GRAPHS FOR BIVARIATE NORMAL PROBABILITIES

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1. Introduction and summary. Recently there has been much activity dealing with the tabulation of the bivariate normal probability integral. D. B. Owen [3], [4] has summarized many of the properties of the bivariate normal distribution function and tabulated an auxiliary function which enables one to calculate the bivariate normal probability integral. In addition, the National Bureau of Standards [1] has compiled extensive tables of the bivariate normal integral drawn from the works of K. Pearson, Evelyn Fix and J. Neyman, and H. H. Germond. In this same volume, D. B. Owen has contributed an extensive section on applications.

It is the purpose of this paper to present three charts, which will enable one to easily compute the bivariate normal integral to a maximum error of 10^{-2}. This should be sufficient for most practical applications. Owen and Wiesen [5] have also presented charts with a similar objective; however, as pointed out below, we believe the charts presented here lend themselves more easily to visual interpolation. Actually the motivation for these charts came from the Owen and Wiesen work.

2. Notation and formulas. We present here notation and useful formulas relating to the bivariate normal integral. Let X and Y be random variables following a bivariate normal distribution with zero means, unit variances, and correlation coefficient ρ. Then

$$Pr\{X \geqq h, Y \geqq k\} = L(h, k; \rho) = \int_h^\infty dx \int_k^\infty g(x, y; \rho)\, dy, \tag{1}$$

where

$$g(x, y; \rho) = [2\pi\sqrt{1 - \rho^2}]^{-1} \exp - \tfrac{1}{2}[(x^2 - 2\rho xy + y^2)/(1 - \rho^2)]$$

is the bivariate normal probability density function.

Useful relations for $L(h, k; \rho)$ are set out below:

$$L(h, k; \rho) = L(k, h; \rho), \tag{2}$$

$$L(-h, -k; \rho) = \int_{-\infty}^h dx \int_{-\infty}^k g(x, y; \rho)\, dy, \tag{3}$$

$$L(-h, k; -\rho) = \int_{-\infty}^h dx \int_k^\infty g(x, y; \rho)\, dy, \tag{4}$$

Received September 30, 1959.

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$$(5)\qquad L(h, -k; -\rho) = \int_h^\infty dx \int_{-\infty}^{k} g(x, y; \rho)\, dy,$$

$$(6)\qquad 2[L(h, k; \rho) + L(h, k; -\rho) + P(h) - Q(k)] = \int_{-h}^{h} dx \int_{-k}^{k} g(x, y; \rho)\, dy,$$

$$(7)\qquad L(-h, k; \rho) + L(h, k; -\rho) = Q(k),$$

$$(8)\qquad L(-h, -k; \rho) - L(h, k; \rho) = P(k) - Q(h),$$

where

$$P(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{x} e^{-t^2/2}\, dt = 1 - Q(x).$$

Special values of $L(h, k; \rho)$ are

$$(9)\qquad L(h, k; 0) = Q(h)Q(k),$$

$$(10)\qquad L(h, k; -1) = 0 \qquad \text{if } h + k \geqq 0,$$

$$(11)\qquad L(h, k; -1) = P(h) - Q(k) \qquad \text{if } h + k \leqq 0,$$

$$(12)\qquad L(h, k; 1) = Q(h) \qquad \text{if } k \leqq h,$$

$$(13)\qquad L(h, k; 1) = Q(k) \qquad \text{if } k \geqq h,$$

$$(14)\qquad L(0, 0; \rho) = \tfrac{1}{4} + ((\arcsin \rho)/2\pi).$$

3. Discussion of Charts. Owen [4] has shown that

$$(15)\qquad L(h, k; \rho) = L\left(h, 0; \frac{(\rho h - k)\operatorname{sgn} h}{(h^2 - 2\rho hk + k^2)^{\frac{1}{2}}}\right) + L\left(k, 0; \frac{(\rho k - h)\operatorname{sgn} k}{(h^2 - 2\rho hk + k^2)^{\frac{1}{2}}}\right) - \begin{cases} 0 & \text{if } \; hk > 0 \; \text{ or } \; hk = 0 \; \text{ and } \; h + k \geqq 0 \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

This makes it possible to evaluate $L(h, k; \rho)$ as a function of $L(h, 0; \rho)$ which only depends on two parameters. Figures 1, 2, and 3 are plots of h versus ρ with constant contour lines such that $L(h, 0; \rho) = 0.01(.01).10(.02).50$.

