the ideas above and those of Fraser ([2],[3], pp. 23–31), one can demonstrate the following result.

THEOREM. *If* $(X, \mathcal{S})$ *is an arbitrary measurable space, then* (I) $\Omega_0(X)$, $\Omega_1(X)$ *and* $\Omega_2(X)$ *are symmetrically complete for all n.*

If, further, λ *is a nonatomic,* σ*-finite measure on* $\mathcal{S}$ *and* $\mathcal{a}$ *is a semialgebra which generates* $\mathcal{S}$*, then,* (II) $\Omega(\mathcal{A}, \lambda)$, $\Omega(\mathcal{S}, \lambda)$ *and* $\Omega_3(\lambda)$ *are symmetrically complete for all n.*

REFERENCES

[1] C. B. BELL, "On the structure of distribution-free statistics," *Ann. Math. Stat.*, Vol. 31 (1960), pp. 703–709.
[2] D. A. S. FRASER, "Completeness of order statistics," *Can. J. Math.*, Vol. 6 (1954), pp. 42–45.
[3] D. A. S. FRASER, *Non-Parametric Methods in Statistics*, John Wiley and Sons, New York, 1957.
[4] E. L. LEHMANN, *Testing Statistical Hypotheses*, John Wiley and Sons, New York, 1959.
[5] E. L. LEHMANN AND HENRY SCHEFFÉ, "Completeness, similar regions and unbiased estimation," Part I, *Sankhyā*, Vol. 10 (1950), pp. 305–340.
[6] E. L. LEHMANN AND HENRY SCHEFFÉ, "Completeness, similar regions and unbiased estimation," Part II, *Sankhyā*, Vol. 15 (1955), pp. 219–236.
[7] H. SCHEFFÉ, "On a measure problem arising in the theory of non-parametric tests," *Ann. Math. Stat.*, Vol. 14 (1943), pp. 227–233.

ON CENTERING INFINITELY DIVISIBLE PROCESSES[1]

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The concept of centering stochastic processes having independent increments, introduced by Lévy, is applied to processes having both stationary and independent increments. The main purpose of this note is to answer the question as to what centering functions preserve the stationarity of the increments.

In 1934, Lévy [1] proved that any stochastic process with independent increments may be transformed by subtraction of a sure function, called a centering function, into a process whose sample functions possess certain desirable smoothness properties. (cf. Lévy [2] and Doob [3]). It is clear that the transformed process, called the centered process, is also a process possessing independent increments. The purpose of this paper is to show that a process having stationary and independent increments may be centered in such a way so as to preserve the stationarity as well as the independence of the increments.

To be more precise, consider the following definitions (cf. Doob [3] p. 407).

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For a set $T \subset R_1$, let T^* denote the set of limit points of T except that the supremum and infimum of T are to be included in T^* only if they belong to T.

Definition 1: A stochastic process $\{X_t : t \varepsilon T\}$ is said to be centered if and only if

(a) for every $\{t_n\} \subset T$ satisfying $t_n \nearrow t \varepsilon T^* (t_n \searrow t \varepsilon T^*)$ there exists a random variable $X_{t-}(X_{t+})$, independent of the particular sequence, such that

$$X_{t_n} \xrightarrow{\text{a.s.}} X_{t-}, \; (X_{t_n} \xrightarrow{\text{a.s.}} X_{t+})$$

(b) there exists a function g defined and continuous on the closure of T such that any difference $X_t - X_s$, $t, s \varepsilon T^*$, or any such difference with t replaced by $t+$ or $t-$ and/or s replaced by $s+$ or $s-$, is constant a.s. if and only if

$$X_t - X_s = g(t) - g(s) \text{a.s.}$$

(c) $X_{t-} = X_t = X_{t+}$ a.s. for all but at most a countable number of points of T.

This definition differs from that given by Doob only through condition (b). In Doob's definition, the function g was restricted to be constant over T^*. The above modified definition has the advantage of making it unnecessary to distinguish between degenerate and non-degenerate processes in the theorems below, as well as of insuring the truth of the statement that if $\{X_t : t \varepsilon T\}$ is a centered process, then so is $\{X_t + h(t) : t \varepsilon T\}$ for every continuous and bounded function h on T. This statement is not true under the more restrictive definition of Doob.

