RELATIONS AMONG THE BLOCKS OF THE KRONECKER PRODUCT OF DESIGNS

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1. Summary and Introduction. In the case of some incomplete block designs, interesting relations among their blocks have been discovered. For example, Fisher [1] has shown that in the case of a symmetrical BIB (Balanced Incomplete Block) design with parameters $v = b$, $r = k$, λ, any two blocks have exactly λ treatments in common. Similarly, Bose [2] has shown that in the case of an affine resolvable BIB design with parameters

$$v = nk = n^2\{(n-1)t + 1\}, \qquad b = nr = n\{n^2t + n + 1\}, \qquad \lambda = nt + 1,$$

the blocks can be divided into sets of n blocks, such that each set is a complete replication and any two blocks have $(k^2)/v = (nt - t + 1)$ or 0 treatments in common according as they belong to different groups or the same group. Also see Connor [3] and Bose and Connor [4] for similar results.

Confining our attention to PBIB (Partially Balanced Incomplete Block) designs with two or three associate classes, we wish to see how this type of information for blocks of BIB designs can be used to obtain similar information for the blocks of their Kronecker product.

In the next section are given a few general properties of the Kronecker product of designs. In Section 3 the main theorems of the paper are proved and their important particular cases are discussed. Some observations on the interconnection between these results and the theorems on inversion of designs (cf. Roy [5], Shrikhande [6]) are made in Section 4.

2. Some general properties of the Kronecker product of designs. We shall always denote the Kronecker product of matrices A and B by $A \times B$ (cf. Vartak [7]); and the ordinary product of A and B, whenever it exists, will be denoted by $A \cdot B$ or AB. The Kronecker product of designs was defined in [7] as the design whose incidence matrix is the Kronecker product of the incidence matrices of the given designs.

We shall consider throughout this section two designs N_1 and N_2 with v_1 and v_2 treatments and b_1 and b_2 blocks respectively. The design N_1' whose incidence matrix N_1' is the transpose of N_1, is said to be the design obtained from N_1 by inversion [5], or dualization [6]. Similarly for the design N_2'. Since the Kronecker product of matrices satisfies the law

$$(2.1) \qquad (A \times B)' = A' \times B'$$

we get the following result for the inversion of the Kronecker product of designs.

Theorem 2.1. *The design obtained by the inversion of the Kronecker product of*

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two given designs is the same as the Kronecker product of the inversions of the given designs. Thus if N_1 and N_2 are both symmetric (or self-dual), their Kronecker product is also symmetric (or self-dual).

In many cases we are interested in the matrix NN' where N is the incidence matrix of a given design. Let $N = N_1 \times N_2$, where N_1 and N_2 are the given designs. Clearly the Kronecker product of matrices satisfies the relation

$$(AB) \times (CD) = (A \times C) \cdot (B \times D) \tag{2.2}$$

where A, B, C and D are matrices of orders $m \times k$, $k \times n$, $p \times j$ and $j \times q$ respectively. Both sides of (2.2) are then $mp \times nq$ matrices. Hence we get

THEOREM 2.2. *The matrix NN' for the Kronecker product $N = N_1 \times N_2$ of two given designs is the Kronecker product of the corresponding matrices for the given designs; similarly for the matrix $N'N$.*

Finally, we need the following two results from [7].

2A. The Kronecker product $N = N_1(\text{BIB}) \times N_2(\text{BIB})$ of two BIB designs $N_1(\text{BIB})$ and $N_2(\text{BIB})$ defined by the respective sets of parameters

$$v_1, b_1, r_1, k_1, \lambda_1 \tag{2.3}$$

and

$$v_2, b_2, r_2, k_2, \lambda_2 \tag{2.4}$$

is a PBIB design with at most three associate classes.

The three associate classes of the design N defined above are all distinct if $r_1\lambda_2 \neq r_2\lambda_1$.

In any case, the parameters of the design N can be expressed in terms of those of the BIB designs given by (2.3) and (2.4) by the following equations

$$\begin{gathered}
v' = v_1v_2, \quad b' = b_1b_2, \quad r' = r_1r_2, \quad k' = k_1k_2, \\
n_1' = v_2 - 1, \quad n_2' = v_1 - 1, \quad n_3' = n_1n_2, \\
\lambda_1' = r_1\lambda_2, \quad \lambda_2' = r_2\lambda_1, \quad \lambda_3' = \lambda_1\lambda_2, \\
(p_{yz}'^{1}) = \begin{bmatrix} v_2 - 2 & 0 & 0 \\ 0 & 0 & v_1 - 1 \\ 0 & v_1 - 1 & (v_1 - 1)(v_2 - 2) \end{bmatrix}, \\
(p_{yz}'^{2}) = \begin{bmatrix} 0 & 0 & v_2 - 1 \\ 0 & v_1 - 2 & 0 \\ v_2 - 1 & 0 & (v_1 - 2)(v_2 - 1) \end{bmatrix}, \\
(p_{yz}'^{3}) = \begin{bmatrix} 0 & 1 & v_2 - 2 \\ 1 & 0 & v_1 - 2 \\ v_2 - 2 & v_1 - 2 & (v_1 - 2)(v_2 - 2) \end{bmatrix}
\end{gathered} \tag{2.5}$$

where $y, z = 1, 2, 3$.

