PROBABILITY CONTENT OF REGIONS UNDER SPHERICAL NORMAL DISTRIBUTIONS, II: THE DISTRIBUTION OF THE RANGE IN NORMAL SAMPLES[1]

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1. Introduction and summary. Numerous investigations, both theoretical and numerical, have been made of the distribution of the range in normal samples. One of the first investigators was Student [1] who examined the distribution on an empirical basis. Somewhat earlier, Tippett [2] presented tables and charts for the mean, standard deviation, and of the measures of skewness and kurtosis (β_1 and β_2) for the range; further studies of the moments were made by E. S. Pearson [3], Hartley and Pearson [6] and by Ruben [7]. Tables of the moment constants are available in [6] and in Pearson and Hartley's book of statistical tables [10] (Tables 20 and 27), while, more recently, tables of the moment constants were provided by Harter and Clemm [11].

Approximations to the probability integral and percentage points of the distribution were suggested by E. S. Pearson [4], Cox [12], Patnaik [14] and Tukey [15]. Pearson's approximations were based on Pearsonian distributions of Type I and VI, while Cox used the Gamma Function (i.e., a multiple of χ^2 with fractional degrees of freedom) and Patnaik a multiple of χ with fractional degrees of freedom as the basis for their approximations. These last two approximations have been compared by Pearson [5]. An approximation of a different type was derived by Johnson [16] who gave a series expansion for the probability integral suitable for low sample sizes and low values of the range. The behavior of the distribution for large sample size has been studied by Gumbel [17], [18], [19], Elfving [20], Cox [13] and Harley and Pearson [21].

For theoretical studies of the exact distribution reference is made to McKay and Pearson [22], Pillai [23], [24] and Cadwell [25] (see also Hartley [26]). Finally, tables of the probability integral and percentage points of the distribution have been provided by Hartley and Pearson [27], [10] (Tables 23 and 22), as well as by Harter and Clemm [11].

In the present paper, the following new results relating to the range distribution will be obtained: (i) The latter function may be expressed as the product of the sample size and the probability content of a certain parallelotope relative to a hyperspherical normal distribution, and (ii) the function can be evaluated as an infinite series involving the even moments of a sum of independent truncated

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normal variables. The distribution function may also be directly related to the moment generating function of the square of the sum of these truncated variables.

2. Distribution of the range in normal samples. It is well known (see e.g., [10], p. 43) that the distribution function of the range in normal samples is given by

$$P_n(w) = n \int_{-\infty}^{\infty} f(x)[F(x+w) - F(x)]^{n-1}\,dx, \tag{1}$$

where

$$f(x) = (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}x^2}, \qquad F(x) = \int_{-\infty}^{x} f(t)\,dt.$$

Accordingly,

$$P_n(w) = n \int_{-\infty}^{\infty} (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}x^2} \int_x^{x+w} (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}u_1^2}\,du_1 \cdots \int_x^{x+w} (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}u_{n-1}^2}\,du_{n-1}\,dx$$

$$= n \int_{-\infty}^{\infty} \int_x^{x+w} \cdots \int_x^{x+w} (2\pi)^{-\frac{1}{2}n} \exp\left(-\frac{1}{2}\left[x^2 + \sum_1^{n-1} u_i^2\right]\right) dx\,du_1 \cdots du_{n-1}.$$

On setting $y_i = u_i - x$ $(i = 1, 2, \cdots, n-1)$, $x' = x$,

$$P_n(w) = n \int_{-\infty}^{\infty} \int_0^{w} \cdots \int_0^{w} (2\pi)^{-\frac{1}{2}n} \cdot \exp\left\{-\frac{1}{2}\left[x'^2 + \sum_1^{n-1} (y_i + x')^2\right]\right\} dx'\,dy_1 \cdots dy_{n-1} \tag{2}$$

$$= n^{\frac{1}{2}} \int_0^{w} \cdots \int_0^{w} (2\pi)^{-\frac{1}{2}(n-1)} e^{-\frac{1}{2}Q}\,dy_1 \cdots dy_{n-1},$$

after integration with respect to x', where Q is a definite positive quadratic form in the y_i,

$$Q \equiv Q(y_1, y_2, \cdots, y_{n-1}) = \sum_1^{n-1} y_i^2 - n^{-1}\left(\sum_1^{n-1} y_i\right)^2 \tag{3}$$

