PÓLYA TYPE DISTRIBUTIONS OF CONVOLUTIONS[1]

By Samuel Karlin and Frank Proschan

Stanford University

1. Introduction. The theory of totally positive kernels and Pólya type distributions has been decisively and extensively applied in several domains of mathematics, statistics, economics and mechanics. Totally positive kernels arise naturally in developing procedures for inverting, by differential polynomial operators [7], integral transformations defined in terms of convolution kernels. The theory of Pólya type distributions is fundamental in permitting characterizations of best statistical procedures for decision problems [8] [9] [13]. In clarifying the structure of stochastic processes with continuous path functions we encounter totally positive kernels [11] [12]. Studies in the stability of certain models in mathematical economics frequently use properties of totally positive kernels [10]. The theory of vibrations of certain types of mechanical systems (primarily coupled systems) involves aspects of the theory of totally positive kernels [5].

In this paper, we characterize new classes of totally positive kernels that arise from summing independent random variables and forming related first passage time distributions.

A function $f(x, y)$ of two real variables ranging over linearly ordered one dimensional sets X and Y respectively, is said to be *totally positive of order* k (TP_k) if for all $x_1 < x_2 < \cdots < x_m$, $y_1 < y_2 < \cdots < y_m$, $(x_i \varepsilon X; y_j \varepsilon Y)$ and all $1 \leqq m \leqq k$,

$$(1)\qquad f\begin{pmatrix} x_1, x_2, \cdots, x_m \\ y_1, y_2, \cdots, y_m \end{pmatrix} = \begin{vmatrix} f(x_1, y_1) & f(x_1, y_2) & \cdots & f(x_1, y_m) \\ f(x_2, y_1) & f(x_2, y_2) & \cdots & f(x_2, y_m) \\ \vdots & \vdots & \ddots & \vdots \\ f(x_m, y_1) & f(x_m, y_2) & \cdots & f(x_m, y_m) \end{vmatrix} \geqq 0.$$

Typically, X is an interval of the real line, or a countable set of discrete values on the real line such as the set of all integers or the set of non-negative integers; similarly for Y. When X or Y is a set of integers, we may use the term "sequence" rather than "function."

A related, weaker property is that of sign regularity. A function $f(x, y)$ *is sign regular of order* k, if for every $x_1 < x_2 < \cdots < x_m$, $y_1 < y_2 < \cdots < y_m$, and $1 \leqq m \leqq k$, the sign of

$$f\begin{pmatrix} x_1, x_2, \cdots, x_m \\ y_1, y_2, \cdots, y_m \end{pmatrix}$$

depends on m alone.

If a TP_k function $f(x, y)$ is a probability density in one of the variables, say x, with respect to a σ-finite measure $\mu(x)$, for each fixed value of y, then $f(x, y)$

Received August 24, 1959; revised April 1, 1960.

[1] This work was supported by the Office of Naval Research under Task NR 047-019.

is said to be *Pólya type of order k* (PT_k). The concepts of PT_1 and PT_2 densities are familiar ones. Every density characterized by a parameter is PT_1; while the PT_2 densities are those having a monotone likelihood ratio [13].

A further specialization occurs if a PT_k kernel may be written as a function $f(x - y)$ of the difference of x and y where x and y traverse the real line; $f(u)$ is then said to be a *Pólya frequency density of order k* (PF_k).

Finally, if the subscript ∞ is written in any of the definitions, then the property in question will be understood to hold for all positive integers.

2. Summary of Results. From Lemma 3 below we trivially obtain the result that if $f_1, f_2, \cdots$ are density functions of non-negative random variables with each f_i a PF_k, then $g(n, x) = f_1 * f_2 * \cdots * f_n(x)$ ($*$ indicates convolution) is PF_k in differences of x for each $n > 0$. One of the key results of this paper is that under the same hypothesis $g(n, x)$ is PT_k in the variables n and x, where n ranges over the positive integers and x traverses the positive real line. That is, total positivity in translation variables (differences of the argument) for each density implies total positivity in the *pair*: the argument and the order of the convolution. (Theorem 1 of Section 4.)

As an easy consequence, we obtain that

$$h(n, x) = P[\sum_{i=1}^{n} X_i \geqq x],$$

where the X_i are independent observations from the corresponding f_i, $i = 1, 2, \cdots$, is TP_k in the variables n and x. The kernel $h(n, x)$ can be interpreted as the probability that first passage into the set $[x, \infty)$ occurs at or before the nth transition where the successive partial sums $S_n = \sum_{i=1}^{n} X_i$, $n = 0, 1, 2, \cdots$ ($S_0 = 0$) describe a discrete time real valued Markov process. If X_i are not identically distributed then the process is not time homogeneous. In this formulation the statement concerning the first passage probability function can be extended to the case of random variables ranging over the whole real line. Thus Theorem 2 of Section 4 asserts that for Pólya frequency densities of a given order, the probability that first passage into the set $[x, \infty)$ for the stochastic process of successive partial sums occurs at the nth transition, is a totally positive function in the variables n, x of the same order. In this framework, Theorem 1 can be deduced from Theorem 2 by employing a suitable limiting argument. Further results of this sort are given in Section 4 and Section 5.

A different kind of characterization is given in Theorem 8 of Section 6. There it is shown that $g(n, x)$, the n-fold convolution of a PF_k density extending over the whole real line, although not possessing the full variation diminishing property of a TP_k function, does possess a restricted variation limiting property. Specifically, $\sum_{i=1}^{m} a_i g(n_i, x)$ has at most $2(m - 1)$ sign changes, where

$$n_1 < n_2 < \cdots < n_m, \; m \leqq (k + 1)/2,$$

and the a_i are real non-zero constants.[2]

[2] The number of sign changes $V(f)$ of a real valued function f is $\sup_{1 \leqq i \leqq m} V(f(x_i))$ where

In Section 7 we establish several smoothening properties possessed by the kernel $f^{(n)}(x)$ (the n-fold convolution of f), when it defines a linear transformation. In particular, we prove that if $f(x)$ is PF_3 and $g(x)$ is convex (concave) then $h(n) = \int f^{(n)}(x)g(x)\,dx$ is convex (concave): This fact is useful in applications.

In Sections 8 and 9 various applications of these results are noted. The inventory problem discussed in Section 8 originally motivated the theoretical results of the present paper; it is exposed here to illustrate the kind of applications made available by exploiting the theorems of Section 4. It is possible to show with the aid of Theorem 1 that the objective function of the inventory problem is concave, so that its maximization becomes a relatively easy task and can be reduced to a rather standard non-linear programming calculation.

In Section 9, a number of totally positive functions are constructed by forming successive convolutions of Pólya frequency densities and then applying Theorem 1. As an illustration of the theory we obtain that $g(n, x) = (x - A_n)^{K_n}$ for $x > A_n$ and 0 for $x \leqq A_n$ is TP_∞ in x and n, provided A_n is any increasing function of n and K_n is any strictly increasing integer-valued function of n.

