PROBABILITY DISTRIBUTIONS RELATED TO RANDOM MAPPINGS

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1. Introduction and Summary. A Random Mapping Space $(X, \mathfrak{J}, P)$ is a triplet, where X is a finite set of elements x of cardinality n, $\mathfrak{J}$ is a set of transformations T of X into X, and P is a probability measure over $\mathfrak{J}$.

In this paper, four choices of $\mathfrak{J}$ are considered

(I) $\mathfrak{J}$ is the set of all transformations of X into X.

(II) $\mathfrak{J}$ is the set of all transformations of X into X such that for each $x \varepsilon X$ $Tx \neq x$.

(III) $\mathfrak{J}$ is the set of one-to-one mappings of X onto X.

(IV) $\mathfrak{J}$ is the set of one-to-one mappings of X onto X, such that for each $x \varepsilon X$, $Tx \neq x$.

In each case P is taken as the uniform probability distribution over $\mathfrak{J}$.

If $x \varepsilon X$ and $T \varepsilon \mathfrak{J}$, we will define $T^k x$ as the kth iteration of T on x, where k is an integer, i.e. $T^k x = T(T^{k-1}x)$, and $T^0 x = x$ for all x. The reader should note that, in general, $T^k x$, $k < 0$, may not exist or may not be uniquely determined.

If for some $k \geqq 0$, $T^k x = y$, then y is said to be a kth image of x in T. The set of successors of x in T, $S_T(x)$ is the set of all images of x in T, i.e.,

$$S_T(x) = \{x, Tx, T^2x, \cdots, T^{n-1}x\},$$

which need not be all distinct elements.

If for some $k \leqq 0$, $T^k x = y$, y is said to be a kth inverse of x in T. The set of all kth inverses of x in T is $T^{(k)}(x)$ and

$$P_T(x) = \bigcup_{k=-n}^{0} T^{(k)}(x)$$

is the set of predecessors of x in T.

If there exists an $m > 0$, such that $T^m x = x$, then x is a cyclical element of T and the set of elements $x, Tx, T^2x, \cdots, T^{m-1}x$ is the cycle containing x, $C_T(x)$. If m is the smallest positive integer for which $T^m x = x$, then $C_T(x)$ has cardinality m.

We note further an interesting equivalence relation induced by T. If there exists a pair of integers k_1, k_2 such that

$$T^{k_1}x = T^{k_2}y,$$

then $x \backsim y$ under T.

It is readily seen that this is in fact an equivalence, and hence decomposes X

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into equivalence classes, which we shall call the components of X in T; and designate by $K_T(x)$ the component containing x.

We define $s_T(x)$ to be the number of elements in $S_T(x)$, $p_T(x)$ to be the number of elements in $P_T(x)$, and $l_T(x)$ to be the number of elements in the cycle contained in $K_T(x)$ (i.e. $l(x)$ = the number of elements in $C_T(x)$ if x is cyclical). We designate by q_T the number of elements of X cyclical in T, and by r_T the number of components of X in T.

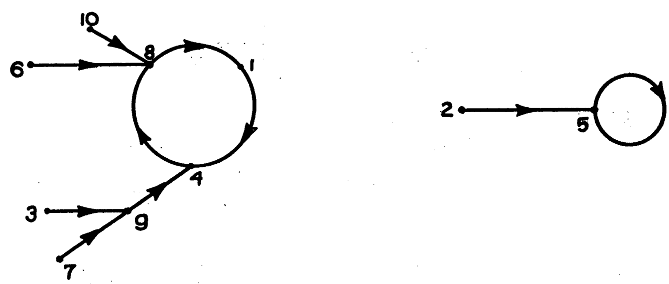
Rubin and Sitgreaves [9] in a Stanford Technical Report have obtained the distributions of s, p, l, q, and have given a generating function for the distribution of r in case I. Folkert [3], in an unpublished doctoral dissertation has obtained the distribution of r in cases I and II. The distribution of r in case III is classical and may be found in Feller [2], Gontcharoff [4], and Riordan [8]. In the present paper, a number of these earlier results are rederived and extended. Specifically, for cases I and II, we compute the probability distributions of s, p, l, q and r. In cases III and IV the distributions of l and r are given. In addition some asymptotic distributions and low order moments are obtained.