(cf., [8]). The orthogonal transformation

$$\begin{aligned} \xi_1 &= (n-1)^{-\frac{1}{2}} \sum_1^{n-1} y_i, \\ \xi_i &= \sum_{j=1}^{n-1} a_{ij} y_j \qquad (i = 2, 3, \cdots, n-1), \end{aligned} \tag{4}$$

reduces Q to a sum of squares,

$$Q(y_1, y_2, \cdots, y_{n-1}) \equiv n^{-1}\xi_1^2 + \sum_2^{n-1} \xi_i^2. \tag{5}$$

Finally, on applying the scaling transformation

$$\begin{aligned} \xi_1^* &= n^{-\frac{1}{2}}\xi_1, \\ \xi_i^* &= \xi_i \qquad (i = 2, 3, \cdots, n-1), \end{aligned} \tag{6}$$

$$Q(y_1, y_2, \cdots, y_{n-1}) = \sum_1^{n-1} \xi_i^{*2}. \tag{7}$$

The relationship between the y_i and the ξ_i^* is provided by

$$y_j = \{n/(n-1)\}^{\frac{1}{2}}\xi_1^* + \sum_{i=2}^{n-1} a_{ij}\xi_i^* \qquad (j = 1, 2, \cdots, n-1), \tag{8}$$

and (2) reduces to

$$P_n(w) = n \int \underset{R}{\cdots} \int (2\pi)^{-\frac{1}{2}(n-1)} \exp\left(-\frac{1}{2}\sum_1^{n-1} \xi_i^{*2}\right) d\xi_1^* \cdots d\xi_{n-1}^*, \tag{9}$$

where the region R is the polytope defined by

$$0 \leqq \{n/(n-1)\}^{\frac{1}{2}}\xi_1^* + \sum_{i=2}^{n-1} a_{ij}\xi_i^{*2} \leqq w \qquad (j = 1, 2, \cdots, n-1). \tag{10}$$

R is a parallelotope since it is bounded by $2(n-1)$ flats, each of dimensionality $n-2$, which are parallel in pairs. Note further that one of the 2^{n-1} vertices is at the center of the distribution of the ξ_i^*, and that the diagonal of the parallelotope having the latter vertex as one of its end-points lies along the ξ_1^*-axis. The length of this diagonal is $\{(n-1)/n\}^{\frac{1}{2}}w$.

It now remains to determine the angles at the various vertices between the flats which bound the parallelotope. Each vertex is characterized by the equations

$$\begin{aligned} y_{i_\alpha} &= w \qquad (\alpha = 1, 2, \cdots, k), \\ y_{j_\beta} &= 0 \qquad (\beta = 1, 2, \cdots, n-1-k) \end{aligned} \tag{11}$$

for $k = 0, 1, 2, \cdots, n-1$, where $(i_1, i_2, \cdots, i_k)$ is a subset of k integers from the set $(1, 2, \cdots, n-1)$ while $(j_1, j_2, \cdots, j_{n-1-k})$ is the complementary subset. It is understood in (11) that the y's are to be expressed in terms of the ξ^*'s by means of (8). To determine the $(n-1)(n-2)/2$ angles between the flats in (11) which are *interior* to the region R, the equation (11) must be replaced by the inequalities

$$\begin{aligned} -y_{i_\alpha} &\geqq -w \qquad (\alpha = 1, 2, \cdots, k), \\ y_{j_\beta} &\geqq 0 \qquad (\beta = 1, 2, \cdots, n-1-k). \end{aligned} \tag{12}$$

For convenience, refer to the first k flats in (11) as w-flats and to the remaining $n-1-k$ flats as 0-flats. The angle θ_{ij} between the ith and jth flats at the vertex defined by (10) is then given by

$$\cos\theta_{ij} = \mp \frac{\dfrac{n}{n-1} + \sum_{m=2}^{n-1} a_{mi}a_{mj}}{\left(\dfrac{n}{n-1} + \sum_{m=2}^{n-1} a_{mi}^2\right)\left(\dfrac{n}{n-1} + \sum_{m=2}^{n-1} a_{mj}^2\right)} \qquad (j \neq i), \tag{13}$$

according as to whether the flats are both w-flats or both 0-flats on the one hand, or are of different types on the other. Equation (13) simplifies to

$$\cos\theta = \mp\tfrac{1}{2}, \tag{14}$$

depending on whether the two flats are or are not of the same type, on using the orthogonality property of the matrix in (4). All the dihedral angles are therefore either $2\pi/3$ or $\pi/3$.