Owen and Wiesen [5] have given charts plotting $L(h, 0; \rho)$ versus h with constant contours for ρ. The advantages of the charts presented here are that (i) the contour lines more fully cover the available graph space making interpolation easier and more accurate, and (ii) the Owen and Wiesen charts require visual interpolation on the ρ contour lines which could easily lead to errors larger than $\pm.01$ in reading L. On the other hand, figures 1, 2, and 3 require only visual interpolation on L between successive contour lines differing by .01 or .02. Hence interpolation errors are at most of the order $\pm.01$ throughout.

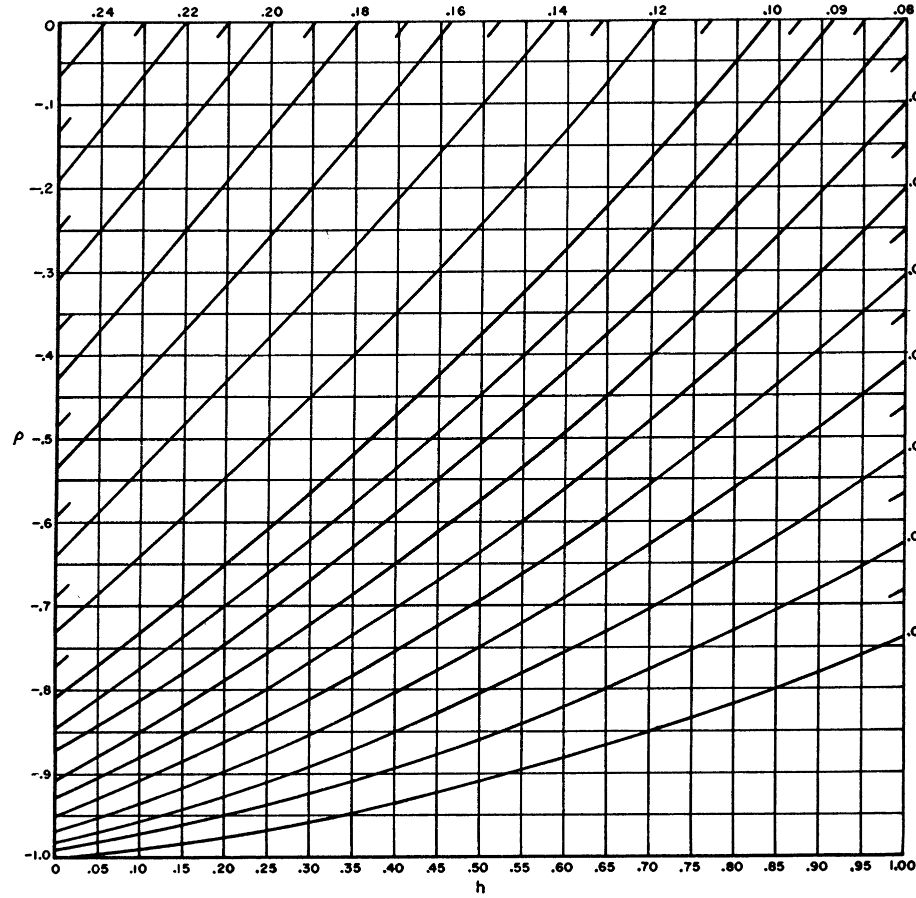

FIG. 1. $L(h, 0; \rho)$ for $0 \leqq h \leqq 1$ and $-1 \leqq \rho \leqq 0$. Values for $h < 0$ can be obtained using

$$L(h, 0; -\rho) = \tfrac{1}{2} - L(-h, 0; \rho)$$

4. Applications of the charts.

EXAMPLE 1. To find $L(0.5, 0.4; 0.8)$. Using (15), we have

$$(h^2 - 2\rho hk + k^2)^{\frac{1}{2}} = 0.3$$

$$L(0.5, 0.4; 0.8) = L(0.5, 0; 0) + L(0.4, 0; -.6) = 0.15 + 0.08 = 0.23.$$

The correct answer to 3D from [1] is $L(0.5, 0.4; 0.8) = 0.233$.

EXAMPLE 2. Let X and Y follow a bivariate normal distribution with means and variances $m_x = 3$, $m_y = 2$, $\sigma_x^2 = 16$, $\sigma_y^2 = 4$ and correlation $\rho = -0.125$. To find the value of $Pr\{X \geqq 2, Y \geqq 4\}$. Since

$$Pr\{X \geqq h, Y \geqq k\} = L[(h - m_x)/\sigma_x, (k - m_y)/\sigma_y; \rho],$$

