THE LIFE DISTRIBUTION AND RELIABILITY OF A SYSTEM WITH SPARE COMPONENTS[1]

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0. Summary. The distribution of the operating life of a series system of like elements supplied with a set of spare components has been obtained for the situation when failed elements are in turn replaced by spares. This distribution has been evaluated for some common types of component life densities, and tables of expected total system life have been constructed. These expectations have been compared with those of systems with no spares as a measure of the efficacy of the additional spare components. The reliability of systems with spares has also been studied.

1. Introduction. Consider a system which is made up of several components in such a way that failure of any component causes the system to fail. With the system is associated a fixed number of spare elements. System failures are corrected by successively replacing failed elements from this store until it is empty. Upon this "final failure" the entire aggregate is discarded.

A convenient example of a system with two components and a single spare is provided by the sale of identical nylon stockings in triplets rather than pairs. Similarly, the four original tires and single spare of an automobile form such a system. In both of these examples, however, protection afforded by the extra elements is directed against accidental failures (runs in a relatively new stocking caused by abrasion with a piece of furniture, or punctures in an otherwise solid tire due to hazards distributed randomly along the highway). The replacement scheme with only a small number of spares offers substantially less protection against system failures due to fatigue or wear, for when one element has failed from these causes, its mates are usually worn as well, and a fresh component will provide relatively little increase in system life.

Component life will be taken to be a positive random variable with density function $f(x)$ and absolutely continuous cumulative distribution function $F(x)$. Independence of the lives of all components and spares will be assumed. We shall write the total operating life of a system of n components and k spares as $L(n, k)$; clearly $L(n, 0)$ is the first order statistic from a sample of n independent component lives.

Cox [4] has recently considered systems with spare components as constituting a problem in renewal theory. From a result of Cox and Smith [5], the approxi-

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mate expected system life may be obtained when the number of spare elements is large relative to the number required in the system. Other methods of renewal theory yield the exact expectations of total life for small systems with simple component life densities.

Black and Proschan [3], [11] have also investigated an aspect of a more general form of the spare components problem. Their system is a collection of subsystems, each with a different form of component life density. Spares of each component type are provided, and system operation continues in the usual manner until a failure occurs in a subsystem whose store of spares has been exhausted. Proschan develops the properties of Polya type distributions to establish the optimality of solutions to the nonlinear programming problem of allocating spares among the different subsystems under a budgetary constraint.

The system discussed in this paper also has an alternative statement in terms of queuing theory. The positions of the system's components may be thought of as the n servers of a single queue. The totality of $n + k$ original components and assigned spares constitute waiting members of the queue, n of whom will be served at once, while the remaining k will successively take the place of any member who has been served. The random variable component life is equal to serving time. Total system life is then the time from the start of service until that point when some server finds himself without a waiting member to serve.

2. Systems with Two Components. It is convenient to distinguish between the two components of the system which are in actual use at any one time and the two positions of these components. In view of the pictorial representation of the possible failure configurations of the system, as displayed in Fig. 2.1, sequences of components that successively replace each other will be called "arms" of the system. That sequence which terminates in final failure on the interval $(L, L + dL)$ (henceforth taken as "L") will be designated as "Arm 1." Corresponding outcomes arise when the arms are reversed. Fig. 2.1 shows all modes of failure of a system of two components and four spares under this definition of Arm 1.

Quite generally, the various ways in which the system can fail at time L may be enumerated as follows:

1. The first component in one arm has life L, while all k spares replace the initial elements in the other arm and each other in succession.

2. One arm has a single failure replaced by one spare, followed by failure at time L, while the remaining $k - 1$ spares replace failures stemming from the first component in the other arm.