In a subsequent publication, Karlin will indicate other generalizations and applications of the results of this paper to the theory of stochastic processes and orthogonal polynomials. For example, we will extend the results from a discrete time formulation corresponding to integer convolutions to a continuous time stochastic process structure. In this framework the present theory bears a close relationship to some recent studies of Karlin and McGregor [11] concerned with totally positive kernels and diffusion processes. We will also develop further the connections of total positivity and absorption and recurrence probabilities for the state variable of certain kinds of stochastic processes.

In [15], Proschan has discussed in detail the inventory model described in Section 8 with applications to some concrete examples. Theorem 1 plays a crucial role in this study.

3. Preliminaries. Many of the structural properties of TP_k functions are deducible from the following identity, which appears in [14], p. 48, problem 68:

LEMMA 1: *If $r(x, w) = \int p(x, t)q(t, w)\,d\sigma(t)$ and the integral converges absolutely, then*

$$r\begin{pmatrix} x_1, x_2, \cdots, x_k \\ w_1, w_2, \cdots, w_k \end{pmatrix} = \underset{t_1<t_2<\cdots<t_k}{\int\int\cdots\int} \cdot p\begin{pmatrix} x_1, x_2, \cdots, x_k \\ t_1, t_2, \cdots, t_k \end{pmatrix} q\begin{pmatrix} t_1, t_2, \cdots, t_k \\ w_1, w_2, \cdots, w_k \end{pmatrix} d\sigma(t_1)\, d\sigma(t_2). \quad \cdots d\sigma(t_k). \tag{2}$$

In particular, we secure from Lemma 1, the following useful result:

LEMMA 2: *If $f(x, t)$ is TP_m and $g(t, w)$ is TP_n, then $h(x, w) = \int f(x, t)\, g(t, w)d\sigma(t)$ is $TP_{\min(m,n)}$* provided $\sigma(t)$ is a regular σ finite measure.

$V(f(x_i))$ is the number of sign changes of the sequence $f(x_1), f(x_2), \cdots, f(x_m)$ with x_i chosen arbitrarily from the domain of definition of f and arranged so that $x_1 < x_2 < \cdots < x_m$ and m any positive integer.

We shall exploit this result principally in the case when f and g are Pólya frequency densities: Therefore,

LEMMA 3: *If $f(x)$ is PF_m and $g(x)$ is PF_n, then $h(x) = \int f(x-t)\, g(t)\, dt$ is $PF_{\min(m,n)}$.*

An important feature of totally positive functions is their variation diminishing property: If $f(x, w)$ is TP_k and $g(w)$ changes sign $j \leqq k-1$ times, then $h(x) = \int f(x, w)\, g(w)\, d\sigma(w)$ changes sign at most j times; moreover, if $h(x)$ actually changes sign j times, then it must change sign in the same order as $g(w)$ as x and w traverse the real line from left to right [8] [9]. This distinctive property underlies many of the applications mentioned above. The variation diminishing property is essentially equivalent to the determinantal inequalities (1).

4. Convolution of Non-Negative Random Variables. We first prove

THEOREM 1: *Let $f_1, f_2, \cdots$ be any sequence of densities of non-negative random variables, with each f_i a PF_k. Then the n-fold convolution $g(n, x) = f_1 * f_2 * \cdots * f_n(x)$ is PT_k in the variables n and x, where n ranges over $1, 2, \cdots$ and x traverses the positive real line.*

PROOF: The proof employs induction. First note that $g(n, x)$ is PT_1 since $g(n, x) \geqq 0$ for each real x and each positive integer n.

Assume now that for *every* sequence of densities satisfying the hypothesis, the corresponding n-fold convolution has been proven PT_{r-1} for $r \leqq k$. We prove that this implies $g(n, x)$ is PT_r.

(a) First consider the case $n_1 = 1$. Given $1 < n_2 < n_3 < \cdots < n_r$, $0 \leqq x_1 < x_2 < \cdots < x_r$, we may write

$$(3) \quad g\begin{pmatrix} 1, n_2, \cdots, n_r \\ x_1, x_2, \cdots, x_r \end{pmatrix} = \sum_{\nu=1}^{r} (-1)^{\nu-1} f_1(x_\nu)\, g\begin{pmatrix} n_2, n_3, \cdots & & , n_r \\ x_1, \cdots, x_{\nu-1}, x_{\nu+1}, \cdots, x_r \end{pmatrix}$$

simply by expanding the determinant on the left by its first row. Next note that for $n = 2, 3, \cdots$ and $x \geqq 0$,

$$(4) \qquad g(n, x) = \int g_1(n-1, \xi) f_1(x - \xi)\, d\xi,$$

where $g_1(n-1, \xi)$ is defined as $f_2 * f_3 * \cdots * f_n(\xi)$. Applying (2) in (4), we may write

$$(5) \quad \begin{aligned} g\begin{pmatrix} n_2, n_3, \cdots & & , n_r \\ x_1, \cdots, x_{\nu-1}, x_{\nu+1}, \cdots, x_r \end{pmatrix} &= \int\int_{0 \leqq \xi_1 < \xi_2 < \cdots < \xi_{r-1}} \cdots \int \\ &\cdot g_1\begin{pmatrix} n_2 - 1, n_3 - 1, \cdots, n_r - 1 \\ \xi_1, \xi_2, \cdots \quad , \xi_{r-1} \end{pmatrix} f_1\begin{pmatrix} x_1, \cdots, x_{\nu-1}, x_{\nu+1}, \cdots, x_r \\ \xi_1, \xi_2, \quad \cdots \quad , \xi_{r-1} \end{pmatrix} \cdot \\ &\qquad d\xi_1\, d\xi_2 \cdots d\xi_{r-1}. \end{aligned}$$

Inserting (5) into (3), we get immediately,

$$g\begin{pmatrix}1, n_2, \cdots, n_r\\ x_1, x_2, \cdots, x_r\end{pmatrix} = \iint_{0<\xi_1<\xi_2<\cdots<\xi_{r-1}} \cdots \int g_1\begin{pmatrix}n_2-1, n_3-1, \cdots, n_r-1\\ \xi_1, \xi_2, \cdots \quad , \xi_{r-1}\end{pmatrix}.$$

(6)

$$\sum_{\nu=1}^{r} (-1)^{\nu-1} f_1(x_\nu) f_1\begin{pmatrix}x_1, \cdots, x_{\nu-1}, x_{\nu+1}, \cdots, x_r\\ \xi_1, \xi_2, \cdots \quad , \xi_{r-1}\end{pmatrix} d\xi_1\, d\xi_2 \cdots d\xi_{r-1} =$$

$$\iint_{0\leqq\xi_1<\xi_2<\cdots<\xi_{r-1}} \cdots \int g_1\begin{pmatrix}n_2-1, n_3-1, \cdots, n_r-1\\ \xi_1, \xi_2, \cdots \quad , \xi_{r-1}\end{pmatrix} \cdot f_1\begin{pmatrix}x_1, x_2, \cdots \quad , x_r\\ 0, \xi_1, \xi_2, \cdots, \xi_{r-1}\end{pmatrix} d\xi_1\, d\xi_2 \cdots d\xi_{r-1}.$$

