NOTES

A CONSERVATIVE PROPERTY OF BINOMIAL TESTS[1]

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Consider n independent binomial trials with common probability of success π. We shall be concerned with the three binomial tests of the null hypothesis H_0: $\pi = \pi_0 (0 < \pi_0 < 1)$ corresponding to the alternative hypotheses (i) ${}_1H_1$: $\pi > \pi_0$, (ii) ${}_2H_1$: $\pi < \pi_0$, and (iii) ${}_3H_1$: $\pi \neq \pi_0$.

There are many situations when the probability of success does in fact vary from trial to trial, being π_i for the ith trial ($i = 1, 2, \cdots, n$). One may then wish to test the modified null hypothesis H_0': $\pi_. = \pi_0$, where $\pi_.$ is the mean of the π_i.

It is the purpose of this note to show that the ordinary tests of H_0 are conservative tests of H_0'. More precisely, letting S_n denote the number of successes in n trials, we shall prove that the inequality

$$\Pr(S_n \geqq a_n \mid H_0) \geqq \Pr(S_n \geqq a_n \mid H_0') \tag{1}$$

holds for any integer a_n such that $n\pi_0 + 1 \leqq a_n \leqq n$. This result is relevant to case (i). At the ordinary levels of significance the fact that a_n has to exceed the expected value of S_n by at least one is no limitation. The corresponding result for case (ii) follows by symmetry, viz.,

$$\Pr(S_n \leqq b_n \mid H_0) \geqq \Pr(S_n \leqq b_n \mid H_0'), \tag{2}$$

where $0 \leqq b_n \leqq n\pi_0 - 1$. Since (2) implies

$$\Pr(S_n > b_n \mid H_0) \leqq \Pr(S_n > b_n \mid H_0')$$

it may be noted on taking $a_n = b_n + 1$ that the inequality in (1) is reversed if $a_n \leqq n\pi_0$. Adding (1) and (2) we obtain the inequality appropriate for the two-sided alternative (iii).

These results are obtained in the course of an ingenious but complicated argument by Hoeffding [2]. The proof given here may, however, be of interest in view of its relative simplicity.

To prove (1) we proceed by induction. For $n = 2$, a_2 must equal 2 and

$$P \equiv \Pr(S_2 = 2 \mid H_0') = \pi_1\pi_2$$

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is a maximum for $\pi_1 = \pi_2 = \pi_0$, i.e., under H_0. Suppose next that (1) is true for $n - 1$ trials. Then

$$(3) \qquad P \equiv \Pr(S_n \geqq a_n \mid H_0') = \sum_{x_n=0}^{1} \Pr(S_{n-1} \geqq a_n - x_n) \Pr(x_n),$$

where x_n is the characteristic random variable describing the nth trial and taking the values 1 and 0 with probabilities π_n and $(1 - \pi_n)$, respectively. For simplicity of writing we omit showing the dependence of the right hand side of (3) on H_0'. With this understanding it follows that

$$\begin{aligned} P &= (1 - \pi_n) \Pr(S_{n-1} \geqq a_n) + \pi_n \Pr(S_{n-1} \geqq a_n - 1) \\ &= \Pr(S_{n-1} \geqq a_n) + \pi_n \Pr(S_{n-1} = a_n - 1). \end{aligned}$$

Since $a_n > (n - 1)\pi_0 + 1$, we have by hypothesis that $\Pr(S_{n-1} \geqq a_n)$ is a maximum, for a given value of π_n, if

$$(4) \qquad \pi_1 = \pi_2 = \cdots = \pi_{n-1} = (n\pi_0 - \pi_n)/(n - 1) = \pi^* \quad \text{(say)}.$$

P now takes the form

$$P = \sum_{r=a_n}^{n-1} \binom{n-1}{r} \pi^{*r}(1 - \pi^*)^{n-r-1} + \pi_n \binom{n-1}{a_n - 1} \pi^{*a_n - 1}(1 - \pi^*)^{n-a_n},$$

and may be regarded as a function of π_n only, n, a_n, π_0 being specified. We have

$$\begin{aligned} \frac{dP}{d\pi_n} = \sum_{r=a_n}^{n-1} &\left[-\binom{n-2}{r-1} \pi^{*r-1}(1 - \pi^*)^{n-r-1} + \binom{n-2}{r} \pi^{*r}(1 - \pi^*)^{n-r-2} \right] \\ &+ \binom{n-1}{a_n - 1} \pi^{*a_n-1}(1 - \pi^*)^{n-a_n} \\ &- \pi_n \left[\binom{n-2}{a_n-2} \pi^{*a_n-2}(1 - \pi^*)^{n-a_n} - \binom{n-2}{a_n-1} \pi^{*a_n-1}(1 - \pi^*)^{n-a_n-1} \right] \\ = -&\binom{n-2}{a_n-1} \pi^{*a_n-1}(1 - \pi^*)^{n-a_n-1}(1 - \pi_n) \\ &+ \binom{n-1}{a_n-1} \pi^{*a_n-1}(1 - \pi^*)^{n-a_n} - \binom{n-2}{a_n-2} \pi_n \pi^{*a_n-2}(1 - \pi^*)^{n-a_n} \\ = &\frac{(n-2)!}{(a_n - 1)!(n - a_n)!} \pi^{*a_n-2}(1 - \pi^*)^{n-a_n-1} F, \end{aligned}$$

where

$$F = -(n - a_n)\pi^*(1 - \pi_n) + (n - 1)\pi^*(1 - \pi^*) - (a_n - 1)\pi_n(1 - \pi^*).$$

By (4), $\pi^* = 1$ gives $n\pi_0 = n - 1 + \pi_n$. But

$$a_n \geqq n\pi_0 + 1 = n + \pi_n,$$

which leaves n as the only possible value of a_n, so that $\pi^* = 1$ does not lead to a zero of $dP/d\pi_n$. The case $\pi^* = 0$ is discussed below.

Turning to the zeros of F we note that this is a quadratic in π_n, viz.,

$$(n - 1)F = -n(\pi_n - \pi_0)(\pi_n - n\pi_0 - 1 + a_n).$$

The condition $a_n \geqq n\pi_0 + 1$ ensures that the root $\pi_n = \pi_0$ corresponds to a local maximum of P, a continuous function of π_n. The derivative $dP/d\pi_n$ vanishes also at $\pi^* = 0$, i.e., at $\pi_n = n\pi_0$ and, for $a_n = n\pi_0 + 1$, at $\pi_n = 0$. Since $\pi^* \geqq 0$ implies by (4) that $\pi_n \leqq \min(n\pi_0, 1)$ it follows that $dP/d\pi_n = 0$ at $\pi_n = \pi_0$ and possibly at extreme values of π_n. Thus the local maximum of P must be a true maximum, so that by (4) P is a maximum for $\pi_i = \pi_0$ (all i), which proves (1).

The question of what approximate corrections to make to the probabilities under H_0 to obtain the corresponding probabilities under H_0' has been considered by Walsh [3]. He also points out the known results (Cramér [1]) that S_n is asymptotically normal provided $\sum_{i=1}^{n} \pi_i(1 - \pi_i)$ diverges as $n \to \infty$. In this case, therefore, since modifying H_0 to H_0' leaves the expectation of S_n unchanged but reduces its variance, the above three results are to be expected in large samples.

I am grateful to the Editor for drawing my attention to references [2] and [3].

REFERENCES

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