$(k + 1)$. The original component of one arm has life in excess of L, while k spares replace the initial component and themselves on the other arm, until final failure at life L. The probability of the $(i + 1)$th of these out-

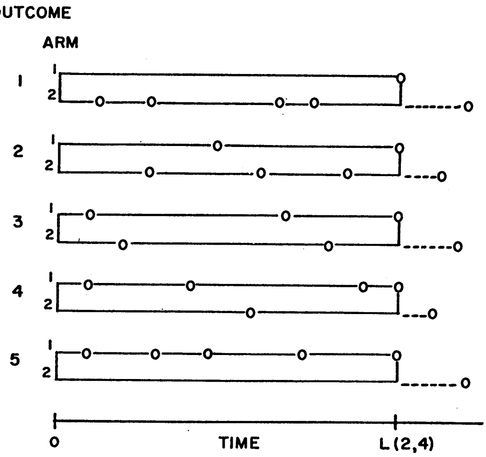

FIG. 2.1 *Possible Outcomes of* $L(2, 4)$

comes, $i = 0, \cdots, k$, in which $i + 1$ failures fall on Arm 1 and $k - i$ failures on Arm 2 prior to time L, is

$$(2.1)\quad 2\Pr(L \leqq x_1 + \cdots + x_{i+1} \leqq L + dL) \cdot \Pr(y_1 + \cdots + y_{k-i} \leqq L \leqq y_1 + \cdots + y_{k-i+1}).$$

The factor 2 arises from the fact that either arm may be designated as "first." To avoid confusion with the lives in Arm 1, those in Arm 2 are written "y_i." The first probability is equal to $f_{i+1}(L)\,dL$, the density of the convolution of $i + 1$ independent variates, each distributed according to $f(x)$. The second can be written as $F_{k-i}(L) - F_{k-i+1}(L)$, where $F_j(L) = \int_0^L f_j(u)\,du$. If these differences of cumulative distributions are abbreviated as $P_i(L) = F_i(L) - F_{i+1}(L)$, the required density of $L(2, k)$ is

$$(2.2)\qquad p(L) = 2\sum_{i=0}^{k} f_{i+1}(L)P_{k-i}(L), \qquad 0 \leqq L < \infty.$$

$p(L)$ and $EL(2, k)$ will now be evaluated for certain component life densities. The density of $L(n, k)$ in the case $f(x) = e^{-x}$ can be derived from the properties of the exponential distribution without recourse to the previous arguments; it is well known to be

$$(2.3)\qquad p(L) = (k!)^{-1}n^{k+1}L^k\exp(-nL), \qquad 0 \leqq L < \infty.$$

For $f(x)$ a gamma distribution with integer power parameter m and unit scale parameter,

$$(2.4)\qquad \begin{aligned} f(x) &= (m!)^{-1}x^m e^{-x}, && 0 \leqq x < \infty, \\ f_i(L) &= [(im + i - 1)!]^{-1}L^{im+i-1}e^{-L}, && 0 \leqq L < \infty, \\ F_i(L) &= 1 - e^{-L}\sum_{j=0}^{im+i-1} L^j/j!, && m \text{ an integer}, \\ P_i(L) &= e^{-L}\sum_{j=0}^{m} L^{j+i(m+1)}/[j + i(m + 1)]!. \end{aligned}$$

Inserting these in (2.2) gives

$$p(L) = 2e^{-2L} \sum_{i=0}^{k} \sum_{j=0}^{m} \frac{L^{j+k(m+1)+m}}{[i(m+1)+m]![j+(k-i)(m+1)]!}, \tag{2.5}$$

while the rth moment of total life is

$$EL^r(2, k) = \sum_{i=0}^{k} \sum_{j=0}^{m} \frac{[j+k(m+1)+m+r]!2^{-j-k(m+1)-m-r}}{[i(m+1)+m]![j+(k-i)(m+1)]!}. \tag{2.6}$$

Theoretical and empirical justifications for the gamma life density may be found in [1], [2], and [6].

Tables 2.1 and 2.2 contain values of $EL(2, k)$ for certain m and k. Except for small m, no simpler expression for the mean appears to be available. If $m = 1$, or component life a $\frac{1}{2}\chi_4^2$ variate, the sums of (2.6) are the odd-index terms in the binomial expansion of $(\frac{1}{2} + \frac{1}{2})^n$ with $n = 2k + 1, 2k + 2$, respectively, and it follows that

$$EL(2, k) = k + 5/4. \tag{2.7}$$

Cox [4] has also obtained this expectation by way of renewal theory.