But $g_1\begin{pmatrix}n_2-1, n_3-1, \cdots, n_r-1\\ \xi_1 \quad, \xi_2 \quad, \cdots, \xi_{r-1}\end{pmatrix} \geqq 0$ by the induction assumption, while $f_1\begin{pmatrix}x_1, x_2, x_3, \cdots, x_r\\ 0, \xi_1, \xi_2, \cdots, \xi_{r-1}\end{pmatrix} \geqq 0$ since f_1 is PF_k by the hypothesis of the theorem since $0 \leqq \xi_1 < \xi_2 < \cdots < \xi_{r-1}$. Hence

$$g\begin{pmatrix}1, n_2, \cdots, n_r\\ x_1, x_2, \cdots, x_r\end{pmatrix} \geqq 0. \tag{7}$$

(b) Now suppose $n_1 > 1$. Then for any $n_1 < n_2 < \cdots < n_k$ and $0 \leqq x_1 < x_2 < \cdots < x_k$, we may write, using (2) and (4):

$$g\begin{pmatrix}n_1, n_2, \cdots, n_r\\ x_1, x_2, \cdots, x_r\end{pmatrix} = \iint_{\xi_1<\xi_2<\cdots<\xi_r} \cdots \int g_1\begin{pmatrix}n_1-1, n_2-1, \cdots, n_r-1\\ \xi_1 \quad, \xi_2 \quad, \cdots, \xi_r\end{pmatrix} \cdot f_1\begin{pmatrix}x_1, x_2, \cdots, x_r\\ \xi_1, \xi_2, \cdots, \xi_r\end{pmatrix} d\xi_1\, d\xi_2 \cdots d\xi_r. \tag{8}$$

From (8) we see that for every sequence of densities satisfying the hypothesis, the corresponding functions g_1, g satisfy

$$g_1\begin{pmatrix}n_1-1, n_2-1, \cdots, n_r-1\\ \xi_1 \quad, \xi_2 \quad, \cdots, \xi_r\end{pmatrix} \geqq 0 \Rightarrow g\begin{pmatrix}n_1, n_2, \cdots, n_r\\ x_1, x_2, \cdots, x_r\end{pmatrix} \geqq 0. \tag{9}$$

Using (7) and (9), it follows by induction that $g\begin{pmatrix}n_1, n_2, \cdots, n_r\\ x_1, x_2, \cdots, x_r\end{pmatrix} \geqq 0$.

Since $g(n, x)$ has thereby been proven PT_r, we have established the validity of the induction step, and the theorem follows.

It is important to emphasize the distinction between Lemma 3 and Theorem 1. Under the hypothesis of Theorem 1, Lemma 3 states that for each fixed positive integer n, $g(n, x)$ is PT_k in differences of x, while Theorem 1 states that $g(n, x)$ is PT_k in the variables n and x.

Will Theorem 1 hold if the random variables are not restricted to be non-negative? In general, the answer is no, as the following example shows.

EXAMPLE: Let $f_1(x) = f_2(x) = \cdots = (1/\sqrt{2\pi})\, e^{-x^2/2}$, a PF_∞. Then

$$g(n, x) = f^{(n)}(x) = (1/\sqrt{2\pi n})\, e^{-x^2/2n}.$$

For $1 \leqq n_1 < n_2$, $x_1 < x_2$, the second order determinant is positive for $0 \leqq x_1 < x_2$ and negative for $x_1 < x_2 \leqq 0$. Thus $g(n, x)$ is *not* PT_2.

However a generalization of Theorem 1 to the case of random variables ranging over the whole real line *is* possible, as developed in Theorem 2 below. In the more general case, total positivity holds, not for the n-fold convolution, but rather for the first passage time probabilities of the partial sum process.

THEOREM 2: *Let $f_1, f_2, \cdots$ be any sequence of PT_k densities of random variables $X_1, X_2, \cdots$ respectively, which are not necessarily non-negative. Consider the first passage probability for x positive:*

$$h(n, x) = P\left[\sum_{i=1}^{n} X_i \geqq x; \quad \sum_{i=1}^{j} X_i < x, j = 1, 2, \cdots, n-1\right] \qquad \text{for } n = 1, 2, \cdots.$$

Then $h(n, x)$ is TP_k, where n ranges over $1, 2, \cdots$ and x traverses the positive axis.

PROOF: The proof proceeds in a similar fashion to that of Theorem 1. We employ induction. First we note that $h(n, x)$ is TP_1 since $h(n, x) \geqq 0$ by its very meaning.

Assume now that for every sequence of densities satisfying the hypothesis, the associated first passage time probability function is TP_{r-1} for $r \leqq k$. We shall prove that this implies $h(n, x)$ is TP_r. From this the conclusion of the theorem will follow.

We clearly have for x positive that

$$h(n, x) = \begin{cases} \int_x^\infty f(\xi)\, d\xi & \text{for } n = 1 \\ \int_0^\infty f(x - \xi) h_1(n-1, \xi)\, d\xi & \text{for } n \geqq 2 \end{cases} \tag{10}$$

where

$$h_1(n-1, \xi) = P\left[\sum_{i=2}^{n} X_i \geqq \xi; \quad \sum_{i=2}^{j} X_i < \xi, j = 2, 3, \cdots, n-1\right].$$

We consider first the case $n_1 = 1$. Given $1 < n_2 < n_3 < \cdots < n_r$, $x_1 < x_2 < \cdots < x_r$, we may write, using (10),

$$h\begin{pmatrix} 1, n_2, n_3, \cdots, n_r \\ x_1, x_2, x_3, \cdots, x_r \end{pmatrix}$$

$$= \begin{vmatrix} \int_0^\infty f_1(x_1 + \xi)\, d\xi & \int_0^\infty f_1(x_2 + \xi)\, d\xi & \cdots & \int_0^\infty f_1(x_r + \xi)\, d\xi \\ h(n_2, x_1) & h(n_2, x_2) & \cdots & h(n_2, x_r) \\ \vdots & \vdots & \ddots & \vdots \\ h(n_r, x_1) & h(n_r, x_2) & \cdots & h(n_r, x_r) \end{vmatrix}$$

$$= \sum_{\nu=1}^{r} (-1)^{\nu-1} \int_0^\infty f_1(x_\nu + \xi) h\begin{pmatrix} n_2, n_3, \cdots & & , n_r \\ x_1, \cdots, x_{\nu-1}, & x_{\nu+1}, \cdots & , x_r \end{pmatrix} d\xi.$$