Definition 2: A function $c : T \to R_1$ is said to be a centering function of a stochastic process $\{X_t : t \varepsilon T\}$ if and only if the process $\{X_t - c(t) : t \varepsilon T\}$ is centered.

It is clear that one may always find a centering function, such that the resulting centered process satisfies (b) of Definition 1 with $g = 0$.

A stochastic process, $\{X_t : t \varepsilon T\}$, having stationary and independent increments and for which $T = [0, +\infty)$ and $X_0 = 0$ a.s. is said to be an Infinitely Divisible (I.D.) process. As is evident from Lemma 1 below, a correspondence may be defined in a natural way between the class of infinitely divisible random variables (r.v.) and the class of I.D. processes. For properties of infinitely divisible r.v.'s used in this paper, the reader is referred to [4] and [5].

In the case of a centering function for processes with independent increments, uniqueness is clearly impossible. One possible centering function is that used by Doob ([3], p. 408), namely the solution to $E\{\arctan [X_t - c(t)]\} = 0$. It should be noted that this particular centering function would not preserve stationarity of increments in case the given process were a non-degenerate I.D. process.

Define for all $\omega \varepsilon R_1$ and $t \geqq 0$, $f(\omega : t) = E\{e^{i\omega X_t}\}$.

Lemma 1: *A stochastic process* $\{X_t : t \geqq 0\}$ *having independent increments is an I.D. process if and only if there exist unique functions* $c : [0, \infty) \to R_1$ *and* $\psi : R_1 \to$ *complex plane, satisfying* (i) *for all* $s, t \geqq 0$, $c(s) + c(t) = c(s + t)$, (ii) *for all rational* $r \geqq 0$, $c(r) = 0$, (iii) *for all* $t \geqq 0$, $\omega \varepsilon R_1$, $\log f(\omega : t) = i\omega c(t) + t\psi(\omega)$.

PROOF: The proof of the sufficiency is left to the reader. The main interest is in the necessity of these conditions. A straightforward proof of this is possible using the Lévy-Khintchine representation of $f(\omega:t)$, namely

$$(1)\quad \log f(w:t) = i\omega\mu(t) + \int_{R_1} (e^{i\omega x} - 1 - i\omega x/(1 + x^2))(1 + x^2)/x^2\, dG(x:t).$$

where $G(\cdot:t)$ is a bounded non-decreasing right-continuous function, since clearly

$$(2)\qquad f(w:s + t) = f(w:s)f(w:t)$$

The purpose of the proof given here is to demonstrate that the powerful tool (1) is not essential for proving the necessity of the conditions of Lemma 1. This is important, it is felt, because the result stated as Lemma 1 should logically be proven very shortly after an I.D. process is defined, and because such a definition may well precede any discussion of infinitely divisible r.v.'s.

For each n, $[f(\omega:t)]^{1/n}$ is a characteristic function. It is well known, and easily proven, that therefore, for all $p > 0$, $[f(\omega:t)]^p$, properly defined, is a characteristic function and that for all $\omega \varepsilon R_1$ and $t \geqq 0$, $f(\omega:t) \neq 0$. Because of (2), $|f(\omega:s)|\,|f(\omega:t)| = |f(\omega:s + t)|$. Since $0 < |f(\omega, s)| \leqq 1$, the solution of this functional equation is given by $2 \log |f(\omega:t)| = t[\psi(\omega) + \psi(-\omega)]$ where $\psi(\omega) = \log f(\omega:1)$. Consequently, upon defining $q(\omega:t) = e^{t\psi(\omega)}[f(\omega:t)]^{-1}$, it follows that $|q(\omega:t)| = 1$, and that $q(\cdot:t)$ is a continuous function for each $t \geqq 0$. Moreover, since for rational r, $f(\omega:rt) = [f(\omega:t)]^r$, one has

$$q(\omega:t) = \lim e^{t\psi(\omega)}[f(\omega:t)]^{-rt} = \lim [f(\omega:1)/f(\omega:rt)]^t$$

where the limit is taken as $r \nearrow t^{-1}$ over the rationals. However, by (2),

$$f(\omega:1)/f(\omega:rt) = f(\omega:1 - rt)$$

is a characteristic function and hence so is $q(\omega:t)$. Because $|q(\omega:t)| = 1$, the proof is then complete since for each $t \geqq 0$, $q(\omega:t) = e^{i\omega c(t)}$ for some real number $c(t)$. It may be easily checked that the function c and ψ thus defined satisfy the required conditions.