We shall define the *relative advantage* of a system with k spares over one with none as

$$a(n, k) = nEL(n, k)/[(n + k)EL(n, 0)]. \tag{2.8}$$

From the density of the $(k + 1)$th order statistic $u = x_{(k+1|n)}$ in a sample of n independent and identically distributed random variables, viz.,

$$g(u) = n\binom{n-1}{k}[1 - F(u)]^{n-k+1} F^k(u)f(u), \tag{2.9}$$

it follows for the gamma population (2.4) and $k = 0$, $n = 2$, that $u \equiv L(2, 0)$ has density

$$g(u) = 2(m!)^{-1}u^m e^{-2u} \sum_{j=0}^{m} u^j/j!, \qquad 0 \leqq u < \infty, \tag{2.10}$$

with rth moment

$$Eu^r = (m!)^{-1}2^{-m-r} \sum_{j=0}^{m} (m + r + j)!2^{-j}/j!. \tag{2.11}$$

Since the zeroth moment of any random variable must be unity,

$$1 = (m!)^{-1}2^{-m} \sum_{j=0}^{m} (m + j)!2^{-j}/j!,$$

so that for $r = 1$, (2.11) may be summed to give

$$\begin{aligned} Ex_{(1|2)} &= m + 1 - 2^{-2m-1}(2m + 1)!/(m!)^2, \\ &\sim m + 1 - \tfrac{1}{2}(2m + 1)/(\pi m)^{\frac{1}{2}}. \end{aligned} \tag{2.12}$$

Likewise,

$$(2.13) \qquad Ex_{(2|2)} \sim m + 1 + \tfrac{1}{2}(2m + 1)/(\pi m)^{\frac{1}{2}}.$$

The values of that quantity in Table 2.1 reflect the approach of $EL(2, 1)$ to the mean of the second order statistic.

The evaluation of (2.2) and subsequent computation of EL(2, k) for the

TABLE 2.1

Expected Total Life, EL(2, 1), Relative Advantage, a(2, 1), and Expectations of Related Order Statistics: 2 Components, 1 Spare.

$f(x) = (m!)^{-1}x^m e^{-m}$

m	$EL(2, 1)$	$a(2, 1)$	$Ex_{(1\|2)}$	$Ex_{(2\|2)}$
0	1.000	1.33	0.500	1.500
1	2.250	1.20	1.250	2.750
2	3.492	1.13	2.062	3.938
3	4.711	1.08	2.906	5.094
4	5.906	1.04	3.770	6.230
5	7.081	1.02	4.646	7.388
10	12.740	.93	9.150	12.873

TABLE 2.2

Expected Total Life, EL(2, k) and Relative Advantage, a(2, k): 2 Components, k Spares.

$f(x) = (m!)^{-1}x^m e^{-x}$

m	$EL(2, 2)$	$a(2, 2)$	$EL(2, 3)$	$a(2, 3)$
0	1.500	1.50	2.000	1.60
1	3.250	1.30	4.250	1.36
2	5.001	1.21	6.500	1.26
3	6.762	1.16	8.748	1.20
4	8.532	1.13	10.989	1.17
5	10.321	1.11	13.220	1.14
10	19.591	1.07	24.341	1.06

m	$EL(2, 4)$	$a(2, 4)$	$EL(2, 5)$	$a(2, 5)$
0	2.500	1.67	3.000	1.71
1	5.250	1.40	6.250	1.43
2	8.000	1.29	9.500	1.32
3	10.750	1.23	12.750	1.25
4	13.504	1.19	15.998	1.21
5	16.262	1.17	19.245	1.18
10	30.192	1.10	35.378	1.10

translated exponential life density $f(x) = \exp[-(x - \mu)]$ is slightly more involved, although the expressions for the expectations are straightforward:

$$
\begin{aligned}
EL(2,1) &= \mu + 3/2 - 1/2e^{-\mu}, \\
EL(2,2) &= 2\mu + 5/4 - 1/2e^{-\mu} + e^{-2\mu}(\mu + 3/4), \\
EL(2,3) &= 2\mu + 11/4 - e^{-\mu}(7/8 + \mu/4) + e^{-2\mu}/4 - e^{-3\mu}/8, \\
EL(2,4) &= 3\mu + 33/16 + e^{-\mu}(7/8 + \mu/4) - e^{-2\mu}(23/32 + 9\mu/16) \\
&\qquad + e^{-3\mu}/8 + e^{-4\mu}(5/32 + 5\mu/16).
\end{aligned} \tag{2.14}
$$