For the convenience of the reader, an index of notations having a fixed meaning is provided in the appendix to the paper.

2. Representation of T as a directed graph. It will be convenient to represent elements of $\mathfrak{I}$ as directed graphs. For example, if $n = 10$, $X = \{1, 2, 3, 4, \cdots, 10\}$, and

$$T(1) = 4, \quad T(2) = 5, \quad T(3) = 9, \quad T(4) = 8, \quad T(5) = 5,$$
$$T(6) = 8, \quad T(7) = 9, \quad T(8) = 1, \quad T(9) = 4, \quad T(10) = 8,$$

Then T has the representation below:



3. Probability Distribution for Case I. In case I, $P(T) = 1/n^n$ for all $T \varepsilon \mathfrak{I}$. We now turn to the computation of the probability distributions of s and l, the number of elements in $S_T(x)$ and the number of elements in the cycle contained in $K_T(x)$ respectively.

Then, for any choice of x, we have:

$$P(s = k, l = j) = P\{T^r x \neq x, Tx, \cdots, T^{r-1}x (0 < r \leq k - 1); T^k x = T^{k-j} x\}$$

Hence

$$P(s = k, l = j) = \frac{(n-1)!}{(n-k)!n^k}, \qquad 1 \leqq j \leqq k \leqq n, \tag{3.1}$$

and summing over j, we have

$$P(s = k) = \frac{(n-1)!k}{(n-k)!n^k}, \tag{3.2}$$

$$P(l = j) = \sum_{k=j}^{n} \frac{(n-1)!}{(n-k)!n^k}. \tag{3.3}$$

From consideration of symmetry, we note trivially that

$$E(l) = E[(s+1)/2]. \tag{3.4}$$

We now obtain the asymptotic probability densities of s and l. In (3.1) let $k = \sqrt{n}x$, $j = \sqrt{n}y$, and replace factorials by Stirling's approximation. Then we have

$$\begin{aligned} P(s = \sqrt{n}x, l = \sqrt{n}y) &\sim \frac{n^{n-\sqrt{n}x-\frac{1}{2}} e^{-\sqrt{n}x}}{(n - \sqrt{n}x)^{n-\sqrt{n}x+\frac{1}{2}}} \\ &= \frac{n^{n-\sqrt{n}x-\frac{1}{2}} e^{-\sqrt{n}x}}{n^{n-\sqrt{n}x+\frac{1}{2}}\left(1 - \frac{x}{\sqrt{n}}\right)^{n-\sqrt{n}x+\frac{1}{2}}}. \end{aligned} \tag{3.5}$$

Write

$$(1 - (x/\sqrt{n})^{n-\sqrt{n}nx+\frac{1}{2}} = \exp[(n - \sqrt{n}x + \tfrac{1}{2}) \log(1 - x/\sqrt{n})],$$

and expand $\log(1 - x/\sqrt{n})$ in a power series, obtaining

$$P(s = \sqrt{n}x, l = \sqrt{n}y) \backsim n^{-1}e^{-\frac{1}{2}x^2}.$$

Thus, the asymptotic density of $(s/\sqrt{n}, l/\sqrt{n})$ is

$$f(x, y) = e^{-\frac{1}{2}x^2}, \qquad 0 < y \leqq x < \infty. \tag{3.6}$$

The marginal distributions $f_1(x)$, $f_2(y)$ give the asymptotic densities of $s/\sqrt{n}$, $l/\sqrt{n}$ respectively and are easily obtained by integration.

$$f_1(x) = xe^{-\frac{1}{2}x^2}, \qquad 0 < x, \tag{3.7}$$

$$f_2(y) = \sqrt{2\pi}\,(1 - \Phi(y)), \qquad 0 < y, \tag{3.8}$$

where $\Phi(y) = \int_{-\infty}^{y} (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}x^2}\, dx$.