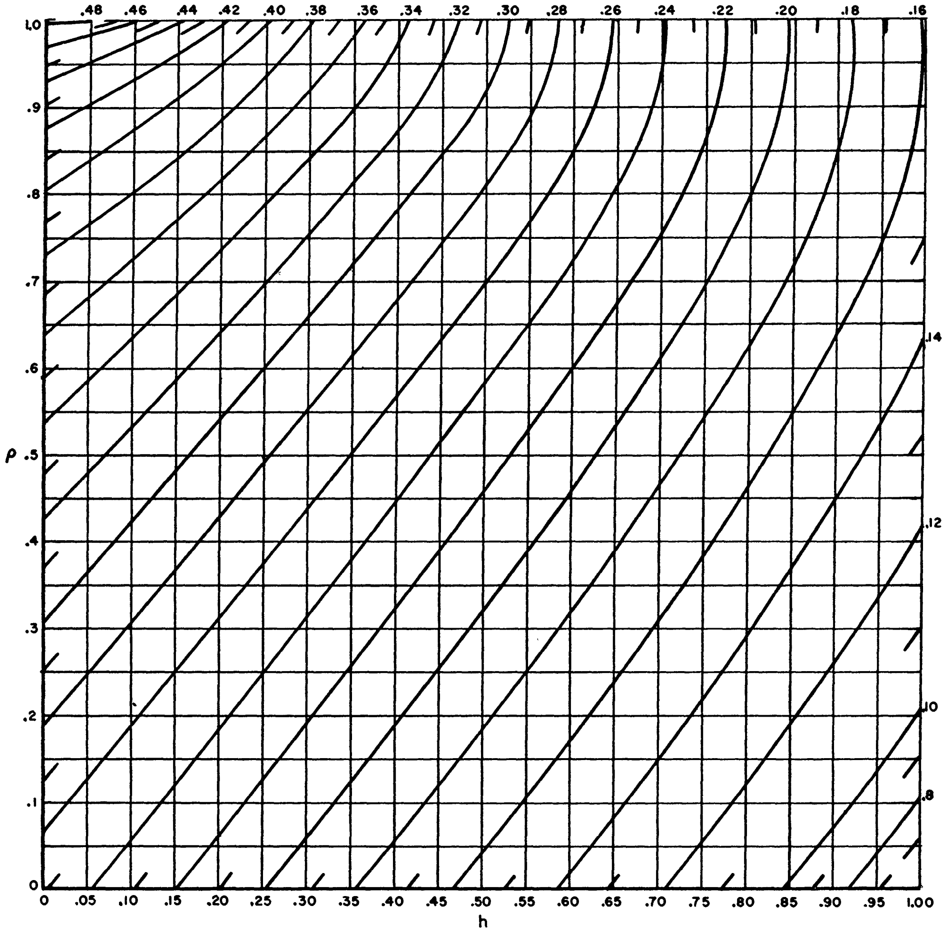

FIG. 2. $L(h, 0; \rho)$ for $0 \leqq h \leqq 1$ and $0 \leqq \rho \leqq 1$. Values for $h < 0$ can be obtained using

$$L(h, 0; -\rho) = \tfrac{1}{2} - L(-h, 0; \rho)$$

We have $Pr\{X \geqq 2, Y \geqq 4\} = L(-0.25, 1.0; -0.125)$. Therefore using (15), $L(-0.25, 1.00; -0.125) = L(-0.25, 0; 0.969) + L(1.0, 0; 0.125) - \tfrac{1}{2}$. The charts only give values for $h > 0$; however using (7) with $k = 0$ we have

$$L(-h, 0, \rho) = \tfrac{1}{2} - L(h, 0; -\rho).$$

Hence $L(0.25, 0; .969) = \tfrac{1}{2} - L(0.25, 0; -.969)$ and thus

$$L(-0.25, 1.0; -.125) = -\mathrm{L}(0.25, 0; -.969) + L(1.0, 0; 0.125)$$

$$= -0.01 + 0.09 = 0.08.$$

The correct answer to 3D from [1] is $L(-0.25, 1.0; -.125) - 0.080$.