3. The Distribution of $L(n, k)$. Although it is possible to obtain the density function of $L(n, k)$ by generalizing the argument of Section 2 to n arms, it will be of greater utility to find instead the cumulative distribution function of system life in the guise of *system reliability*, or the probability that system life will exceed x units of time. Where switching from failed to spare components is instantaneous and perfect, the notion of reliability provides a measure of the advantage of carrying a certain number of spares. Furthermore, the expected system life is merely the integral of the reliability over all positive x.

We shall write the reliability function of a system with n components and k spares as $R_{n,k}(x) = \Pr(L(n, k) > x)$. In the sequel, it will often be convenient to abbreviate reliability, the $P_i(x)$ of Section 2, and the cumulative distribution function of component life as $R_{n,k}$, P_i, and F, respectively. It is well known that $R_{n,0} = (1 - F)^n$. Now a system with a single spare component can operate throughout the interval $(0, x)$ in either of two ways:

(1) The original n components function without failure.

(2) $n - 1$ of the original components and the single spare function without failure.

The probability of the first of these disjoint events is of course $R_{n,0}(x)$. In the second, it will be helpful to visualize an extension of the *arm* concept of Section 2 to n arms. The probability that the length of the arm with the single failure exceeds x is $P_1(x) = F_1(x) - F_2(x)$; obviously the probability that the remaining $n - 1$ arms each has life greater than x is $R_{n-1,0}(x)$. Since the arm with the single failure can be chosen in n ways, the probability of the second event is $nR_{n-1,0}(x)P_1(x)$. Hence,

$$R_{n,1} = R_{n,0} + nR_{n-1,0}P_1 . \tag{3.1}$$

The reliability of a system with $k + 1$ spares is likewise expressible as the reliability of one with k spares, plus the probabilities of the ways in which exactly $k + 1$ elements can fail. Each of these probabilities is associated with a partition of the integer $k + 1$, and is a product of the number of ways that particular partition may be assigned to the n arms of the system and certain terms in $R_{..}$ and P_i. Since the partitions of $k = 2$ are 1^2, 2, the reliability of a system with the two spares is

$$R_{n,2} = R_{n,1} + nR_{n-1,0}\, P_2 + \binom{n}{2} R_{n-2,0}\, P_1^2 . \tag{3.2}$$

Similarly,

$$R_{n,3} = R_{n,2} + \binom{n}{3} R_{n-3,0} P_1^3 + 2\binom{n}{2} R_{n-2,0} P_1 P_2 + nR_{n-1,0} P_3, \tag{3.3}$$

$$\begin{aligned} R_{n,4} = R_{n,3} &+ \binom{n}{4} R_{n-4,0} P_1^4 + n\binom{n-1}{2} R_{n-3,0} P_1^2 P_2 \\ &+ \binom{n}{2} R_{n-2,0} P_2^2 + 2\binom{n}{2} R_{n-2,0} P_1 P_3 + nR_{n-1,0} P_4. \end{aligned} \tag{3.4}$$

Note that the affixes of the P-symbols give the various partitions of $k = 3, 4$. Additional reliabilities for $k = 5(1)8$ are given in the technical report [10].