In numerical computations, the cumulative distribution function $F_2(y)$ is probably more useful than the density function $f_2(y)$ and is therefore given below:

$$F_2(y) = P(Y \leqq y) = 1 - e^{-y^2/2} + y\sqrt{2\pi}(1 - \Phi(y)). \tag{3.9}$$

We note further that

$$EY^r = \frac{1}{r+1} EX^r = \frac{2^{r/2}}{r+1} \Gamma\left(\frac{r+2}{2}\right). \tag{3.10}$$

Hence

$$E(l) \backsim \tfrac{1}{4}(2\pi n)^{\frac{1}{2}}, \qquad \sigma_l^2 \backsim n[(2/3) - (2\pi/16)]. \tag{3.11}$$

Formulas (3.1), (3.2), (3.3), and (3.7) have been obtained by Rubin and Sitgreaves [9].

Rubin and Sitgreaves have also shown that

$$P\{q = j\} = \frac{(n-1)!\,j}{(n-j)!\,n^j}, \qquad j = 1, 2, \cdots, n. \tag{3.12}$$

We now prove this using a partition argument due to Katz [7].

Consider the directed graph representation of T and partition X as follows. Let $M_0(T)$ be those elements of X cyclical under T. Define $M_1(T)$ to be those elements of X whose images are cyclical under T, but are not themselves cyclical. Let $M_2(T)$ be those elements of X whose images under T are in $M_1(T)$. Continuing in this manner until X is exhausted, the $n - j$ non-cyclical elements of X are partitioned into $m(T)$ sets each non-empty for $j \neq n$. Designate the cardinality of $M_j(T)$ by $n_j(T)$, $j = 1, 2, \cdots, m(T)$.

The number of decompositions of X for $n_1, n_2, \cdots, n_m$ fixed is

$$\frac{n!}{j!\,n_1!\,n_2!\cdots n_m!}\, j!\, j^{n_1} n_1^{n_2} \cdots n_{m-1}^{n_m}, \tag{3.13}$$

where $\sum_{i=1}^m n_i = n - j$. Hence

$$P(q = j) = n^{-n} \sum \frac{n!}{n_1!\,n_2!\cdots\cdots n_m!} j^{n_1} n_1^{n_2} \cdots n_{n-1}^{n_m}, \qquad j \neq n, \tag{3.14}$$

where the sum is taken over all non-empty m-part partitions of $n - j$.

Katz [7] has shown that

$$\sum \frac{n!}{j!\,n_1!\,n_2!\cdots n_m!} j^{n_1} n_1^{n_2} \cdots n_{m-1}^{n_m} = \frac{n!\,n^{n-j-1}}{(j-1)!\,(n-j)!}, \tag{3.15}$$

from which we obtain (3.12) for $j \neq n$. We have $j = n$, if and only if T is one-to-one and onto; hence

$$P(q = n) = n!/n^n, \tag{3.16}$$

which coincides with (3.12) for $j = n$.

It is curious to note that this is exactly the same as the distribution of s given in (3.2), and hence has the same asymptotic distribution and asymptotic moments.

The distribution of p has been obtained by Rubin and Sitgreaves [9],

$$P\{p = j\} = \frac{(n-1)!\,j^{j-2}(n-j)^{n-j}}{(n-j)!\,(j-1)!\,n^{n-1}}. \tag{3.17}$$

We establish this as follows:

Let X_{j-1} be $j - 1$ specified elements of X say $x_1, x_2, \cdots, x_{j-1}$. Let x be a distinguished element of X not in X_{j-1}. Then define $\mathfrak{J}_1$ as those transformations T in $\mathfrak{J}$ such that $T(X - (X_{j-1} \cup x)) = X - (X_{j-1} \cup x)$. Define $\mathfrak{J}_2$ as those transformations in $\mathfrak{J}$ such that $T(X_{j-1}) = X_{j-1} \cup x$, and $T^k x_i = x$ for some $k > 0$ and $i = 1, 2, \cdots, j - 1$. We further define $\mathfrak{J}^* = \mathfrak{J}_1 \cap \mathfrak{J}_2$. Then

$$\binom{n-1}{j-1} P(T \,\varepsilon\, \mathfrak{J}^*) = P(p = j), \tag{3.18}$$

and

$$P(T \,\varepsilon\, \mathfrak{J}^*) = P(T \,\varepsilon\, \mathfrak{J}_1) P(T \,\varepsilon\, \mathfrak{J}_2). \tag{3.19}$$