A formal expression for the probability content of the parallelotope may be obtained using the method of sections [9], the sections being conveniently chosen in this instance to be orthogonal to the ξ_1^*-axis. The probability content of that portion of the parallelotope between the sections distant z and $z + dz$ (a slab of infinitesimal thickness dz) from the center of the distribution is $(2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}z^2}\, dz \cdot Q_{n-2}(z; w)$. Here $Q_{n-2}(z; w)$ is the probability content of the $(n - 2)$-dimensional polytope, $T_{n-2}(z; w)$, the intersection of the flat $\xi_1^* = z$ with the parallelotope, relative to the $(n - 2)$-dimensional spherical normal distribution in the linear subspace $\xi_1^* = z$ with center at the centroid of $T_{n-2}(z; w)$ and with unit variance in any direction. Thus,

$$P_n(w) = n\int_0^{\{(n-1)/n\}^{\frac{1}{2}}w} (2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}z^2}Q_{n-2}(z; w)\, dz \qquad (n = 2, 3, \cdots), \tag{15}$$

where $Q_0(\cdot\,; \cdot) \equiv 1$.

For $n = 2$, equation (15) reduces to

$$P_2(w) = 2\int_0^{2^{-\frac{1}{2}}w} (2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}z^2}\, dz, \tag{16}$$

so that the density function is $\pi^{-\frac{1}{2}} \exp(-\frac{1}{4}w^2)$, and we obtain the otherwise obvious result that $\frac{1}{2}w^2$ is distributed as a chi-square with 1 degree of freedom. For $n = 3$, (15) is easily shown to be equivalent to

$$P_3(w) = 12V(2^{-\frac{1}{2}}w, 6^{-\frac{1}{2}}w), \tag{17}$$

where $V(\cdot\,; \cdot)$ is Nicholson's function [28] defined by

$$V(h, k) = \frac{1}{2\pi}\int_0^h \int_0^{kx/h} e^{-\frac{1}{2}(x^2+y^2)}\, dy\, dx$$

and tabulated in [28] and [29] (cf., [22] and [29], p. XXXIII). For $n > 3$, the right-hand member of (15) cannot be expressed in terms of elementary functions. The difficulty arises essentially because the nature of the cross-sectional polytope $T_{n-2}(z; w)$ varies as z increases from 0 to $\{(n - 1)/n\}^{\frac{1}{2}}w$; specifically, the number of faces of the parallelotope intersected by the cutting flat varies with its location along the diagonal of the parallelotope lying on the ξ_1^*-axis. (For values of z near the lower and upper limits, 0 and $\{(n - 1)/n\}^{\frac{1}{2}}w$, respectively, the cross-sections are simplices and the corresponding Q-functions are then the K-functions discussed previously [9].) Nevertheless, (15) should provide a useful point of departure for future further study of the range distribution.

We now obtain a series expansion for $P_n(w)$ from equation (2) as follows:

$$P_n(w) = n^{\frac{1}{2}} \int_0^w \cdots \int_0^w (2\pi)^{-\frac{1}{2}(n-1)} \exp\left(-\frac{1}{2}\sum_1^{n-1} y_i^2\right)$$

$$(18) \qquad \cdot \sum_{r=0}^{\infty} \left\{ \left(\sum_1^{n-1} y_i\right)^{2r} \Big/ [(2n)^r r!] \right\} dy_1 \cdots dy_{n-1}$$

$$= \frac{n^{\frac{1}{2}}}{2^{n-1}\pi^{\frac{1}{2}(n-1)}} \sum_{r=0}^{\infty} \frac{(2r)!}{n^r r!} C_r^{(n)}(\tfrac{1}{2}w^2),$$

where

$$(19) \qquad C_r^{(n)}(\tfrac{1}{2}w^2) = \sum_{k_1+\cdots+k_{n-1}=2r} \prod_{j=1}^{n-1} \Gamma_{\frac{1}{2}w^2}\left(\frac{k_j+1}{2}\right) \Big/ k_j!$$

and $\Gamma_m(x)$ is the Incomplete Gamma Function, $\Gamma_m(x) = \int_0^x e^{-u} u^{m-1}\, du$, $(m > 0)$. In general, however, it will be difficult to evaluate the $C_r^{(n)}$, even for moderate n, by direct enumeration of the partition integers k_i and reference to tables of the Gamma Function [30]. An alternative and more convenient series development for $P_n(w)$ follows directly from (2) in the form

$$(20) \qquad P_n(w) = n^{\frac{1}{2}} G^{n-1}(w) \sum_{r=0}^{\infty} \frac{1}{(2n)^r r!} M_{n-1}^{(2r)}(0; w),$$