Analytic treatment of these expressions for system reliability seems feasible for only the simplest $f(x)$. It can be shown [10] for $f(x) = xe^{-x}$, 2 components, k spares, that

$$\begin{aligned} R_{2,k}(x) - R_{2,k-1}(x) = \tfrac{1}{2}[(2k)!]^{-1}e^{-2x}(2x)^{2k} &+ [(2k+1)!]^{-1}e^{-2x}(2x)^{2k+1} \\ &+ \tfrac{1}{2}[(2k+2)!]^{-1}e^{-2x}(2x)^{2k+2}. \end{aligned} \tag{3.5}$$

Charts of system reliability may be found in [10] for small n, k, and component densities $f^{(1)}(x) = xe^{-x}$, $f^{(2)}(x) = (3!)^{-1}x^3e^{-x}$.

Since the expectation of the random variable with cumulative distribution function $F(x)$ is merely $Ex = \int_0^\infty (1 - F(x))\, dx$, it will be convenient to compute the expected system lives $EL(n, 1)$, $EL(n, 2)$ in that manner. With the aid of the well-known relation for the expected values of order statistics [8],

$$Ex_{(i+1|n)} = Ex_{(i|n)} + \binom{n}{k} \int_0^\infty (1 - F(x))^{n-k} F^k(x)\, dx, \tag{3.6}$$

it is possible to express $EL(n, 1)$ and $EL(n, 2)$ conveniently in terms of expectations of the first few order statistics and certain integrals in P_i and $R_{\cdot\cdot}$:

$$\begin{aligned} EL(n, 1) &= Ex_{(1|n)} + n \int_0^\infty P_1 R_{n-1,0}\, dL \\ &= Ex_{(2|n)} - n \int_0^\infty F_2 R_{n-1,0}\, dL. \end{aligned} \tag{3.7}$$

$$\begin{aligned} EL(n, 2) &= Ex_{(2|n)} + \binom{n}{2} \int_0^\infty P_1^2 R_{n-2,0}\, dL \\ &\quad - n \int_0^\infty F_3 R_{n-1,0}\, dL \\ &= Ex_{(3|n)} - n \int_0^\infty F_3 R_{n-1,0}\, dL \\ &\quad - n(n-1) \int_0^\infty F_2 F R_{n-2,0}\, dL \\ &\quad + \binom{n}{2} \int_0^\infty R_{n-2,0} F_2^2\, dL. \end{aligned} \tag{3.8}$$

4. The Evaluation of $EL(n, 1)$ and $EL(n, 2)$ for Certain $f(x)$.

a. *Exponential Life with Guarantee Period* μ. Substitution of $f(x) = \exp[-(x - \mu)]$, $\mu \leqq x < \infty$, in (3.7) and (3.8) yields, with proper modification of the limits of integration,

$$EL(n, 1) = \mu + (2n - 1)[n(n - 1)]^{-1} - \exp[-\mu(n - 1)][n(n - 1)]^{-1}, \tag{4.1}$$

$$\begin{aligned} EL(n, 2) = \mu &+ (3n^2 - 6n + 2)[n(n - 1)(n - 2)]^{-1} \\ &- \exp[-(n - 2)\mu]\{n[(n - 1)(n - 2)]^{-1} \\ &- (3n - 2)[n^2(n - 1)(n - 2)]^{-1}\} + \exp[-(n - 1)\mu]n^{-1} \\ &- \exp[-2(n - 1)\mu][n^2(n - 1)]^{-1}. \end{aligned} \tag{4.2}$$

Tables of these expectations and their associated relative advantages may be found in [10].

b. $f(x) = xe^{-x}$. $EL(n, 1)$, $EL(n, 2)$, and associated order statistic expectations can be expressed in terms of a finite sum

$$\begin{aligned} S(n) &= \int_0^\infty (1 + x)^{n-1} \exp[-(n - 1)x]\, dx \\ &= (n - 1)!(n - 1)^{-n} \sum_{j=0}^{n-1} (n - 1)^j/j!. \end{aligned} \tag{4.3}$$

With the aid of the recursion relation (3.6), integration of (2.9) gives the expectations of the first three order statistics in terms of $S(\cdot)$:

$$\begin{aligned} Ex_{(1|n)} &= S(n + 1), \\ Ex_{(2|n)} &= nS(n) - (n - 1)S(n + 1), \\ Ex_{(3|n)} &= \binom{n}{2}S(n - 1) - n(n - 2)S(n) + \binom{n-1}{2}S(n + 1). \end{aligned} \tag{4.4}$$