We readily see that

$$P(T \,\varepsilon\, \mathfrak{J}_1) = [(n - j)/n]^{n-j}. \tag{3.20}$$

Hence we have only to compute $P(T \,\varepsilon\, \mathfrak{J}_2)$. For any $T \,\varepsilon\, \mathfrak{J}_2$, we can, by restricting attention to X_{j-1} define an associated transformation T_{j-1}^* which has $T_{j-1}^* x_i = Tx_i$, $i = 1, 2, \cdots, j - 1$, and $T_{j-1}^* x = x$. Let N_{j-1} be the member of distinct transformations which can be constructed in this manner from $T \,\varepsilon\, \mathfrak{J}_2$. Since, in $\mathfrak{J}_2$, x has n equally likely images under T, we have

$$P(T \,\varepsilon\, \mathfrak{J}_2) = N_{j-1}/n^{j-1} \tag{3.21}$$

and N_{j-1} is readily obtained by Katz's Lemma and the partition argument used in (3.13). Hence

$$N_{j-1} = \frac{1}{j} \sum \frac{j!}{1!\, n_1! \cdots n_m!} 1^{n_1} n_1^{n_2} \cdots n_{m-1}^{n_m} = j^{j-2}, \qquad j > 1, \tag{3.22}$$

and, trivially, $N_0 = 1$.

In (3.22), the sum is over all non-empty m-part partitions of $j - 1$, and the factor $(1/j)$ is obtained by distinguishing the element x. Hence

$$P(p = j) = \binom{n-1}{j-1} \frac{j^{j-2}}{n^{j-1}} \left(\frac{n-j}{n}\right)^{n-j}$$

and (3.17) is established.

We now note an interesting relationship,

$$E(S) = E(p).^{1} \tag{3.23}$$

This is established at once by symmetry. For any $T \,\varepsilon\, \mathfrak{J}$ such that y is a successor of x, there is a corresponding $T \,\varepsilon\, \mathfrak{J}$ with x a predecessor of y; the correspondence is accomplished by interchanging x and y in the directed graphs.

We may also note an interesting physical property of directed graphs of this type, which holds for every $T \,\varepsilon\, \mathfrak{J}$. For any $T \,\varepsilon\, \mathfrak{J}$, let r_j be the number of ele-

[1] This was pointed out by D. Blackwell in a private conversation with the author.

ments x for which $T^{-1}x$ has j elements. Then,

$$\sum_{j=0}^{n} jr_j = n. \tag{3.24}$$

Also,

$$\sum_{j=0}^{n} r_j = n. \tag{3.25}$$

Thus,

$$r_0 = \sum_{j=1}^{n-1} jr_{j+1}. \tag{3.26}$$

From this it follows at once that

$$E(p^{(1)}) = 1, \tag{3.27}$$

where $p^{(1)}$ is the number of elements in $T^{-1}(x)$.

The distribution of $p^{(1)}$ is readily seen to be

$$P\{p^{(1)} = j\} = \binom{n}{j}\left(\frac{1}{n}\right)^j\left(\frac{n-1}{n}\right)^{n-j}. \tag{3.28}$$

We proceed now to the question of the probability distribution of r, the number of components of X in T. Folkert [3] has obtained the distribution and has shown

$$P\{r = j\} = \frac{1}{n^n}\sum_{\mu=j}^{n}\frac{S_\mu^j}{\mu!}\sum_{k_1,k_2,\cdots,k_\mu}\frac{n!}{k_1!\,k_2!\cdots k_\mu!}\,k_1^{k_1}k_2^{k_2}\cdots k_\mu^{k_\mu}, \tag{3.29}$$

where S_μ^r are Stirling's Numbers of the First Kind, and the sum over $k_1, k_2, \cdots, k_\mu$ is over all choices of $k_1, k_2, \cdots, k_\mu$ with $k_i > 0$ $(i = 1, 2, \cdots, \mu)$ and $\sum_{i=1}^{\mu} k_i = n$.

In this paper we obtain a probability generating function for the number of components, which has a good deal of intrinsic interest because of its relation to Faa de Bruno's formula (Jordan [6]) and the exponential polynomials of Bell [1].

