if any, are relatively small, or are confined to a fixed bounded set in the parameter space.

30. **Normal Approximation to the Chi-square and Non-central F Probability Functions.** Norman C. Severo and Marvin Zelen, University of Buffalo and National Bureau of Standards. (By title)

Let x_p denote the $100p\%$ percentage point of the normal distribution, i.e., $\Phi(x_p) = \int_{-\infty}^{x_p} (2\pi)^{\frac{1}{2}} \exp(-\frac{1}{2}t^2)dt = 1 - p$. It is shown that the $100p\%$ percentage point of the Chi-square distribution with ν degrees of freedom may be approximated by

$$\chi_p^2(\nu) \doteq \nu\{1 - [2/(9\nu)] + (x_p - h_\nu)[2/[9\nu)]^{\frac{1}{2}}\}^3$$

where h_ν is an auxiliary function whose value may be obtained by linear interpolation in one of two short tables (one for $\nu \geqq 30$, and one for $5 \leqq \nu < 30$) consisting of only 15 entries each. For values of p between .005 and .995, this improved "Wilson-Hilferty" approximation gives results in error by at most .01 for $\nu \geqq 30$, and at most .05 for $5 \leqq \nu < 30$. Let $P(F' \mid \nu_1, \nu_2, \lambda)$ denote the probability distribution function of the non-central F distribution with degrees of freedom ν_1 and ν_2, and non-centrality parameter λ. It is shown that $P(F' \mid \nu_1, \nu_2, \lambda) \doteq \Phi(x)$, where

$$x = \frac{\left\{\left(\frac{\nu_1 F'}{\nu_1 + \lambda}\right)^{1/3}\left(1 - \frac{2}{9\nu_2}\right) - \left(1 - \frac{2(\nu_1 + 2\lambda)}{9(\nu_1 + \lambda)^2}\right)\right\}}{\left\{\left[\frac{2(\nu_1 + 2\lambda)}{9(\nu_1 + \lambda)^2} + \frac{2}{9\nu_2}\left(\frac{\nu_1 F'}{\nu_1 + \lambda}\right)^{2/3}\right]^{1/2}\right\}}$$

For values of the parameters investigated, the error of the approximation is at most .01.

31. **On a Geometrical Method of Construction of Cyclic PBIB** (Preliminary Report). Esther Seiden, Northwestern University. (By title)

An effective method of construction of cyclic PBIB is found provided that the number of treatments is $2^{2m} - 1$, m a positive integer. Using the notation of R. C. Base and T. Shimamoto ("Classification and analysis of partially balanced incomplete block designs with two associate classes," *J.A.S.A.*, Vol. 47, 1952, the parameters are as follows: $v = 2^{2m} - 1$ $b = (2^m + 2)(2^m + 1)/2$ $r = (2^m + 2)/2$ $k = 2^m - 1$ $\lambda_1 = 1$ $\lambda_2 = 0$ $n_1 = 2^{2m}/2 - 2$ $n_2 = 2^{2m}/2$ $\alpha = 2^{2m}/4 - 3$ $\beta = 2^{2m}/4 - 1$ $v = 2^{2m} - 1$ $b = 2^m(2^m - 1)/2$ $r = 2^m/2$ $k = 2^m + 1$ $\lambda_1 = 1$ $\lambda_2 = 0$ $n_1 = 2^{2m}/2$ $n_2 = 2^{2m}/2 - 2$ $\alpha = 2^{2m}/4$ $\beta = 2^{2m}/4$. The construction makes use of the fact that in a projective plane with $2^m + 1$ points on a line there exists an effective construction of a Desarguessian plane based on a set of $2^m + 2$ points of which no three are on one line. The problem whether such a construction is possible if based on a non-Desarguessian plane is under investigation.

32. **Distribution of Quantiles in Samples from a Bivariate Population.** M. M. Siddiqui, Boulder Laboratories.