where

$$(21) \qquad G(w) = (2\pi)^{-\frac{1}{2}} \int_0^w e^{-\frac{1}{2}x^2}\, dx,$$

$$M_{n-1}(\theta; w) = \{G^{n-1}(w)\}^{-1} \int_0^w \cdots \int_0^w (2\pi)^{-\frac{1}{2}(n-1)}$$

$$(22) \qquad \cdot \exp\left(-\frac{1}{2}\sum_1^{n-1} y_i^2 + \theta \sum_1^{n-1} y_i\right) dy_1 \cdots dy_{n-1}$$

$$= [e^{\frac{1}{2}\theta^2}\{F(w-\theta) - F(-\theta)\}/G(w)]^{n-1},$$

and

$$(23) \qquad M_{n-1}^{(2r)}(0; w) = \left.\frac{\partial^{2r} M_{n-1}(\theta; w)}{\partial \theta^{2r}}\right|_{\theta=0} \equiv \mu_{2r}'\left(\sum_1^{n-1} y_i\right).$$

The series in equation (20) seems to be computationally convenient for low or moderate n. It should be remarked that $M_{n-1}(\theta; w)$ is the moment generating function of the sum of $n-1$ independent and standardized normal variates $y_1, y_2, \cdots, y_{n-1}$, each of which has been truncated at 0 and at w, while the $M_{n-1}^{(2r)}(0; w)$ then give a representation of $P_n(w)$ in terms of the *even* moments of the sum variate. The $\mu_{2r}'(\sum_1^{n-1} y_i)$ can be obtained as polynomial functions of $u_k'(y_i)$, the moments of a standardized normal variate truncated at 0 and at w. The moment generating function of the latter variate is

$$(24) \qquad M_1(\theta; w) = e^{\frac{1}{2}\theta^2}\{F(w-\theta) - F(-\theta)\}/G(w),$$

from which the recursive relation

$$M_1^{(m+1)}(0; w) = mM_1^{(m-1)}(0; w) - w^m f(w)/G(w) \qquad (m = 1, 2, \cdots) \tag{25}$$

is readily adduced. (Observe that $M_1^{(k)}(0; w) \equiv \mu_k'(y)$.) An explicit formula for $\mu_k'(y)$ may also be obtained in the form

$$\mu_k'(y) = -\frac{1}{G(w)} \sum_{j=0}^{k}{}' \binom{k}{j} \frac{(k-j)!}{((k-j)/2)!\,2^{\frac{1}{2}(k-j)}} (I_{j-1}(w) - I_{j-1}(0)) \tag{26}$$

where $\sum_{j=0}^{\prime k}$ indicates that summation is to be effected over those j from the set $(0, 1, 2, \cdots, k)$ for which $k - j$ is even, and

$$\begin{aligned} I_{j-1}(x) &= f(x)H_{j-1}(x) \qquad (j = 1, 2, \cdots), \\ I_{-1}(x) &= -F(x), \end{aligned} \tag{27}$$

$H_{j-1}(x)$ being the Tchebycheff-Hermite polynomial of degree $j - 1$ in x. In particular, for the even moments of y,

$$\begin{aligned} \mu_{2r}'(y) = {} & 1\cdot3 \cdots (2r-1) - \{G(w)\}^{-1} \\ & \cdot \sum_{m=1}^{r} [2^{m-r}(2m+1)(2m+2)\cdots(2r)/(r-m)!]f(w)H_{2m-1}(w) \\ & \qquad (r = 1, 2, \cdots).^3 \end{aligned} \tag{28}$$

However, equation (25) seems to be more convenient for the purpose of computation. The first ten moments of y are given by equation (25) as

$$\mu_1'(y) = \{f(0) - f(w)\}/G(w), \tag{29}$$

$$\mu_2'(y) = 1 - wf(w)/G(w), \tag{30}$$

$$\mu_3'(y) = \{2f(0) - 2f(w) - w^2f(w)\}/G(w), \tag{31}$$

$$\mu_4'(y) = 3 - (w^3 + 3w)f(w)/G(w), \tag{32}$$

$$\mu_5'(y) = \{8f(0) - 8f(w) - 4w^2f(w) - w^4f(w)\}/G(w), \tag{33}$$

$$\mu_6'(y) = 15 - (w^5 + 5w^3 + 15w)f(w)/G(w), \tag{34}$$

$$\mu_7'(y) = \{48f(0) - 48f(w) - 24w^2f(w) - 6w^4f(w) - w^6f(w)\}/G(w), \tag{35}$$

$$\mu_8'(y) = 105 - (w^7 + 7w^5 + 35w^3 + 105w)f(w)/G(w), \tag{36}$$

$$\begin{aligned} \mu_9'(y) = {} & \{384f(0) - 384f(w) - 192w^2f(w) - 48w^4f(w) \\ & - 8w^6f(w) - w^8f(w)\}/G(w), \end{aligned} \tag{37}$$

$$\mu_{10}'(y) = 945 - (w^9 + 9w^7 + 63w^5 + 315w^3 + 945)f(w)/G(w). \tag{38}$$

The even moments of $\sum_1^{n-1} y_i$ required in formula (20) may be obtained from the moments of y by using the relationship