The additional integrals in (3.7) and (3.8) may be evaluated from the expressions (2.4) for F_i and P_i, $m = 1$:

$$EL(n, 1) = S(n + 1)[3/2 + 1/(3n)] + 1/(3n), \tag{4.5}$$

$$\begin{aligned} EL(n, 2) = S(n + 1)&[15/8 + 5/(6n) + 1/(18n^2) - 2/(15n^3)] \\ &+ 3/(4n) + 11/(90n^2) - 2/(15n^3). \end{aligned} \tag{4.6}$$

These quantities and their relative advantages $a(n, 1)$, $a(n, 2)$ have been evaluated in Table 4.1.

It is possible to compute approximate values of the expectations derived from $S(n)$ for large n. In (4.3), replace $(n - 1)!$ with its Stirling approximate:

$$S(n) \sim (2\pi/(n - 1))^{\frac{1}{2}} \sum_{j=0}^{n-1} \exp[-(n - 1)](n - 1)^j/j!. \tag{4.7}$$

TABLE 4.1

Expected Total Life, $EL(n, k)$, Relative Advantage, $a(n, k)$, and Expectations of Related Order Statistics: n Components, k Spares.

$f(x) = xe^{-x}$

n	$EL(n, 1)$	$a(n, 1)$	$EL(n, 2)$	$a(n, 2)$	$Ex_{(1\|n)}$	$Ex_{(2\|n)}$	$Ex_{(3\|n)}$
2	2.250	1.20	3.250	1.30	1.250	2.750	—
3	1.663	1.30	2.333	1.45	.963	1.824	2.597
4	1.357	1.35	1.871	1.55	.805	1.438	2.210
5	1.167	1.38	1.588	1.62	.702	1.215	1.772
6	1.034	1.41	1.395	1.66	.629	1.067	1.512
7	.936	1.43	1.254	1.70	.574	.960	1.335
8	.860	1.44	1.146	1.73	.531	.878	1.206
9	.798	1.45	1.060	1.75	.495	.813	1.106
10	.748	1.46	.989	1.77	.466	.760	1.026
11	.705	1.47	.930	1.78	.441	.715	.960
12	.669	1.47	.879	1.80	.420	.677	.904
13	.637	1.48	.836	1.81	.401	.644	.857
14	.607	1.48	.797	1.82	.384	.616	.816
15	.585	1.48	.764	1.82	.370	.590	.780

The expression within the summation represents the probability of $n - 1$ or less occurrences of a Poisson variate with parameter $n - 1$ Since the standardized variate $u = (x - n + 1)/(n - 1)^{\frac{1}{2}}$ is distributed as $N(0, 1)$ for large n, the sum may be replaced by the normal probability integral $\frac{1}{2} - \Phi(-\sqrt{n-1})$; this approaches $\frac{1}{2}$ as n increases, so that

$$S(n) \sim \tfrac{1}{2}(2\pi/(n-1))^{\frac{1}{2}}, \qquad (4.8)$$
$$Ex_{(1|n)} \sim \tfrac{1}{2}(2\pi/n)^{\frac{1}{2}},$$

a result which also can be obtained by standard methods. The approximate expected values of the next two order statistics follow in turn from this result and several applications of the asymptotic expression $(n - 1)^{\frac{1}{2}} \sim (n)^{\frac{1}{2}} - \frac{1}{2}(n)^{\frac{1}{2}}/n$:

$$Ex_{(2|n)} \sim 3/2\, Ex_{(1|n)}, \qquad (4.9)$$
$$Ex_{(3|n)} \sim 15/8\, Ex_{(1|n)}.$$

These expressions imply that the relative advantages $a(n, 1)$, $a(n, 2)$ approach 3/2 and 15/8, respectively, as n increases.