Now using (10) and (2), we obtain

$$h\begin{pmatrix} n_2, n_3, \cdots & , n_r \\ x_1, \cdots, x_{\nu-1}, x_{\nu+1}, \cdots, x_r \end{pmatrix} = \iint_{0 \leqq \xi_1 < \xi_2 < \cdots < \xi_{r-1}} \cdots \int \tag{11}$$

$$\cdot f_1\begin{pmatrix} x_1, \cdots, x_{\nu-1}, x_{\nu+1}, \cdots, x_r \\ \xi_1, \xi_2, \cdots & , \xi_{r-1} \end{pmatrix} h_1\begin{pmatrix} n_2 - 1, n_3 - 1, \cdots, n_r - 1 \\ \xi_1, \xi_2, \cdots & , \xi_{r-1} \end{pmatrix}.$$

$$d\xi_1\, d\xi_2 \cdots d\xi_{r-1}.$$

Inserting (11) in the equation above and replacing $-\xi$ by ξ, gives

$$h\begin{pmatrix} 1, n_2, n_3, \cdots, n_r \\ x_1, x_2, x_3, \cdots, x_r \end{pmatrix} = \iint_{\xi < 0 \leqq \xi_1 < \xi_2 < \cdots < \xi_{r-1}} \cdots \int$$

$$\cdot h_1\begin{pmatrix} n_2 - 1, n_3 - 1, \cdots, n_r - 1 \\ \xi_1, \xi_2, \cdots & , \xi_{r-1} \end{pmatrix} \sum_{\nu=1}^{r} (-1)^{\nu-1} f_1(x_\nu - \xi).$$

$$f_1\begin{pmatrix} x_1, \cdots, x_{\nu-1}, x_{\nu+1}, \cdots, x_r \\ \xi_1, \xi_2, \cdots & , \xi_{r-1} \end{pmatrix} d\xi\, d\xi_1\, d\xi_2 \cdots d\xi_{r-1}$$

$$= \iint_{\xi < 0 \leqq \xi_1 < \xi_2 < \cdots < \xi_{r-1}} \cdots \int h_1\begin{pmatrix} n_2 - 1, n_3 - 1, \cdots, n_r - 1 \\ \xi_1, \xi_2, \cdots & , \xi_{r-1} \end{pmatrix}$$

$$\cdot f_1\begin{pmatrix} x_1, x_2, \cdots, x_r \\ \xi, \xi_1, \xi_2, \cdots, \xi_{r-1} \end{pmatrix} d\xi\, d\xi_1\, d\xi_2. \qquad \cdots d\xi_{r-1}.$$

But $h_1\begin{pmatrix} n_2 - 1, n_3 - 1, \cdots, n_r - 1 \\ \xi_1 \quad, \xi_2 \quad, \cdots, \xi_{r-1} \end{pmatrix} \geqq 0$ by the inductive assumption, while $f_1\begin{pmatrix} x_1, x_2, \cdots & , x_r \\ \xi, \xi_1, \xi_2, \cdots, \xi_{r-1} \end{pmatrix} \geqq 0$ since $\xi < \xi_1 < \xi_2 < \cdots < \xi_{r-1}$, $x_1 < x_2 < \cdots < x_r$, and $f_1(x)$ is PF_k by hypothesis. Hence $h\begin{pmatrix} 1, n_2, n_3, \cdots, n_r \\ x_1, x_2, x_3, \cdots, x_r \end{pmatrix} \geqq 0$.

The remainder of the proof parallels the corresponding portion of the proof of Theorem 1; simply replace g by h.

It is appropriate to compare Theorem 1 and Theorem 2. For this purpose we sketch an argument which shows that Theorem 1 is actually a limiting case of Theorem 2. A careful examination of the preceding argument reveals that in the case of non-negative PF_k random variables, the probability of first passage at time n into any positive interval, not only the interval $[x, \infty]$, is TP_k. In view of this fact we shrink the interval to a point, and it readily follows that the first passage time probability converges to the density corresponding to the n-fold convolution. Since total positivity is preserved under this limiting operation, Theorem 1 follows.

We now develop a series of consequences of Theorems 1 and 2. Let $F_i(x)$ be the cumulative distribution function corresponding to $f_i(x)$, $i = 1, 2, \cdots$. Then as a direct corollary of Theorem 2, we have

THEOREM 3: *Under the assumptions of Theorem* 1,

$$h(n, x) = F_1 * F_2 * \cdots * F_{n-1}(x) - F_1 * F_2 * \cdots * F_n(x) \quad is \quad TP_k,$$

where n ranges over $1, 2, \cdots$ and $x > 0$. In particular, if $f_i = f$, $i = 1, 2, \cdots$, then $h(n, x) = F^{(n-1)}(x) - F^{(n)}(x)$ is TP_k.

PROOF: Simply note that

$$h(n, x) = P[\textstyle\sum_{i=1}^{n} X_i \geqq x; \quad \sum_{i=1}^{n-1} X_i < x]$$

since the random variables are non-negative.

Actually we can say more about $h(n, x)$ in the situation where the X_i are non-negative, independent, and identically distributed random variables; Theorem 4 asserts that for each fixed $x > 0$, $h(n + m, x)$ is sign regular in the variables $n \geqq 0$ and $m \geqq 0$.

THEOREM 4: *Suppose $f(x)$ is PF_k with $f(x) = 0$ for $x < 0$. We define $h(n, x)$ by $h(n, x) = F^{(n-1)}(x) - F^{(n)}(x)$ for $n = 1, 2, \cdots$; $x \geqq 0$, and for fixed $x \geqq 0$ we define*

$$c(n) = \begin{cases} h(n, x) & \text{for } n \geqq 1 \\ 0 & \text{for } n \leqq 0. \end{cases}$$

Then $c(n + m)$ is sign regular of order k in $n \geqq 1$ and $m \geqq 1$; moreover, for $1 \leqq n_1 < n_2 < \cdots < n_r$, $1 \leqq m_1 < m_2 < \cdots < m_r$, the sign of

$$c_+\begin{pmatrix} n_1, n_2, \cdots, n_r \\ m_1, m_2, \cdots, m_r \end{pmatrix} \quad \textit{is} \quad (-1)^{r(r-1)/2}, \quad \textit{where} \quad c_+\begin{pmatrix} n_1, n_2, \cdots, n_r \\ m_1, m_2, \cdots, m_r \end{pmatrix}$$

$$= \begin{vmatrix} c(n_1 + m_1) & \cdots & c(n_1 + m_r) \\ \vdots & \ddots & \vdots \\ c(n_r + m_1) & \cdots & c(n_r + m_r) \end{vmatrix}.$$

PROOF: For $m \geqq 1$ and $n \geqq 1$, we have

$$c(n + m) = \int g(m, \xi) h(n, x - \xi)\, d\xi; \tag{12}$$

where $g(m, \xi) = f^{(m)}(\xi)$. (12) simply states that if the partial sum first exceeds x at the $n + m$th stage, then this can occur by having the mth partial sum equal to some non-negative $\xi < x$, while the partial sum starting with the $m + 1$st variable first exceeds $x - \xi$ at the nth stage. From (12) and (2), we get, for $1 \leqq n_1 < n_2 < \cdots < n_r$, $1 \leqq m_1 < m_2 < \cdots < m_r$, $r \leqq k$,

$$c_+\begin{pmatrix} n_1, n_2, \cdots, n_r \\ m_1, m_2, \cdots, m_r \end{pmatrix} = \iint_{0 \leqq \xi_1 < \xi_2 < \cdots < \xi_r < x} \cdots \int g\begin{pmatrix} m_1, m_2, \cdots, m_r \\ \xi_1, \xi_2, \cdots, \xi_r \end{pmatrix} \cdot h\begin{pmatrix} n_1, & n_2, & \cdots, & n_r \\ x - \xi_1, & x - \xi_2, & \cdots, & x - \xi_r \end{pmatrix} d\xi_1, d\xi_2, \cdots, d\xi_r. \tag{13}$$