[3] For $r = m$, $(2m + 1)(2m + 2) \cdots (2r)$ is to be interpreted as 1.

$$E\left[\left(\sum_1^{n-1} y_i\right)^{2r}\right] = E\left[(2r)! \sum_{k_1+\cdots+k_{n-1}=2r} \prod_{j=1}^{n-1} y_j^{k_j}/k_j!\right] = (2r)! \sum_{k_1+\cdots+k_{n-1}=2r} \prod_{j=1}^{n-1} \mu'_{k_j}/k_j! \tag{39}$$

where μ'_{k_j} refers to $\mu'_{k_j}(y)$.

This gives for the first three even moments of $\sum_1^{n-1} y_i$,

$$\mu'_2(\textstyle\sum y_i) = (n-1)u'_2 + (n-1)_2(\mu'_1)^2, \tag{40}$$

$$\mu'_4(\textstyle\sum y_i) = (n-1)\mu'_4 + 4(n-1)_2\mu'_3\mu'_1 + 3(n-1)_2(\mu'_2)^2 + 6(n-1)_3(\mu'_1)^2\mu'_2 + (n-1)_4(\mu'_1)^4, \tag{41}$$

$$\begin{aligned}\mu'_6(\textstyle\sum y_i) &= (n-1)\mu'_6 + 6(n-1)_2\mu'_5\mu'_1 + 15(n-1)_2\mu'_4\mu'_2 \\ &+ 10(n-1)_2(\mu'_3)^2 + 15(n-1)_3(\mu'_1)^2\mu'_4 + 60(n-1)_3\mu'_1\mu'_2\mu'_3 \\ &+ 15(n-1)_3(\mu'_2)^3 + 20(n-1)_4(\mu'_1)^3\mu'_3 + 45(n-1)_4(\mu'_1)^2(\mu'_2)^2 \\ &+ 30(n-1)_5(u'_1)^4\mu'_2 + (n-1)_6(\mu'_1)^6,\end{aligned} \tag{42}$$

where $(n-1)_r = (n-1)(n-2)\cdots(n-r)$.

3. Concluding remarks. It may be useful to conclude with some remarks concerning further possible research on the distribution of the range in normal samples.

The distribution function of the range has been expressed in terms of the probability content of a parallelotope relative to a centered spherical normal distribution with unit standard deviation in any direction (equation (9)). Here the latter probability content was subsequently represented as an infinite series (equation (20)). It would be desirable to obtain alternative expressions for the probability content and, in particular, suitable approximations for both moderate and large n. It seems likely, for example, that (15) should provide good approximation formulae for the distribution and its percentage points. One such approximation is obtained by replacing $T_{n-2}(z; w)$ by the $(n-2)$-dimensional sphere of equal volume-content and, correspondingly, the Q-function by the distribution function of a chi-square with $n-2$ degrees of freedom (or an Incomplete Gamma Function).

Two further approximations worthy of further study may be obtained in the following manner:

(a) The volume-content of the parallelotope R in (9) is $n^{-\frac{1}{2}}w^{n-1}$, and the center C or R is at the point $(\{(n-1)/n\}^{\frac{1}{2}}w/2, 0, \cdots, 0)$ in ξ^*-space. This suggests that R may be replaced approximately by a sphere of volume $n^{-\frac{1}{2}}w^{n-1}$ and with center at C. It will be noted that this is equivalent to approximating the distribution of w^2 by a non-central χ^2 with $n-1$ degrees of freedom.

(b) Replace R by a spherical sector of equivalent volume-content with its pole at the center O of the spherical normal distribution and with its $n-1$ bounding flats determined by the $n-1$ faces of R which meet at O (the dihedral angle between any two of the bounding flats is then $2\pi/3$). This is equivalent to approximating w by a multiple of χ with $n - 1$ degrees of freedom (cf., [14]). Preliminary computation indicates that this approach yields good approximations for the *moments* of w, provided n is not too large.

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