As a direct consequence of 2A and Theorems 2.1 and 2.2, we get the following corollary

COROLLARY 2.1.1. *If a symmetrical* PBIB *design (i.e., one with $v = b$ and hence $r = k$) with three associate classes and parameters* (2.5) *is the Kronecker product of two symmetrical* BIB *designs, then with respect to any block B in it, the other blocks fall into three groups* (α), (β) *and* (γ) *such that the group* (α) *contains n_1' blocks each having λ_1' treatments in common with B, the group* (β) *contains n_2' blocks each having λ_2' treatments in common with B, and the group* (γ) *contains n_3' blocks each having λ_3' treatments in common with B.*

PROOF. Let the given symmetrical PBIB design N be the Kronecker product of the symmetrical BIB designs N_1(BIB) and N_2(BIB) with respective sets of parameters

$$v_1 = b_1, \qquad r_1 = k_1, \qquad \lambda_1$$

and

$$v_2 = b_2, \qquad r_2 = k_2, \qquad \lambda_2 .$$

By the well known result in [1], it follows that

$$N_1'(\text{BIB}) \cdot N_1(\text{BIB}) = (r_1 - \lambda_1) I_{v_1} + \lambda_1 E_{v_1 v_1}$$

where I_{v_1} is the identity matrix of order v_1 and $E_{v_1 v_1}$ is the matrix of order $v_1 \times v_1$ with all elements equal to 1. Similarly for the design N_2(BIB). Since

$$N = N_1(\text{BIB}) \times N_2(\text{BIB}),$$

it follows from Theorem 2.2 that

$$N'N = \{(r_1 - \lambda_1) I_{v_1} + \lambda_1 E_{v_1 v_1}\} \times \{(r_2 - \lambda_2) I_{v_2} + \lambda_2 E_{v_2 v_2}\},$$

which, in virtue of (2.5), simplifies to

$$N'N = \begin{bmatrix} A & B & \cdots & B \\ B & A & \cdots & B \\ \cdots & \cdots & \cdots & \cdots \\ B & B & \cdots & A \end{bmatrix}$$

where

$$A = (r' - \lambda_1') I_{v_2} + \lambda_1' E_{v_2 v_2}$$

and

$$B = (\lambda_2' - \lambda_3') I_{v_2} + \lambda_3' E_{v_2 v_2} .$$

The result of Corollary 2.1.1 follows from the fact that the element in the ith row and the jth column of $N'N$ equals the number of treatments common to the ith and the jth blocks.

2B. A set of necessary and sufficient conditions for the Kronecker product N of the two BIB designs given by (2.3) and (2.4) to have only two distinct associate classes is given by

$$v_1 = v_2 = v \text{ say}, \quad \text{and} \quad k_1 = k_2 = k \text{ say}. \tag{2.6}$$

If these conditions are fulfilled, then it follows that

$$b_2/b_1 = r_2/r_1 = \lambda_2/\lambda_1 = \mu, \quad \text{say} \tag{2.7}$$

where μ is a positive fraction; and in this case the parameters of N can be expressed in terms of those of the BIB designs by the equations

$$\begin{gathered} v' = v^2, \quad b' = \mu b_1^2, \quad r' = \mu r_1^2, \quad k' = k^2, \\ n_1' = 2(v-1), \quad n_2' = (v-1)^2, \\ \lambda_1' = \mu r_1 \lambda_1, \quad \lambda_2' = \mu \lambda_1^2, \\ (p_{yz}'^1) = \begin{bmatrix} v-2 & v-1 \\ v-1 & (v-1)(v-2) \end{bmatrix}, \quad (p_{yz}'^2) = \begin{bmatrix} 2 & 2(v-2) \\ 2(v-2) & (v-2)^2 \end{bmatrix}, \end{gathered} \tag{2.8}$$

where $y, z = 1, 2$.

Both the results 2A and 2B are particular cases of a general result, Theorem 4.2 of [7].

3. The Main Theorems. Let N be the incidence matrix of a given design with parameters v, b, r, k. Then the matrix

$$N'N = (n_{ij}'); \quad i, j = 1, 2, \cdots, b; \tag{3.1}$$

is such that its general element n_{ij}' gives the number of treatments common to the ith and the jth blocks of N. If N_1 and N_2 are two designs with parameters v_1, b_1, r_1, k_1 and v_2, b_2, r_2, k_2 respectively, then the matrix $N'N$ for the Kronecker product $N = N_1 \times N_2$ is given by

$$N'N = (N_1'N_1) \times (N_2'N_2). \tag{3.2}$$

From this we get the following theorem.

