Yates' estimate can now be written as

$$(4) \quad \bar{\tau}_i = [1 - \phi(\hat{\sigma}_\beta^2)][x_i + \gamma(u_i - x_i)] + \phi(\hat{\sigma}_\beta^2)[x_i + (\lambda t/rk)(u_i - x_i)]$$

Clearly (4) is equivalent to (1). Rearranging and simplifying (4) we get

$$\bar{\tau}_i = [x_i + \gamma(u_i - x_i)] + \phi(\hat{\sigma}_\beta^2)[(\lambda t/rk) - \gamma](u_i - x_i)$$

Graybill and Weeks have shown in [2] that $E[x_i + \gamma(u_i - x_i)] = \tau_i$. Therefore in order to show that Yates' estimate is unbiased we need only show that

$$E[\phi(\hat{\sigma}_\beta^2)((\lambda t/rk) - \gamma)(u_i - x_i)] = 0$$

Let $z_i = (u_i - x_i)$ where $i = 1, 2, \cdots, t - 1$. Now $\hat{\sigma}_\beta^2$ is a function of z_i, S^{*2}, and S^2. So let

$$\hat{\sigma}_\beta^2 = g(z_1, z_2, \cdots, z_{t-1}, S^{*2}, S^2).$$

γ is also a function of z_i, S^{*2}, and S^2. Therefore, let

$$\gamma = h(z_1, z_2, \cdots, z_{t-1}, S^{*2}, S^2).$$

Denote the joint density of the $t + 1$ random variables $z_1, z_2, \cdots, z_{t-1}, S^{*2}, S^2$ by $f(z_1, z_2, \cdots, z_{t-1}, S^{*2}, S^2)$. From (2) it is clear that γ is an even function of the z_i and from (3) we see that $\hat{\sigma}_\beta^2$ is also an even function of the z_i. Therefore, $\phi(\hat{\sigma}_\beta^2)$ is an even function of z_i, $(i = 1, 2, \cdots, t - 1)$ and $\phi(\hat{\sigma}_\beta^2)[(\lambda t/rk) - \gamma]$ is also an even function of z_i. Hence $\phi(\hat{\sigma}_\beta^2)[(\lambda t/rk) - \gamma](u_i - x_i)$ is an odd function of z_i. Therefore,

$$E[\phi(\hat{\sigma}_\beta^2)((\lambda t/rk) - \gamma)(u_i - x_i)] = 0,$$

since z_i are independent normal variables with mean zero and are independent of S^2 and S^{*2}. Thus Yates' estimator, which is based on intrablock and interblock information, is unbiased.

REFERENCES

[1] Yates, F. "The recovery of interblock information in balanced incomplete block designs," *Ann. Eugenics*. Vol. 10, (1940), pp. 317–325.
[2] Graybill, F. A., and D. L. Weeks, "Combining interblock and intrablock information in balanced incomplete blocks," *Ann. Math. Stat.* Vol. 30, (1959), pp. 799–805.

ON THE BLOCK STRUCTURE OF CERTAIN PBIB DESIGNS WITH TWO ASSOCIATE CLASSES HAVING TRIANGULAR AND L_2 ASSOCIATION SCHEMES

By Damaraju Raghavarao

University of Bombay

0. Summary. The PBIB designs [2] with two associate classes are classified in [3] as 1. Group Divisible, 2. Simple, 3. Triangular, 4. Latin Square type with i

Received November 6, 1959.

constraints, and 5. Cyclic. Group Divisible designs are divided into three types [1]: 1. Singular, 2. Semi-regular, and 3. Regular. It has been proved [1] that every block of a Semi-regular Group Divisible design contains k/m treatments from each of the m groups of the association scheme. In this note we prove analogous results in the case of certain PBIB designs with triangular and L_2 association schemes.