By Theorem 1, $g\begin{pmatrix} m_1, m_2, \cdots, m_r \\ \xi_1, \xi_2, \cdots, \xi_r \end{pmatrix} \geqq 0$. Since the $x - \xi_1, x - \xi_2, \cdots, x - \xi_r$ are in decreasing order of magnitude it follows, invoking Theorem 3, that

$h\binom{n_1 \quad , n_2 \quad , \cdots , n_r}{x - \xi_1 , x - \xi_2 , \cdots , x - \xi_r}$ has the sign $(-1)^{r(r-1)/2}$. Thus

$$c_+ \binom{n_1, n_2, \cdots, n_r}{m_1, m_2, \cdots, m_r}$$

has the sign $(-1)^{r(r-1)/2}$ as was to be proved.

We prove below that $c(n)$ has the property of being PF_2 provided $f(x)$ is PF_2 (Theorem 5). This is the property required for the analysis of the inventory model of Section 8. In contrast, this relationship between $c(n)$ and $f(x)$ does not persist beyond the second order.

THEOREM 5: *If $f(x)$ is PF_2 with $f(x) = 0$ for $x < 0$, then $c(n)$ (defined in Theorem 4)* is PF_2.

PROOF: Let $n_1 < n_2$, $m_1 < m_2$. Write

$$c\binom{n_1, n_2}{m_1, m_2} = \begin{vmatrix} c(n_1 - m_1) & c(n_1 - m_2) \\ c(n_2 - m_1) & c(n_2 - m_2) \end{vmatrix}.$$

(a) If $n_1 \leqq m_2$, then $c(n_1 - m_2) = 0$, so that $c\binom{n_1 , n_2}{m_1 , m_2} \geqq 0$.

(b) If $n_1 > m_2$, we must have $m_1 < m_2 < n_1 < n_2$. Hence

$$c\binom{n_1, n_2}{m_1, m_2} = \begin{vmatrix} h(n_1 - m_1, x) & h(n_1 - m_2, x) \\ h(n_2 - m_1, x) & h(n_2 - m_2, x) \end{vmatrix}$$

$$= \int \begin{vmatrix} h(n_1 - m_2, \xi) & h(n_1 - m_2, x) \\ h(n_2 - m_2, \xi) & h(n_2 - m_2, x) \end{vmatrix} g(m_2 - m_1, x - \xi)\, d\xi.$$

Since $\xi < x$, $n_1 - m_2 < n_2 - m_2$, and $h(n, x)$ is TP_2 by Theorem 3, then $c\binom{n_1 , n_2}{m_1 , m_2} \geqq 0$ and the proof is finished.

5. Compound Distributions. As an easy corollary of Theorem 1, we have corresponding determinantal properties for compound distributions composed from PT_k densities. Specifically:

THEOREM 6: *Let $X_i \geqq 0$ be distributed with density $f_i(x)$, a PF_k, $i = 1, 2, \cdots$. Define $S_n = \sum_{i=1}^{N} X_i$, where N is a random variable independent of $X_1, X_2, \cdots$, with density $d(n, \mu)$, where μ is a parameter, and $d(n, \mu)$ is PT_k in the variables n and μ. Then $r(x, \mu)$, the probability density for S_N, is PT_k in the variables $x > 0$ and μ.*

PROOF: $r(x, \mu) = \sum_{n=1}^{\infty} P[N = n]\, f_1 * f_2 * \cdots * f_n(x) = \sum_{n=1}^{\infty} d(n, \mu)$ $g(n, x)$. By Theorem 1, $g(n, x)$ is PT_k. Applying Lemma 2 we conclude that $r(x, \mu)$ is also PT_k.

In a similar fashion, we may study transforms of $g(n, x)$ in the variable x; the proof is as in Theorem 6.

THEOREM 7: *In addition to the hypothesis of Theorem 1 assume that $\varphi(x, s)$ is a PT_k function. Then $\phi(n, s) = \int g(n, x)\varphi(x, s)\, dx$ is PT_k.*

As an illustration, let $\varphi(x, s) = e^{xs}$, $-\infty < s \leqq 0$, so that $\varphi(x, s)$ is PT_∞. From Theorem 7, we have that $\phi(n, s)$ is PT_k in the variables n and s. But $\phi(n, s)$ is the Laplace transform of the convolution of n densities and so we have $\phi(n, s) = \phi_1(s)\phi_2(s) \cdots \phi_n(s)$, where $\phi_i(s) = \int f_i(x)e^{xs}\,dx$, $i = 1, 2, \cdots$. In particular, we obtain the interesting set of inequalities:

$$\begin{vmatrix} \phi_1(s_1) & \phi_1(s_1)\phi_2(s_1) & \cdots & \phi_1(s_1)\phi_2(s_1) & \cdots & \phi_m(s_1) \\ \phi_1(s_2) & \phi_1(s_2)\phi_2(s_2) & \cdots & \phi_1(s_2)\phi_2(s_2) & \cdots & \phi_m(s_2) \\ \vdots & \vdots & & \vdots & & \\ \phi_1(s_m) & \phi_1(s_m)\phi_2(s_m) & \cdots & \phi_1(s_m)\phi_2(s_m) & \cdots & \phi_m(s_m) \end{vmatrix} \geqq 0$$

where $s_1 < s_2 < \cdots < s_m \leqq 0$, $m \leqq k$, $\phi_i(s) = \int f_i(x)e^{xs}\,dx$, and $f_i(x)$ is a PF_k density with $f_i(x) = 0$ for $x < 0$, $i = 1, 2, \cdots, k$.

6. Convolution of Random Variables Ranging Over the Real Line.

We have seen on the basis of the example following Theorem 1, that the n-fold convolution $g(n, x)$ of a PF_k density whose possible values extend over the whole real line, is not necessarily PT_k. Thus, in generalizing Theorem 1 to densities whose possible values extend throughout the real line, it was necessary to formulate the problem in terms of first passage probabilities rather than n-fold convolutions. However, the question remains: what smoothening properties are possessed by the n-fold convolution of a PF_k density, which has possible values ranging over the full real line. We can answer this query in terms of a weakened version of the variation diminishing property possessed by totally positive functions. Recall that if $p(x, w)$ is TP_k and $q(w)$ changes sign $j \leqq k - 1$ times, then

$$r(x) = \int p(x, w)q(w)\,dF(w)$$

changes sign at most j times; moreover, if $r(x)$ actually changes sign j times, then it must change sign in the same order as does $q(w)$ [9]. This variation diminishing property may be compared with the following result.