THEOREM 3.1. *If in the design N_1 there exists a pair of blocks having m_1 treatments in common and in the design N_2 a pair of blocks having m_2 treatments in common, then in their Kronecker product $N = N_1 \times N_2$ there exists a pair of blocks having m_1m_2 treatments in common.*

PROOF. It is clear that m_1 will be an element of $N_1'N_1$ and m_2 of $N_2'N_2$; so that by (3.2) $N'N$ will contain m_1m_2 as an element. This proves Theorem 3.1.

Now consider a block $B^{(1)}$ of the design N_1 and let $b_i^{(1)}$ of the totality of the blocks of N_1 have each i treatments in common with $B^{(1)}$; $i = 0, 1, 2, \cdots, k_1$. Clearly $\sum_{i=0}^{k_1} b_i^{(1)} = b_1$. Let $B^{(2)}$ and $b_j^{(2)}$ have similar meanings for the design N_2 so that $\sum_{j=0}^{k_2} b_j^{(2)} = b_2$. Remembering that the blocks of N_1 are of size k_1 and those of N_2 are of size k_2 we get the following theorem.

THEOREM 3.2. *If there exist blocks $B^{(1)}$ and $B^{(2)}$ in the designs N_1 and N_2 respectively having the above properties, then there exists a block B in the Kronecker product $N = N_1 \times N_2$ such that $b_u^{(0)}$ blocks of N have each u treatments in common with B, where $b_u^{(0)}$ is the coefficient of $\{u\}$ in the expression*

$$\left(\sum_{i=0}^{k_1} b_i^{(1)}\{i\}\right)\left(\sum_{j=0}^{k_2} b_2^{(2)}\{j\}\right) \tag{3.3}$$

where the symbols $\{u\}$ obey the ordinary laws of algebra, viz.,

$$a\{u\} + b\{u\} = (a + b)\{u\},$$
$$\{u\}\{v\} = \{v\}\{u\} = \{uv\}, \tag{3.4}$$
$$(a\{u\})(b\{v\}) = ab\{uv\}.$$

PROOF. From the conditions satisfied by the block $B^{(1)}$ of N_1 we find that the matrix $N_1'N_1$ contains a row of the form

$$\rho_1 = (0, 0, \cdots, 0, \quad 1, 1, \cdots, 1, \cdots, \quad k_1, k_1, \cdots, k_1), \tag{3.5}$$

where the integer i is repeated $b_i^{(1)}$ times; $i = 0, 1, \cdots, k_1$. Similarly from the properties of the block $B^{(2)}$ of N_2 we find that the matrix $N_2'N_2$ contains a row of the form

$$\rho_2 = (0, 0, \cdots, 0, \quad 1, 1, \cdots, 1, \cdots, \quad k_2, k_2, \cdots, k_2), \tag{3.6}$$

where the integer j occurs $b_j^{(2)}$ times; $j = 0, 1, \cdots, k_2$. The matrix $N'N$ for the Kronecker product $N = N_1 \times N_2$ will clearly contain a row $\rho = \rho_1 \times \rho_2$.

Now pick out the integer 0 in ρ. It arises b_2 times when each of the $b_0^{(1)}$ zeros in ρ_1 is the coefficient of ρ_2 in ρ, and also b_1 times when each of the $b_0^{(2)}$ zeros in ρ_2 multiplies the elements of ρ_1. But in this enumeration of zeros, the multiplication of zeros of ρ_1 and ρ_2 has been counted twice, so that actually the number of zeros in ρ is $b_0^{(0)} = b_0^{(2)}b_1 + b_0^{(1)}b_2 - b_0^{(1)}b_0^{(2)}$, which is exactly the coefficent of $\{0\}$ in (3.3) when expanded according to the properties (3.4).

Similarly, the integer 1 occurs only in those places where one of the $b_1^{(1)}$ 1's of ρ_1 multiplies one of the $b_1^{(2)}$ 1's in ρ_2. Hence we must have $b_1^{(0)} = b_1^{(1)} \cdot b_1^{(2)}$ which is exactly the coefficient of $\{1\}$ in the expression (3.3) when expanded according to the properties (3.4).

In the same way, it is easy to verify that the integer u occurs in ρ $b_u^{(0)}$ times where $b_u^{(0)}$ is the coefficient of $\{u\}$ in (3.3). This proves the theorem.

From the block structures of affine resolvable BIB designs [2] and symmetrical BIB designs [1], we can easily deduce the following corollaries of Theorem 3.2.