Let $F(x, y)$ be the distribution function of (X, Y) possessing a pdf $f(x, y)$. Let $F_1(x)$ (pdf $f_1(x)$) and $F_2(y)$ (pdf $f_2(y)$) be the marginal distributions of X and Y respectively. Given two numbers F_1 and F_2 in $(0, 1)$ let α and β be the numbers such that $F_1(\alpha) = F_1$ and $F_2(\beta) = F_2$. Assume that the first and second partial derivatives of $F(x, y)$ are continuous at (α, β) and $f(\alpha, \beta) \neq 0$.

A random sample (X_k, Y_k), $k = 1, \cdots, n$ is drawn and the values of X and Y are ordered separately so that $X'_1 < X'_2 \cdots < X'_n$; $Y'_1 < Y'_2 \cdots < Y'_n$. Let i and j be the integers such that $i/n \leqq F_1 < (i + 1)/n$, $j/n \leqq F_2 < (j + 1)/n$. Let M be the number of sample points (X, Y) such that $X < X'_i$ and $Y < Y'_j$. The joint distribution of (M, X'_i, Y'_j) is obtained and it is shown that it is asymptotically trivariate normal. The asymptotic correlation coefficient between (X'_i, Y'_j) is given by

$$\rho = \{(F - F_1F_2)/[F_1F_2(1 - F_1)(1 - F_2)]^{\frac{1}{2}}\}, \qquad F = F(\alpha, \beta).$$

The statistic M/n has asymptotic mean F and variance of order n^{-1}. This is used to set up confidence limits on ρ. A generalization to the asymptotic distribution of a set of quantiles in samples from a multivariate population is stated.

33. **Power Characteristics of the Control Chart for Means.** Frederick A. Sorensen, United States Steel Corporation Applied Research Laboratory.

Methods are derived for the determination of the Type I error probability and the power of the control chart for sample means (no standard given). Under the null hypothesis, the process is assumed to be $N(\mu, \sigma^2)$, where μ and σ are unknown constants. Under the alternatives considered, the process is assumed to be $N(\mu_i, \sigma^2)$ during the time interval from which the ith subgroup $(i = 1, \cdots, k)$ is taken. For $k = 2, 3, 5, 10$ and 25, and subgroup sizes of 5 and 10, the power is tabulated with respect to two particular types of alternative believed to be typical of those encountered in practice: (1) One of the μ_i differs from the rest by an amount $\delta\sigma$ (single slippage); (2) Two of the μ_i differ from the rest by an equal amount $\delta\sigma$, but in opposite directions (symmetrical double slippage). The effect of using variable-width limits that produce a constant Type I error probability of 0.05 rather than using the traditional "three-sigma" limits is investigated. The power of the control chart is compared with that of the corresponding Model I analysis of variance test.

34. **A Set of Sufficient Statistics for Variance Components in a Two-Way Classification Model With Unequal Numbers in the Subclasses.** David L. Weeks and Franklin A. Graybill, Oklahoma State University. (By title)

One of the important methods of estimating variance components is by the analysis of variance (A.O.V.). The analysis of variance (A.O.V.) method of estimating variance components consists of obtaining an analysis of variance table, equating observed mean squares to expected mean squares and solving these equations for the estimates of the variance components. If the model is Eisenhart's Model II, then the A.O.V. method of estimating variance components gives estimators which are unbiased. If the model also has equal numbers in all subclasses, and all random variables are normally and independently distributed, the A.O.V. method gives unique, minimum variance, unbiased estimators. If the model is Eisenhart's Model II with equal numbers in all subclasses, and if all random variables are independently but not necessarily normally distributed, then the A.O.V. method of estimation gives estimators which are minimum variance, quadratic unbiased. However, if the model has unequal numbers in the subclasses, the problem is more complex. The A.O.V. method of estimation does not give minimum variance unbiased estimators in this case. The purpose of this paper is to exhibit a set of sufficient statistics for the general two-way classification model with unequal numbers in the subclasses. In particular, we show that the row totals, the column totals, and the intra-block error form a set of *sufficient* statistics for the variance components in a two-way classification model with unequal numbers in the subclasses.