c. *Rayleigh Distribution of Component Life*, 1 *Spare*. The Rayleigh density is a particular case of the Weibull failure law,

$$f(x) = px^{p-1} \exp(-x^p), \qquad (4.10)$$

with $p = 2$. Some integrations by parts in (3.7) yield

$$EL(n, 1) = \tfrac{1}{2}(\pi/n)^{\frac{1}{2}}[1 + n/(2n - 1)], \qquad (4.11)$$

while

$$Ex_{(1)} = \tfrac{1}{2}(\pi/n)^{\frac{1}{2}},$$

$$(4.12)\quad Ex_{(2)} = \tfrac{1}{2}(\pi)^{\frac{1}{2}}\{(n)^{\frac{1}{2}}/n + [n(n-1)^{\frac{1}{2}} - (n)^{\frac{1}{2}}(n-1)]/(n-1)\}$$

$$\sim \tfrac{1}{2}(\pi/n)^{\frac{1}{2}}[1 + n/(2(n-1))].$$

5. Bounds on $EL(n, k)$. From the way in which replacements are added, it follows for $k < n$ that $L(n, k)$ cannot be less than the first order statistic of a random sample of size n (all spares have life zero), or larger than the $(k + 1)$th (no spares are available to extend life upon final failure.) Thus,

$$(5.1)\qquad Ex_{(1|n)} \leqq EL(n, k) \leqq Ex_{(k+1|n)}, \qquad k < n.$$

The lower bound is of little practical value for $k > 2$. The upper bound is also of greatest utility for small k, but its sharpness improves with increasing n.

Bounds for more general k that do not require knowledge of the order statistic expectations are available from a different direction. We may write

$$(5.2)\qquad \begin{aligned} L(n, k) &= [\text{Total life of } n + k \text{ components} - \text{remaining life of } n - 1 \text{ survivors}]/n \\ &= \left(\sum_{i=1}^{n+k} x_i - \sum_{i=1}^{n-1} r_i\right)\Big/ n. \end{aligned}$$

TABLE 5.1

Comparison of $EL(2, k)$ with its Lower Bound $(k + 1)n^{-1}Ex$ and Upper Bound $(n + k)n^{-1}Ex$.

$f(x) = (m!)^{-1}x^m e^{-x}$

m	Lower Bound	$EL(2, 2)$	Upper Bound	Lower Bound	$EL(2, 4)$	Upper Bound
0	1.500	1.500	2.000	2.500	2.500	3.000
1	3.000	3.250	4.000	5.000	5.250	6.000
3	6.000	6.762	8.000	10.000	10.750	12.000
5	9.000	10.321	12.000	15.000	16.262	18.000

TABLE 5.2

Comparison of $EL(2, k)$ with the Lower Bound $Eu_{(1|2)}$ for Selected Component Life Densities $f(x)$

k	$f(x) = e^{-x}$		$f(x) = xe^{-x}$	
	Lower Bound $Eu_{(1\|2)}$	$EL(2, k)$	Lower Bound $Eu_{(1\|2)}$	$EL(2, k)$
2	1.250	1.500	2.906	3.250
4	2.062	2.500	4.646	5.250
8	3.770	4.500	8.214	9.250

Assuming non-negative wear, $\sum_{i=1}^{n-1} r_i \geqq 0, E\sum_{i=1}^{n-1} r_i \leqq (n - 1) Ex$, so that

$$(5.3) \qquad (k + 1)n^{-1} Ex \leqq EL(n, k) \leqq (n + k)n^{-1} Ex.$$

The lower bound is exact for exponential life. Some indication of the closeness of these bounds for certain gamma densities is given in Table 5.1.

For a system of two components and an even number k of spares, it is possible to show that

$$(5.4) \qquad EL(2, k) \geqq Eu_{(1|2)},$$

where the variate u is the convolution of $\frac{1}{2}k + 1$ independent random variables, each with density $f(x)$. This lower bound is equal to the expected life of a system wherein each original component position is preassigned $\frac{1}{2}k$ replacements for its sole use. Relative to $EL(2, k)$, this bound appears to become more precise as $f(x)$ departs from exponential form, and as k increases. Table 5.2 compares $EL(2, k)$ with this lower bound for certain $f(x)$.

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