COROLLARY 3.2.1. *If a* PBIB *design with three associate classes and with parameters* (2.5) *is the Kronecker product of the affine resolvable* BIB *design with parameters*

$$v_1 = nk_1 = n^2\{(n - 1)t + 1\}, \qquad b_1 = nr_1 = n\{n^2t + n + 1\}, \qquad \lambda_1 = nt + 1, \tag{3.7}$$

and the symmetrical BIB *design with parameters*

$$v_2 = b_2, \qquad r_2 = k_2, \qquad \lambda_2, \tag{3.8}$$

then with respect to any block B in it, the other blocks fall into four groups (α), (β), (γ), *and* (δ) *such that the group* (α) *contains* $n_1' = b_2 - 1 = v_2 - 1$ *blocks each having* $k_1\lambda_2$ *treatments in common with B, the group* (β) *contains* $b_1 - n$ *blocks each having* mr_2 *treatments in common with B, the group* (γ) *contains* $n_1'(b_1 - n)$

blocks each having $m\lambda_2$ *treatments in common with* B, *and the group* (δ) *contains* $b_2(n - 1)$ *blocks each having zero treatments in common with* B, *where*

$$m = (k_1)^2/v_1 = (n - 1)t + 1.$$

The groups (α), (β), (γ), (δ) *are all distinct if* $n\lambda_2 \neq k_2$.

COROLLARY 3.2.2. *If a* PBIB *design with three associate classes and with parameters* (2.5) *is the Kronecker product of the two affine resolvable* BIB *designs with parameters.*

$$(3.9)\quad \begin{aligned} v_1 &= n_1k_1 = n_1^2\{(n_1 - 1)t_1 + 1\}, \\ b_1 &= n_1r_1 = n_1\{n_1^2t_1 + n_1 + 1\}, \qquad \lambda_1 = n_1t_1 + 1 \end{aligned}$$

and

$$(3.10)\quad \begin{aligned} v_2 &= n_2k_2 = n_2^2\{(n_2 - 1)t_2 + 1\}, \\ b_2 &= n_2r_2 = n_2\{n_2^2t_2 + n_2 + 1\}, \qquad \lambda_2 = n_2t_2 + 1, \end{aligned}$$

then with respect to any block B *in it, the other blocks fall into four groups* (α), (β), (γ) *and* (δ), *such that the group* (α) *contains* $b_2 - n_2$ *blocks each having* m_2k_1 *treatments in common with* B, *the group* (β) *contains* $b_1 - n_1$ *blocks each having* m_1k_2 *treatments in common with* B, *the group* (γ) *contains* $(b_1 - n_1)(b_2 - n_2)$ *blocks each having* m_1m_2 *treatments in common with* B, *and the group* (δ) *contains*

$$b_1(n_2 - 1) + b_2(n_1 - 1) - (n_1 - 1)(n_2 - 1)$$

blocks each having zero treatments in common with B, *where*

$$m_1 = (k_1)^2/v_1 = (n_1 - 1)t_1 + 1 \quad \text{and} \quad m_2 = (k_2)^2/v_2 = (n_2 - 1)t_2 + 1.$$

The groups (α), (β), (γ), (δ) *are all distinct if* $n_1 \neq n_2$.

4. Concluding remarks.

(i) A similar analysis can be carried out for the PBIB designs with two associate classes which are Kronecker product of BIB designs (cf. 2B above).

(ii) It is easy to see that a PBIB design which is the Kronecker product of a resolvable BIB design and another BIB design is also resolvable.

(iii) It is interesting to note clearly the connection between corollaries to Theorem 3.2 on the one hand and Theorem 2.1 on the inversion of designs on the other. For example, from Corollary 3.2.1 one may gather the false impression that the Kronecker product of an affine resolvable BIB design and a symmetrical BIB design would lead on inversion to a PBIB design with four associate classes. Remembering, however, that an affine resolvable BIB design gives on inversion a PBIB design with two associate classes and that a symmetrical BIB design is self-dual, we find from Theorem 2.1 and Theorem 4.2 of [7], that the dual of the Kronecker product under consideration is, in fact, a PBIB design with five associate classes all of which are distinct. This apparent contradiction is resolved if we observe that the number of distinct associate classes in a PBIB design depends not only on its λ parameters but also on the matrices

(p^i_{ju}) of its secondary parameters, the exact relation being given in Lemma 4.1 of [7], whereas for finding relations among the blocks of the inverted design we are concerned only with the number of different λ parameters of the PBIB design. Thus in the example under discussion, the PBIB design with five distinct associate classes has two of its λ parameters equal to zero, and therefore there are only four different λ parameters which determine the four types of relations among the blocks of the inverted design.

Similar remarks apply to the PBIB designs obtained in 2B and their duals.

Further work of this type applicable to PBIB designs in general is under progress and the author hopes to publish a separate paper dealing with it.

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