Let k_i denote the number of components with exactly i elements. Then every $T \varepsilon \mathfrak{J}$ determines an n-tuple $(k_1, k_2, \cdots, k_n)$. Hence, for every specification of $(k_1, k_2, \cdots, k_n)$ we have a set of transformations $\mathfrak{J}_{k_1,k_2,\cdots,k_n}$ in $\mathfrak{J}$.

Then

$$P(T \varepsilon \mathfrak{J}_{k_1,k_2,\cdots,k_n}) = \frac{n!\,I_1^{k_1}I_2^{k_2}\cdots I_n^{k_n}}{1!^{k_1}\,2!^{k_2}\cdots n!^{k_n}\,k_1!\,k_2!\cdots k_n!\,n^n}, \tag{3.30}$$

where I_j/j^j $(j = 1, 2, \cdots, n)$ is the probability that a transformation T_j on j elements X_j is indecomposable, i.e. $K_{T_j}(x) = X_j$ for all $x \varepsilon X_j$, where $0 \leqq k_i \leqq n$

and $\sum_{i=1}^{n} ik_i = n$. We have

$$\frac{I_j}{j^j} = \sum_{i=1}^{j} P(q = i, K_{T_j}(x) = X_j) = \sum_{i=1}^{j} \frac{(j-1)!\,i(i-1)!}{(j-i)!\,j^i\,i!}.$$

Hence

$$I_j = \sum_{i=0}^{j-1} \frac{(j-1)!\,j^i}{i!}. \tag{3.31}$$

This result has been obtained earlier by both Katz [7] and Rubin and Sitgreaves [9]. Then, the generating function of $k_1, k_2, \cdots, k_n$ is given by

$$G(x_1, x_2, \cdots, x_n) = \sum_{k_1,k_2,\cdots,k_n} \frac{n!(I_1x_1)^{k_1}(I_2x_2)^{k_2}\cdots(I_nx_n)^{k_n}}{1!^{k_1}2!^{k_2}\cdots n!^{k_n}k_1!\,k_2!\cdots k_n!\,n^n}, \tag{3.32}$$

since the coefficient of $x_1^{k_1}x_2^{k_2}\cdots x_n^{k_n} = P(T \,\varepsilon\, \mathfrak{I}_{k_1,k_2,\cdots,k_n})$ for $\sum_{i=1}^{n} ik_i = n$.

Since $r = \sum_{i=1}^{n} k_i$,

$$G(x_1, x_2, \cdots, x_n) = \sum_{k_1,k_2,\cdots,k_n} \frac{n!\,r!(I_1x_1)^{k_1}(I_2x_2)^{k_2}\cdots(I_nx_n)^{k_n}}{n^n r!\,k_1!\,k_2!\cdots k_n!\,1!^{k_1}2!^{k_2}\cdots n!^{k_n}},$$

and

$$G(x_1, x_2, \cdots, x_n) \doteq \sum_r \frac{n!}{r!\,n^n}\left(\frac{I_1x_1}{1!} + \frac{I_2x_2}{2!} + \cdots + \frac{I_nx_n}{n!}\right)^r. \tag{3.33}$$

We can extend the definition to $G(x_1, x_2, \cdots)$ with no loss of generality, since this will in no way affect the coefficient of $x_1^{k_1}x_2^{k_2}\cdots x_n^{k_n}$. Hence

$$G(x_1, x_2, \cdots) = \frac{n!}{n^n}\exp\sum_{i=1}^{\infty}\frac{I_ix_i}{i!}. \tag{3.34}$$

If in (3.34), we replace x_i by x^i, the coefficient of x^n in $G(x, x^2, \cdots)$ is 1 for all n. Thus, we have

$$\frac{n!}{n^n}\exp\sum_{i=1}^{\infty}\frac{I_ix^i}{i!} = \frac{n!}{n^n}\sum_{i=0}^{\infty}\frac{i^ix^i}{i!} \tag{3.35}$$

and

$$\sum_{i=1}^{\infty} I_ix^i/i! = \log\sum_{i=0}^{\infty}(i^i/i!)x^i. \tag{3.36}$$

Replacing x_i by t^ix_i in (3.34) we obtain:

$$G(tx_1, t^2x_2, \cdots) = \frac{n!}{n^n}\exp\sum_{i=1}^{\infty}\frac{I_it^ix_i}{i!}. \tag{3.37}$$

In (3.37) the coefficient of t^n gives the probability of any possible decomposition of X into components with the exponents of x_i indexing the decomposition.