1. On the Block Structure of certain PBIB designs with two associate classes having a triangular association scheme. A PBIB design with two associate classes is said to have a triangular association scheme [3] if the number of treatments $v = n(n - 1)/2$ and the association scheme is an array of n rows and n columns with the following properties:

(a) The positions in the principal diagonal are blank.

(b) The $n(n - 1)/2$ positions above the principal diagonal are filled by the numbers $1, 2, \cdots, n(n - 1)/2$ corresponding to the treatments.

(c) The array is symmetric about the principal diagonal.

(d) For any treatment θ, the first associates are exactly those treatments which lie in the same row and same column as θ.

It is then obvious that

(1) the number of first associates of any treatment is $n_1 = 2n - 4$, and

(2) with respect to any two treatments θ_1 and θ_2 which are first associates, the number of treatments which are first associates of both θ_1 and θ_2 is $p_{11}^1(\theta_1, \theta_2) = n - 2$.

We now prove

THEOREM 1.1. *If in a PBIB design with two associate classes having a triangular association scheme*

$$rk - v\lambda_1 = n(r - \lambda_1)/2, \tag{1.1}$$

then $2k$ is divisible by n. Further, every block of the design contains $2k/n$ treatments from each of the n rows of the association scheme.

PROOF. Let e_j^i treatments occur in the jth block from the ith row of the association scheme ($i = 1, 2, \cdots, n; j = 1, 2, \cdots, b$). Then we have

$$\begin{aligned} &\sum_{j=1}^{b} e_j^i = (n - 1)r, \\ &\sum_{j=1}^{b} e_j^i(e_j^i - 1) = (n - 1)(n - 2)\lambda_1, \end{aligned} \tag{1.2}$$

since each of the treatments occurs in r blocks and every pair of treatments from the same row of the association scheme occurs together in λ_1 blocks. From (1.2), we get

$$\sum_{j=1}^{b} (e_j^i)^2 = (n - 1)\{r + (n - 2)\lambda_1\}. \tag{1.3}$$

Define $e_.^i = b^{-1}\sum_{j=1}^{b} e_j^i = (n-1)r/b = 2k/n$. Then

$$\begin{aligned}\sum_{j=1}^{b}(e_j^i - e_.^i)^2 &= (n-1)\{r + (n-2)\lambda_1\} - 4bk^2/n^2 \\ &= 2(n-1)[\{n(r-\lambda_1)/2\} - (rk - v\lambda_1)]/n \\ &= 0,\end{aligned} \tag{1.4}$$

from (1.1). Therefore $e_1^i = e_2^i = \cdots = e_b^i = e_.^i = 2k/n$. Since e_j^i ($i = 1, 2, \cdots, n$; $j = 1, 2, \cdots, b$) must be integral, $2k$ is divisible by n. This completes the proof of the theorem.

It has been proved ([4], [5], [7]) that a PBIB design with two associate classes satisfying the relations (1) and (2) has a triangular association scheme for all n except 8. Using this result and Theorem 1.1, we have

COROLLARY 1.1.1. *A necessary condition for the existence of a PBIB design with two associate classes having the parameters*

$$v = n(n-1)/2, b, r, k, \lambda_1, \lambda_2, n_1 = 2n-4, p_{11}^1 = n-2, \tag{1.5}$$

where $rk - v\lambda_1 = n(r-\lambda_1)/2$ *and* $n \neq 8$, *is that* $2k$ *is divisible by* n.

Now let us consider the PBIB design with parameters

$$\begin{aligned} &v = n(n-1)/2, \quad b = (n-1)(n-2)/2, \quad r = n-2, \\ &k = n, \quad n_1 = 2n-4, \quad n_2 = (n-2)(n-3)/2, \\ &\lambda_1 = 1, \quad \lambda_2 = 2, \quad p_{11}^1 = n-2, \quad p_{11}^2 = 4 \end{aligned} \tag{1.6}$$

This PBIB design has been shown to have a triangular association scheme [8]. Further, the parameters satisfy relation (1.1). Hence every block of this design contains $2k/n = 2$ treatments from each of the n rows of the association scheme.