THEOREM 8: *Let $f(x)$ be a continuous PF_k, with $f(x)$ not necessarily 0 for $x < 0$. Let $r_m(x) = \sum_{i=1}^m a_i g(n_i, x)$, where $n_1 < n_2 < \cdots < n_m$, $m \leqq (k + 1)/2$, and the a_i are real non-zero constants. Then $r_m(x)$ has $\leqq 2(m - 1)$ sign changes.*

PROOF: We proceed by induction. The theorem trivially holds for $m = 1$.

Assume the theorem holds for the case of a sum consisting of $m_0 - 1$ terms, where $m_0 \leqq (k + 1)/2$. Write

$$r_{m_0}(x) = \sum_{i=1}^{m_0} a_i g(n_i, x) = \sum_{i=2}^{m_0} a_i \int g(n_i - n_1, \theta)g(n_1, x - \theta)\,d\theta + a_1 \lim_{R\to\infty} \int g_R(0, \theta)g(n_1, x - \theta)\,d\theta,$$

where

$$g_R(0, \theta) = \begin{cases} R & \text{for } 0 \leqq \theta \leqq \dfrac{1}{R} \\ 0 & \text{otherwise.} \end{cases}$$

Factoring, we get

$$(14)\qquad r_{m_0}(x) = \lim_{R\to\infty} \int \left\{ \sum_{i=2}^{m_0} a_i g(n_i - n_1, \theta) + a_1 g_R(0, \theta) \right\} g(n_1, x - \theta)\, d\theta.$$

By the inductive hypothesis, $\sum_{i=2}^{m_0} a_i g(n_i - n_1, \theta)$ has at most $2(m_0 - 2)$ sign changes as a function of θ. With R sufficiently large, $a_1 g_R(0, \theta)$ can introduce at most 2 additional sign changes. Thus for sufficiently large R,

$$\sum_{i=}^{m_0} a_i g(n_i - n_1, \theta) + a_1 g_R(0, \theta)$$

has at most $2(m_0 - 1)$ sign changes. Since $g(n_1, x - \theta)$ is a PF_k, and therefore, variation diminishing, we obtain that the integral of (14) possesses at most $2(m_0 - 1)$ sign changes as a function of x. Taking the limit as $R \to \infty$, the number of sign changes cannot increase, and thus the number of sign changes of $r_{m_0}(x)$ is $\leqq 2(m_0 - 1)$.

Applying induction, we conclude that the theorem holds for $m = 1, 2, \cdots, (k + 1)/2$ and the proof is finished.

7. Preserving Convexity and Concavity. Let $X_i \geqq 0$ be independent random variables distributed according to $f(x)$, a PF_k. We now describe some further smoothening properties possessed by the transformation which maps functions into sequence, viz.

$$h(n) = \int f^{(n)}(x) g(x)\, dx \qquad \text{for } n = 1, 2, \cdots.$$

We show first that the property of convexity is preserved under this transformation. Explicitly, we prove that convexity in $g(x)$ is carried over into convexity in $h(n)$. This will be demonstrated not only for the ordinary notion of convexity, but for a type of convexity of higher order, which notion is made precise below. Similar results hold for concavity.

Assume $f(x)$ is PF_3 and $g(x)$ is convex (of order 2). Let $\mu_i = \int x^i f(x)\, dx$, $i = 1, 2, \cdots$ represent the moments of X. Note that for arbitrary real constants a_0 and a_1,

$$\int \{g(x) - [(a_0/\mu_1)x + a_1]\} f^{(n)}(x)\, dx = h(n) - (a_0 n + a_1).$$

Since $g(x)$ is convex, then $g(x) - [(a_0/\mu_1)x + a_1]$ has at most 2 changes of sign and if 2 changes of sign actually occur, they occur in the order $+ - +$ as x traverses the real axis from $-\infty$ to $+\infty$. Since f is PF_3, then by Theorem 1, $f^{(n)}(x)$ is PT_3 in the variables n and x.

By the variation diminishing property of Pólya type functions, we infer that $h(n) - (a_0 n + a_1)$ will have at most 2 changes of sign. Moreover, if $h(n) - (a_0 n + a_1)$ has exactly 2 changes of sign, then these will occur in the same order as those of $g(x) - [(a_0/\mu_1)x + a_1]$, namely $+ - +$. Since a_0, a_1 are arbitrary, we easily infer that $h(n)$ is a convex function of n.

In a similar fashion we can show that higher order convexity is preserved under this transformation as follows: A function $g(x)$ is said to be convex of order r if for an arbitrary polynomial $p(x) = a_0x^{r-1} + a_1x^{r-2} + \cdots + a_{r-1}$ of degree $r - 1$, $g(x) - p(x)$ has at most r changes of sign, and if r changes of sign actually occur, they occur in the order $+ - + \cdots$.

Assume that $f(x)$ is PF_{r+1} and $g(x)$ is convex of order r. Note that $\int x^k f^{(n)}(x)\,dx = E(X_1 + \cdots + X_n)^k = \mu_1^k n^k +$ lower powers of n. It follows immediately that for an arbitrary polynomial $q(n) = a_0n^{r-1} + a_1n^{r-2} + \cdots + a_{r-1}$ of degree $r - 1$, there exists a polynomial $p(x) = b_0x^{r-1} + b_1x^{r-2} + \cdots + b_{r-1}$ of degree $r - 1$ such that $\int p(x)f^{(n)}(x)\,dx = q(n)$, and hence $\int \{g(x) - p(x)\} f^{(n)}(x)\,dx = h(n) - q(n)$ with $a_0b_0 > 0$. Since $f(x)$ is PF_{r+1}, then by Theorem 1, $f^{(n)}(x)$ is PT_{r+1} in the variables n and x and again by the variation diminishing property of Pólya type functions, we obtain that $h(n) - q(n)$ will have no more changes of sign than $g(x) - p(x)$. But $g(x) - p(x)$ has at most r changes in sign since $g(x)$ is convex of order r; and so $h(n) - q(n)$ has at most r changes of sign. Moreover, if $h(n) - q(n)$ actually has r changes of sign, then they will occur in the same order as those of $g(x) - p(x)$, namely $+ - + \cdots$. Thus $h(n)$ is convex of order r since $q(n)$ was an arbitrary polynomial of degree $r - 1$.

Similar results apply to concavity of higher order. A function $g(x)$ is concave of order r if for an arbitrary polynomial $p(x) = a_0x^{r-1} + a_1x^{r-2} + \cdots + a_{r-1}$ of degree $r - 1$, $g(x) - p(x)$ has at most r changes of sign, and if r changes of sign happen then they occur in the order $- + - \cdots$.