By using the function c in Lemma 1, one obtains

COROLLARY 1: *For an I.D. process* $\{X_t:t \geqq 0\}$, *there exists a centering function* c *such that the resulting centered process* $\{X_t - c(t):t \geqq 0\}$ *is also an I.D. process.*

It is remarked that a stationarity preserving centering function for an I.D. process is unique up to the addition of straight lines through the origin.

It is evident that portions of Definition 1 are superfluous when applied to I.D. processes. In fact, one can easily prove

LEMMA 2: *An I.D. process is centered if and only if its characteristic function* $f(\omega:t)$ *is continuous in* t.

COROLLARY 2: *An I.D. process is centered if and only if for all sequences*

$$0 \leqq t_n \to t,\ X_{t_n} \xrightarrow{\text{a.s.}} X_t.$$

It is emphasized that the above results are neither difficult nor too surprising. The fact that a *centered* I.D. process has a characteristic function which satisfies $\log f(\omega:t) = t\psi(\omega)$ is well known (e.g., cf. Lévy [2] p. 186, Doob [3] p. 419, Ito [6]). The justification for the presentation of the above material is two-fold;

(i) Corollary 1 has not been located in the literature and

(ii) several recent papers in the literature indicate that Lemma 1 and Corollary 1 are not known.

Concerning (ii), several authors assume that (8) is true for all separable I.D. processes (cf. [7], [11]) while in other papers the exact role played by centering in the case of I.D. processes seems to have been misunderstood (cf. [8]). Furthermore, as a consequence of Lemma 1, the assumption (retaining the notation of the papers referred to) that $\phi(t:\lambda)$ be continuous in λ may be removed from Theorem 1 of [9] and from Theorem 1 of [10]. For example Theorem 1 (iii) of [9] could be strengthened to read: $F(x:\lambda) \varepsilon C_1$ if and only if $\phi(t:\lambda) = [f(t)]^\lambda e^{itc(\lambda)}$ where $f(t)$ is a characteristic function and where c is a function satisfying the conditions of Lemma 1.

As mentioned in the above paragraph separability is sometimes thought to imply that a process is centered. Although this is not true, it is possible to relate these two properties as well as the properties of measurability and of boundedness of sample functions, as stated in

Lemma 3: *For a separable I.D. process* $X = \{X_t:t \geqq 0\}$, *the following conditions are equivalent*: (i) *X is centered,* (ii) *X is measurable,* (iii) *there exists a separating sequence which is a subset of the rational numbers,* (iv) *there exists an open interval in* $[0, \infty)$ *over which almost all sample functions are either bounded from above or below.*

REFERENCES

[1] P. Lévy, "Sur les intégrales dont les éléments sont des variables aléatoires indépendotes," *Ann. Scuola Norm. d. Pisa,* (2) No. 3 (1934), pp. 337–366.

[2] P. Lévy, *Théorie de l'Addition des Variables Aléatoires,* 2nd ed., Gauthier-Villars, Paris, 1954.

[3] J. L. Doob, *Stochastic Processes,* J. Wiley and Sons, New York, 1953.

[4] B. V. Gnedenko and A. N. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables,* Addison-Wesley, Cambridge, Mass., 1954.

[5] M. Loéve, *Probability Theory,* D. Van Nostrand, New York, 1955.

[6] K. Ito, "On stochastic processes (I): infinitely divisible laws of probability," *Jap. J. Math.* Vol 18 (1942) pp. 261–301.

[7] G. Baxter and M. D. Donsker, "On the distribution of the supremum functional for processes with stationary independent increments," *Trans. A.M.S.* Vol. 85, No. 1, (1957) pp. 73–87.

[8] V. E. Beneš, "Characterization and decomposition of stochastic processes with stationary independent increments," (Abstract) Notices, *Ann. Math. Stat.,* Vol. 5 (1958), pp. 246–247.

[9] H. Teicher, "On the convolution of distributions," *Ann. Math. Stat.,* Vol. 25 (1954), pp. 775–778.

[10] E. L. Crow, "A property of additively closed families of distributions," *Ann. Math. Stat.,* Vol. 29 (1958), pp. 892–897.

[11] Herman Rubin and Howard G. Tucker, "Estimating the parameters of a differential process," *Ann. Math. Stat.,* Vol. 30 (1959), pp. 641–658.