2. On the Block Structure of certain PBIB Designs with two associate classes having a L_2 association scheme. A PBIB design is said to have a L_2 association scheme [3], if the number of treatments $v = s^2$, where s is a positive integer, and the treatments can be arranged in an $s \times s$ square such that treatments in the same row or the same column are first associates, while others are second associates. The following results are easily seen to hold in this case:

(i) The number of first associates of any treatment is $n_1 = 2s - 2$.

(ii) With respect to any two treatments θ_1 and θ_2 which are first associates, the number of treatments which are first associates of both θ_1 and θ_2 is $p_{11}^1 = s - 2$.

We now prove

THEOREM 2.1. *If, in a PBIB design with two associate classes having a* L_2 *association scheme,*

$$rk - v\lambda_1 = s(r - \lambda_1), \tag{2.1}$$

then k *is divisible by* s. *Further, every block of the design contains* k/s *treatments from each of the* s *rows (or columns) of the association scheme.*

PROOF. Let f_p^q treatments occur in the pth block from the qth row (or column) of the association scheme ($p = 1, 2, \cdots, b; q = 1, 2, \cdots, s$). We then have

$$\sum_{p=1}^{b} f_p^q = sr, \qquad (2.2)$$
$$\sum_{p=1}^{b} f_p^q(f_p^q - 1) = s(s-1)\lambda_1,$$

since each of the treatments occurs in r blocks and every pair of the treatments from the same row (or column) of the association scheme occurs together in λ_1 blocks.

From (2.2), we get

$$\sum_{p=1}^{b} (f_p^q)^2 = s\{r + (s-1)\lambda_1\}. \qquad (2.3)$$

Define $f_.^q = b^{-1}\sum_{p=1}^{b} f_p^q = sr/b = k/s$. Then

$$\sum_{p=1}^{b} (f_p^q - f_.^q)^2 = s\{n + (s-1)\lambda_1\} - bk^2/s^2 \qquad (2.4)$$
$$= s(r - \lambda_1) - (rk - v\lambda_1) = 0,$$

from (2.1). Therefore $f_1^q = f_2^q = \cdots = f_p^q = f_.^q = k/s$. Since f_p^q ($p = 1, 2, \cdots, b$; $q = 1, 2, \cdots, s$) must be integral, k is divisible by s. Thus the theorem is proved.

It has been proved that a PBIB design with two associate classes satisfying the relations (i) and (ii) has a L_2 association scheme if $s \neq 4$([6], [9]). Using this result and Theorem 2.1 we have

COROLLARY 2.1.1. *A necessary condition for the existence of a PBIB design with two associate classes having the parameters*

$$v = s^2, b, r, k, \lambda_1, \lambda_2, \quad n_1 = 2s - 2, \quad p_{11}^1 = s - 2, \qquad (2.5)$$

where $rk - v\lambda_1 = s(r - \lambda_1)$ *and* $s \neq 4$, *is that* k *is divisible by* s.

Acknowledgement. My sincere thanks are due to Professor M. C. Chakrabarti for his kind interest in this work.

REFERENCES

[1] R. C. BOSE AND W. S. CONNOR, "Combinatorial properties of group divisible incomplete block designs," *Ann. Math. Stat.*, Vol. 23 (1952), pp. 367–383.

[2] R. C. BOSE AND K. R. NAIR, "Partially balanced incomplete block designs," *Sankhyā*, Vol. 4 (1939) pp. 337–372.

[3] R. C. BOSE AND T. SHIMAMOTO, "Classification and analysis of partially balanced designs with two associate classes," *J. Amer. Stat. Assn.*, Vol. 47 (1952), pp. 151–190.

[4] W. S. CONNOR, "The uniqueness of the triangular association scheme," *Ann. Math. Stat.*, Vol. 29 (1958), pp. 262–266.