An application may be made to the inventory model discussed in [2], p. 227. The probability density of demand for each period is $f(\xi)$, a PF_3. The policy followed is to maintain the stock size at a fixed level S which will be suitably chosen so as to minimize appropriate expected costs, or is determined by a fixed capacity restriction. At the end of each period an order is placed to replenish the stock consumed during that period so that a constant stock level is maintained on the books. Delivery takes place a periods later. The expected cost for a stationary period as a function of the lag is

$$L(a) = \int_0^S h(S - z)f^{(a)}(z)\,dz + \int_S^\infty p(z - S)f^{(a)}(z)\,dz$$

where S is fixed.

Assume now that h and p are convex increasing functions with $h(0) = p(0) = 0$. Then we may write $L(a) = \int r(z)f^{(a)}(z)\,dz$, where

$$r(z) = \begin{cases} h(S - z) & \text{for } 0 \leqq z \leqq S \\ p(z - S) & \text{for } S < z. \end{cases}$$

Then $r(z)$ is a convex function. Using the preceding results, we conclude that $L(a)$ is a convex function. Thus, if the length of lag should increase, the marginal expected loss increases.

Similar results hold if p and h are concave. Also, if we assume f is PF_{k+1} and p and h are convex (concave) of order k, we may conclude that $L(a)$ is convex (concave) of order k.

8. Application to an Inventory Problem. We wish to determine the initial spare parts kit for a system, which maximizes assurance of no shortage whatsoever during a period of length t, under a budget for spares c_0. We consider only essential components, and assume that a failed component is instantly replaced by a spare, if available. Only spares initially provided may be used for replacement. The system contains d_i operating components of type i, $i = 1, 2, \cdots, k$. The length of life of the jth operating component of the ith type is an independent random variable with PF_k density f_{ij}, $j = 1, 2, \cdots, d_i$. The unit cost of a component of type i is c_i.

Our problem is to find n_i, the number of spares initially stocked of the ith type, $i = 1, 2, \cdots, k$, such that $\prod_{i=1}^{k} P_i(n_i)$ is maximized subject to

$$\sum_{i=1}^{k} n_i c_i \leqq c_0 \quad \text{and} \quad n_i = 0, 1, 2, \cdots \qquad \text{for} \quad i = 1, 2, \cdots, k,$$

where $P_i(m)$ = probability of experiencing $\leqq m$ failures of type i. (See [3], [15] for a detailed discussion of this model and its application to reliability; our present treatment is confined to aspects of the problem relevant to the present paper.)

In [3] and [15], methods are given for computing the solution when each $\ln P_i(m)$ is concave in m, or equivalently, when each $P_i(n - m)$ is a TP_2 sequence in n and m. To show $P_i(n - m)$ is a TP_2 sequence in n and m, we note:

1. $c_{ij}(n)$, the probability of requiring n replacements of operating component i, j, is a PF_2 sequence in n for each fixed i, j by Theorem 5 above.

2. $p_i(n)$, the probability of requiring n replacements of type i, is a PF_2 sequence in n for each i by Lemma 3, since $p_i(n) = c_{i1} * c_{i2} * \cdots * c_{id_i}(n)$.

3. $P_i(n - m)$ is a TP_2 sequence in n, m for each i, since

(a)
$$P_i(n) = \sum_{m=-\infty}^{\infty} p_i(n - m)q(m), \qquad \text{where } q(m) = \begin{cases} 1 & \text{for} \quad m = 0, 1, 2, \cdots \\ 0 & \text{otherwise,} \end{cases}$$

(b) $q(m)$ is a PF_∞ sequence.

(c) The convolution of PF_k sequences is PF_k, by Lemma 3.

Thus when the underlying densities for the life of components are PF_2, the methods given in [3] and [15] for obtaining optimal kits are applicable.

9. Generating Totally Positive Functions. In this section we give a series of examples of the above theorems. These theorems are written in terms of real valued random variables but it should be emphasized that all our results are equally valid for integer valued random variables. The underlying densities are assumed to be the appropriate PF_k sequences. The first few illustrations involve integer valued random variables.

EXAMPLE 1:

(a) Let

$$f(k) = \begin{cases} q & \text{for } k = 0 \\ p & \text{for } k = 1, \\ 0 & \text{for other } k \end{cases} \qquad \text{where } p + q = 1.$$

then $f(k)$ is a PF_∞ sequence by direct verification. Alternately, we may appeal to a classical result of Schoenberg and Edrei which asserts that a sequence is a PF sequence, if and only if its generating function is of the form

$$e^{\gamma s}\prod_1^\infty \{(1 + \alpha_\nu s)/(1 - \beta_\nu s)\}, \gamma \geqq 0, \alpha_\nu \geqq 0, \beta_\nu \geqq 0; \sum \alpha_\nu \quad \text{and} \quad \sum \beta_\nu$$

convergent. (See p. 305, [6].) Applying Theorem 1, we obtain that the binomial density $g(n, k) = f^{(n)}(k) = \binom{n}{k} p^k q^{n-k}$ is PT_∞. It follows that $\binom{n}{k}$ is TP_∞ in the variables n and k.

A direct proof, in this case, is easy. For some of our further examples the result is less apparent

(b) Let

$$f_i(k) = \begin{cases} 1/(1 + p^i) & \text{for } k = 0 \\ p^i/(1 + p^i) & \text{for } k = 1 \\ 0 & \text{for other } k, \end{cases}$$

$i = 1, 2, \cdots$. As pointed out in (a) above, each $\{f_i(k)\}_{k=0,1,\ldots}$ is a PF_∞ sequence. Hence, by Theorem 1, $g(n, k) = f_1 * f_2 * \cdots * f_n(k)$ is PT_∞. But $g(n, k)$ is simply the coefficient of s^k in the generating function $\prod_{i=1}^n [(1 + p^i s)/(1 + p^i)]$ of the n-fold convolution. Using the Gauss identity

$$\prod_{i=1}^n (1 + p^i s) = \sum_{\nu=0}^n \begin{bmatrix} n \\ \nu \end{bmatrix} p^{(\nu^2+\nu)/2} s^\nu,$$

where, by definition,

$$\begin{bmatrix} n \\ \nu \end{bmatrix} = \{(1 - p^n)(1 - p^{n-1}) \cdots (1 - p^{n-\nu+1})/(1 - p^\nu)(1 - p^{\nu-1}) \cdots (1 - p) \qquad \text{for } \nu \leqq n,$$

we find that the coefficient of s^k in $\prod_{i=1}^n[(1 + p^i s)/(1 + p^i)]$ is $\begin{bmatrix} n \\ k \end{bmatrix} p^{(k^2+k)} / \prod_{i=1}^n (1 + p^i)$. Since $p^{(k^2+k)/2}$ is a function of k alone while $\prod_{i=1}^n (1 + p^i)$ is a function of n alone, we conclude that $\begin{bmatrix} n \\ k \end{bmatrix}$ is TP_∞. Note that $\begin{bmatrix} n \\ k \end{bmatrix}$ is a type of generalization of the binomial coefficient $\binom{n}{k}$ since for $p \to 1$, $\begin{bmatrix} n \\ k \end{bmatrix} \to \binom{n}{k}$.