35. Minimal Sufficient Statistics for Eisenhart's Model II in a Class of Two-Way Classification Models. David L. Weeks and Franklin A. Graybill, Oklahoma State University. (By title)

The class of designs in which the number of experimental units per block is constant, and the number of observations per treatment is constant, is examined in order to determine a set of minimal sufficient statistics. This class of designs includes as a subset, the balanced incomplete block, and the partially balanced incomplete block designs. Eisenhart's Model II under normal theory is assumed. The number of minimal sufficient statistics is expressed as a function of the distinct characteristic roots of the matrix NN' where N is the incidence matrix of the design. The distribution of each statistic is given and pairwise independence investigated. In the case of the BIB and GD-PBIB's, the statistics are defined explicitly in terms of quantities normally calculated in the analysis of variance. Instructions as to how the statistics may be computed easily for the case of the GD-PBIB's is also given.

36. Two New Continuous Sampling Plans. John S. White, General Motors Technical Center Research Labs.

Two new continuous sampling plans are proposed. Both plans are variations of the concepts used by Dodge and Torrey (*Ind. Qual. Cont.*, Jan., 1951) in CSP-2 and CSP-3. CSP-3 differs from CSP-2 in that following the discovery of a defective unit during sampling inspection, the next four units submitted must be inspected and found non-defective if sampling inspection is to continue. A new plan, called CSP-2.1, is proposed which requires only that the next unit after the defective pass inspection rather than the next four, in order that sampling inspection be continued. In this notation, the original CSP-2 might be denoted as CSP-2.0 and CSP-3 as CSP-2.4. A second plan is given which does require the inspection of four units following the discovery of a defective during sampling inspection but which eliminates the spacing number (i.e. $k = 0$). Graphs giving contours of constant sampling frequency in the $AOQL$ and i = clearance number plane are provided for both plans.

37. On a Class of Covariance Kernels Admitting a Power Series Expansion (Preliminary Report). N. Donald Ylvisaker, Columbia University.

Let $\mathcal{K}(T)$ denote the class of covariance kernels defined on $T \times T$ and let $a = (a_0, a_1, \cdots)$ be a sequence of nonnegative real numbers. The mapping $\varphi_a: K \to \sum_{j=0}^{\infty} a_j K^j$ maps $B_a = \{K \varepsilon \mathcal{K}(T) | \sum_{j=0}^{\infty} a_j K^j(s, s) < \infty \text{ for all } s \varepsilon T\}$ into $\mathcal{K}(T)$. This paper discusses these mappings and in particular the reproducing kernel space associated with the kernel $\varphi_a(K)$ is studied relative to the reproducing kernel space associated with the kernel K. Some applications of these results are noted in reference to problems of mean value estimation under the model $Y(t) = m(t, \beta) + X(t)$, $t \varepsilon T$, $\beta \varepsilon \Lambda \subset R_1$, where $X(\cdot)$ is assumed to be Gaussian process with mean function zero and known covariance kernel.

38. A Calculus for Factorial Arrangements. M. Zelen and B. Kurkjian, National Bureau of Standards and Diamond Ordnance Fuze Laboratories.

A calculus complete with special axioms, operations, and rules of formation is formally defined with respect to factorial arrangements. The object of this calculus is to permit easy manipulation of complicated mathematical operations. Its use enables large order matrix operations to be carried out using logical operations.

CORRECTION TO ABSTRACTS

"Semi-Markov Processes: Countable State Space" and "Stationary Probabilities for a Semi-Markov Process with Finitely Many States

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The titles of the above-named abstracts, numbers 55 and 74 on pages 240 and 245–46 respectively in the March 1960 *Ann. Math. Stat.*, were reversed in printing. Therefore, "Semi-Markov Processes: Countable State Space" applies to abstract 55 and "Stationary Probabilities for a Semi-Markov Process with Finitely Many States" applies to abstract 74.