[5] A. J. HOFFMAN, "On the uniqueness of the triangular association scheme," *Ann. Math. Stat.*, Vol. 31 (1960), pp. 492–497.

[6] D. M. MESNER, "An investigation of certain combinatorial properties of partially balanced incomplete block designs and association schemes with a detailed study of designs of Latin Square and related types," Unpublished thesis, Michigan State University, 1956.

[7] S. S. SHRIKHANDE, "On a characterization of the triangular association scheme," *Ann. Math. Stat.*, Vol. 30 (1959), pp. 39–48.
[8] S. S. SHRIKHANDE, "Relation between certain types of incomplete block designs," Institute of Statistics, University of North Carolina, *Mimeograph Series No. 207* (1958).
[9] S. S. SHRIKHANDE, "The uniqueness of the L_2 association scheme," *Ann. Math. Stat.* Vol. 30 (1959), pp. 781–798.

OPTIMALITY CRITERIA FOR INCOMPLETE BLOCK DESIGNS[1]

BY K. R. SHAH[2]

Forest Research Institute, Dehra Dun, India

1. Introduction and Summary. Several optimality criteria have been suggested for the efficiency of incomplete block designs. This note surveys these criteria, extends certain results and puts forward a new and simpler criterion.

2. Existing Criteria. Important aims in experimental design are to estimate the effects of treatment comparisons with maximum precision for a given total number of experimental units, or total cost, and to perform a test of the null hypothesis. These two considerations lead us to different criteria for choosing from among the designs.

Consider the class of incomplete block designs, D_{vkb}, for fixed values of v, k and $b(v > k)$, where v treatments are arranged in b blocks of k plots each, and each treatment is replicated r times. In the usual notation, (see for example, Kempthorne [2]) intra-block estimates of treatment effects are given by

$$\mathbf{C}\hat{\mathbf{t}} = \mathbf{Q}, \tag{2.1}$$

where $\mathbf{C} = r\mathbf{I} - \mathbf{NN}'/k$, $\mathbf{N}$ being the incidence matrix of the design. We consider only connected designs, so that the rank of $\mathbf{C}$ is $v - 1$. Let $\lambda_1, \lambda_2, \cdots, \lambda_v$, be the $v - 1$ non-zero latent roots of $\mathbf{C}$. It is proved in [2] that the average variance of all elementary treatment contrasts is proportional to $\sum \lambda_i^{-1}$. Let $\mathbf{P}_i'\mathbf{t}(i = 1, 2, \cdots, v - 1)$ be any complete set of $v - 1$ orthogonal normalised contrasts. Set

$$\mathbf{P} = [\mathbf{P}_1, \mathbf{P}_2, \cdots, \mathbf{P}_{v-1}], \qquad \mathbf{P}'\mathbf{t} = \boldsymbol{\rho}, \qquad \boldsymbol{\rho} = \{\rho_1, \cdots, \rho_{v-1}\}.$$

It can be shown that $\mathbf{P}'\mathbf{C}\mathbf{P}$ is a non-singular matrix with latent roots $\lambda_1, \cdots, \lambda_{v-1}$, and that (2.1) leads to

$$\mathbf{P}'\mathbf{C}\mathbf{P}\hat{\boldsymbol{\rho}} = \mathbf{P}'\mathbf{Q} \quad \text{or} \quad \hat{\boldsymbol{\rho}} = (\mathbf{P}'\mathbf{C}\mathbf{P})^{-1}\mathbf{P}'\mathbf{Q}. \tag{2.2}$$

Let us denote the dispersion matrix of $\mathbf{x}$ by $V(\mathbf{x})$. Now $V(\mathbf{Q}) = \mathbf{C}\cdot\sigma^2$, which

Received June 8, 1959.

[1] This work was supported by a senior research training scholarship of the Government of India.

[2] Present address: Indian Statistical Institute, Calcutta, India.