(c) Let $f(k) = q^k p$, $p + q = 1$, $k = 0, 1, 2, \cdots$; $f(k)$, the geometric density, is the probability that the first success in a sequence of Bernoulli trials occurs following k successive failures. The corresponding generating function is $p/(1 - qs)$. By [1], p. 305, $f(k)$ is PF_∞. Now

$$g(n, k) = f^{(n)}(k) = \binom{n + k - 1}{k} p^n q^k,$$

so that $g(n, k)$ represents the probability that the nth success occurs at trial $n + k$ in the sequence of Bernoulli trials. By Theorem 1, $g(n, k)$ is PT_∞. Since p^n is a function of n only, while q^k is a function of k only, we obtain that $\binom{n + k - 1}{k}$ is TP_∞.

(d) Next, let $f_i(k) = q^{ik}(1 - q^i)$, $k = 0, 1, 2, \cdots$, $i = 1, 2, \cdots$. As noted in (c), each $\{f_i(k)\}_{k=0,1\cdots}$ is a PF_∞ sequence. Hence

$$g(n, k) = f_1 * f_2 * \cdots * f_n(k)$$

is PT_∞ by Theorem 1. But $g(n, k)$ is simply the coefficient of s^k in the generating function $\prod_{i=1}^n [(1 - q^i)/(1 - q^i s)]$. Using the Heine hypergeometric relation, [6], p. 8,

$$1 \Big/ \prod_{i=1}^{n} (1 - q^i s) = \sum_{k=1}^{\infty} \frac{[n + 1][n + 2] \cdots [n + k]}{[k][k - 1] \cdots [1]} s^k,$$

where the symbol $[m]$ is defined equal to $[(1 - q^m)/(1 - q)]$. We find that the coefficient of s^k in the generating function is

$$\left\{ \prod_{i=1}^{n} (1 - q^i) \right\} \frac{[n + 1][n + 2] \cdots [n + k]}{[k][k - 1] \cdots [1]}.$$

Since $\prod_{i=1}^n (1 - q^i)$ is a function of n alone, we obtain that

$$\frac{[n + 1][n + 2] \cdots [n + k]}{[k][k - 1] \cdots [1]} \text{ is } TP_\infty.$$

Next we consider an example of the application of Theorem 1 to continuous densities

(e) Let

$$f_i(x) = \begin{cases} (x - a_i)^{k_i - 1} e^{-(x - a_i)}/\Gamma(k_i) & \text{for } x \geqq a_i \\ 0 & \text{for } x < a_i, \end{cases}$$

where k_i is a positive integer, $a_i \geqq 0$, $i = 1, 2, \cdots$; thus $f_i(x)$ is a translated gamma density. Then the characteristic function of $f_i(x)$,

$$\varphi_i(t) = \int_{a_i}^{\infty} e^{itx}((x - a_i)^{k_i - 1} e^{-(x - a_i)})/\Gamma(k_i)\, dx = e^{ita_i}/(1 - it)^{k_i}.$$

Defining $g(n, x) = f_1 * f_2 * \cdots * f_n(x)$, we have for its characteristic function $\exp [it\sum_{i=1}^n a_i]1/(1 - it)^{\sum_{i=1}^n k_i}$; and consequently

$$g(n, x) = \begin{cases} ((x - A_n)^{K_n - 1} e^{-(x - A_n)})/\Gamma(K_n) & \text{for } x \geqq A_n \\ 0 & \text{for } x < A_n , \end{cases}$$

where $A_n = \sum_{i=1}^n a_i$ and $K_n = \sum_{i=1}^n k_i$. This means that $g(n, x)$ is also a translated gamma with parameters corresponding to the sum of the individual parameters.

Since each f_i is PF_∞, we may conclude that $g(n, x)$ is PT_∞ in the variables n and x by Theorem 1, or equivalently, factoring out e^{-x} and $e^{A_n}/\Gamma(K_n)$, that $(x - A_n)^{K_n - 1}$ is TP_∞. Note that by appropriate selection of the a_i and the k_i we may achieve for A_n any increasing function of n and $K_n - 1$ may likewise denote any strictly increasing integer-valued function of n.

REFERENCES

[1] AISSEN, M., EDREI, A., SCHOENBERG, I. J., AND WHITNEY, A., "On the generating functions of totally positive sequences," *Proc. Nat. Acad. Sci.*, U.S.A., Vol. 37 (1951), pp. 303–307.

[2] ARROW, K. J., KARLIN, S. AND SCARF, H., *Studies in the Mathematical Theory of Inventory and Production*, Stanford University Press, Stanford, California, 1958.

[3] BLACK, G., AND PROSCHAN, F., "On optimal redundancy," *Operations Research*, Vol. 7 (1959), pp. 581-588.

[4] FELLER, W., *An Introduction to Probability Theory and its Applications*, John Wiley and Sons, New York, 1950.

[5] GANTMACHER, F. AND KREIN, M., *Oscillation Matrices and Kernels and Small Vibrations of Mechanical Systems*, 2nd ed. (in Russian), Moscow, 1950.

[6] HAHN, WOLFGANG, "Über Orthogonalpolynome die q-Differenzengleichungen genügen," *Mathematische Nachrichten*, Vol. 2 (1949), pp. 4–34.

[7] HIRSCHMANN, I. I. AND WIDDER, D. V., *The Convolution Transform*, Princeton University Press, Princeton, New Jersey, 1955.

[8] KARLIN, S., "Decision theory for Pólya type distributions, Case of Two Actions, I," *Proceedings of the Third Berkeley Symposium on Probability and Statistics*, Vol. I, University of California Press, 1956, pp. 115–129.

[9] KARLIN, S., "Pólya type distributions, II," *Ann. Math. Stat.*, Vol. 28 (1957), pp. 281–308.

[10] KARLIN, S., *Mathematical Methods and Theory in Games, Programming and Economics* Addison Wesley, Cambridge, Mass., 1959.

[11] KARLIN, S. AND MCGREGOR, J., "Coincidence probabilities," *Pacific J. Math.*, Vol. 9 (1959), pp. 1141–64.

[12] KARLIN, S. AND MCGREGOR, J., "Coincident properties of birth and death processes," *Pacific J. Math.*, Vol. 9. (1959), pp. 1109–40.

[13] KARLIN, S. AND RUBIN, H., "The theory of decision procedures for distributions with monotone likelihood ratio," *Ann. Math. Stat.*, Vol. 27 (1956), pp. 272–299.

[14] PÓLYA, G. AND SZEGÖ, G., *Aufgaben und Lehrsatze aus der Analysis*, Vol. I, Julius Springer, Berlin, 1925.

[15] PROSCHAN, F., "Pólya type distributions in renewal theory, with an application to an inventory problem," Thesis in the Statistics Department, Stanford University, May, 1959.

[16] SCHOENBERG, I. J., "On Pólya frequency functions I, the totally positive functions and their Laplace transforms," *J. d'Analyse Mathématique*, Vol. I (1951), pp. 331–374.