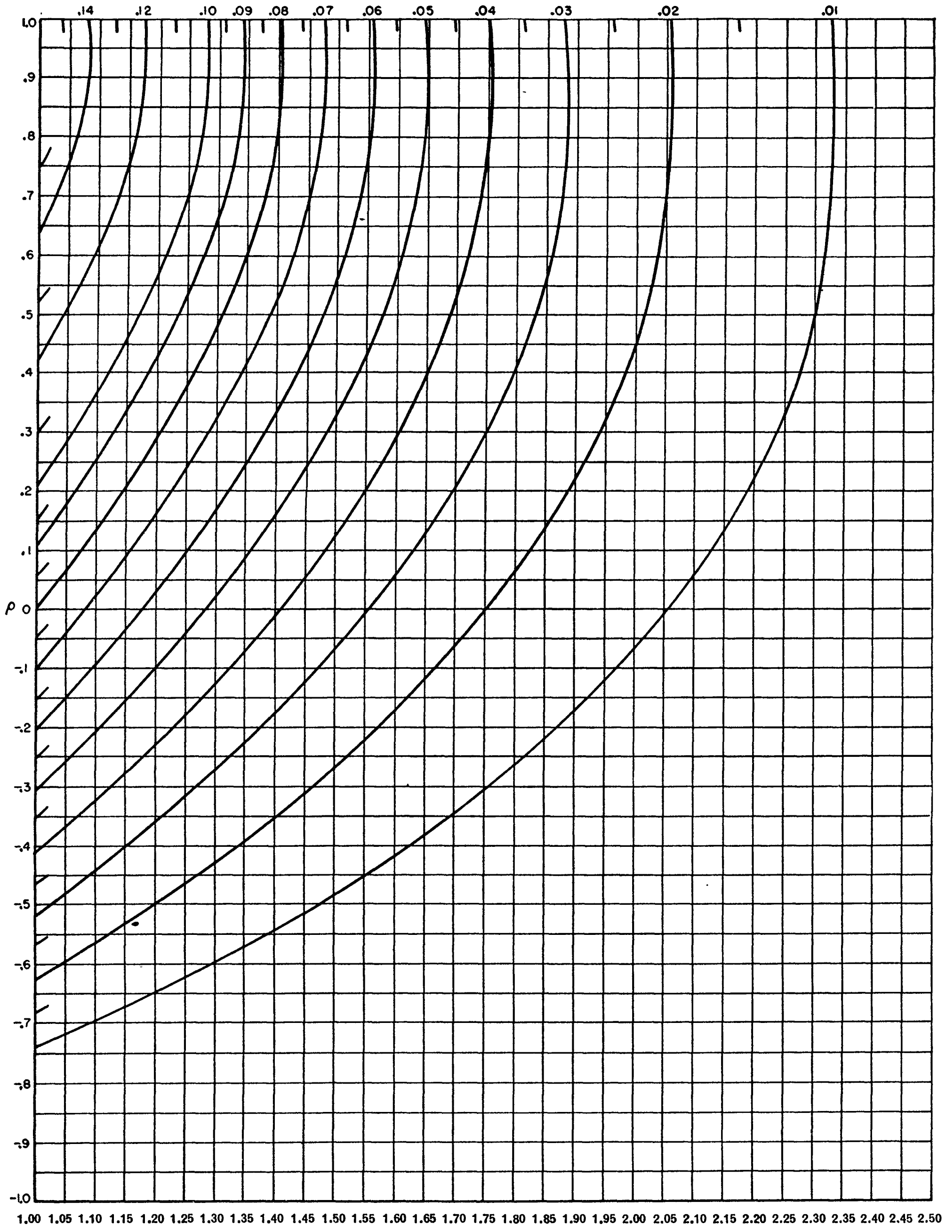

FIG. 3. $L(h, 0; \rho)$ for $h \geqq 1$ and $-1 \leqq \rho \leqq 1$. Values for $h < 0$ can be obtained using

$$L(h, 0; -\rho) = \tfrac{1}{2} - L(-h, 0; \rho)$$

EXAMPLE 3. To find the value of

$$V(h, ah) = (2\pi)^{-1} \int_0^h dx \int_0^{ax} e^{-\frac{1}{2}(x^2+y^2)}\, dy$$

when $a = 2$ and $h = 0.5$.

The function $V(h, ah)$ is sometimes known as Nicholson's function [2] and is useful in finding integrals taken over polygons with respect to the circular normal distribution. The relation between $V(h, ah)$ and $L(h, 0; \rho)$ can be shown to be

$$V(h, ah) = \tfrac{1}{4} + L(h, 0; \rho) - L(0, 0; \rho) - \tfrac{1}{2}Q(h)$$

where $\rho = -[a/(1 + a^2)^{\frac{1}{2}}$. Hence for this example

$$V(0.5, 1.0) = 0.25 + L(0.5, 0; -.894) - L(0, 0; -.894) - \tfrac{1}{2}Q(0.5)$$
$$= 0.25 + 0.01 - 0.07 - \tfrac{1}{2}(.29) = 0.05$$

The correct answer to 3D from [1] is $V(0.5, 1.0) = 0.047$.

5. Acknowledgments. We would like to acknowledge the help of David S. Liepman and Miss M. Carroll Dannemiller who carried out the computations for the charts.

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