Finally we observe that, replacing x_i by tx^i, we get

$$G(tx, tx^2, \cdots) = \frac{n!}{n^n} \exp t \sum_{i=1}^{\infty} \frac{I_i x^i}{i!}, \tag{3.38}$$

or equivalently

$$G(tx, tx^2, \cdots) = \frac{n!}{n^n} \left[\sum_{i=0}^{\infty} \frac{i^i x^i}{i!} \right]^t, \tag{3.39}$$

and the coefficient of $t^k x^n$ in $G(tx, tx^2, \cdots) = P\{r = k\}$.

We now employ the generating function given above to obtain Folkert's formula (3.29). From (3.38), we have

$$\text{coefficient of } t^k = \frac{n!}{n^n k!} \left(\sum_{i=1}^{\infty} \frac{I_i x^i}{i!} \right)^k, \tag{3.40}$$

and from (3.36) we have

$$\text{coefficient of } t^k = \frac{n!}{n^n k!} \left[\log \left(1 + \sum_1^{\infty} \frac{i^i}{i!} x^i \right) \right]^k, \tag{3.41}$$

Since

$$\log (1 + u)^k = \sum_{\mu=k}^{\infty} \frac{k!}{\mu!} S_\mu^k u^\mu;$$

see, for example, Jordan [6], p. 146. Employing this in (3.41), we get

$$\text{coefficient of } t^k = \frac{n!}{n^n k!} \sum_{\mu=k}^{\infty} \frac{k!}{\mu!} S_\mu^k \left[\sum_1^{\infty} \frac{i^i}{i!} x^i \right]^\mu, \tag{3.42}$$

and, expansion by the multinomial theorem gives

$$\begin{aligned} \text{coefficient of } t^k &= \frac{n!}{n^n k!} \sum_{\mu=k}^{\infty} \frac{k!}{\mu!} S_\mu^k \\ &\cdot \sum_{\substack{k_1, k_2, \cdots, k_n \geq 0 \\ \sum_1^n k_1 = \mu}} \frac{\mu!}{k_1! k_2! \cdots k_n!} \left(\frac{1^1}{1!} x \right)^{k_1} \left(\frac{2^2}{2!} x^2 \right)^{k_2} \cdots \left(\frac{n^n}{n!} x^n \right)^{k_n}. \end{aligned} \tag{3.43}$$

To find the coefficient of x^n in (3.43) it suffices to restrict the second sum to non-negative n-tuples $(k_1, k_2, \cdots, k_n)$ with $\sum_1^n k_i = \mu$, $\sum_{i=1}^n ik_i = n$; hence

$$P(r = k) = \frac{n!}{n^n k!} \sum_{\mu=k}^{n} S_\mu^k \sum_{k_1, k_2, \cdots k_n} \frac{1}{k_1! k_2! \cdots k_n!} \left(\frac{1}{1!} \right)^{k_1} \left(\frac{2^2}{2!} \right)^{k_2} \cdots \left(\frac{n^n}{n!} \right)^{k_n} \tag{3.44}$$

which coincides with (3.29), except that partitions of n are enumerated without regard to order in (3.44), and thus we have obtained an alternate form of Folkert's formula. Rubin and Sitgreaves [9] noted that $n^{-1}E(s) = E(s^{-1})$. We remark, further, that it is even more curious that

$$n^{-1}E(s) = P(r = 1) = P(l = 1) = n^{-1}E(p) = n^{-1}E(q). \tag{3.45}$$

4. Probability Distribution for Case II. In case II, $P(T) = (n - 1)^{-n}$ for all $T \varepsilon \mathfrak{J}$. As in case I, we first consider the probability distribution of s and l.

Computing exactly as in Section 3, we obtain

$$P(s = k, l = j) = \frac{(n-2)!}{(n-1)^{k-1}(n-k)!}, \qquad 2 \leqq j \leqq k \leqq n, \tag{4.1}$$

and

$$P(s = k) = \frac{(n-2)!(k-1)}{(n-1)^{k-1}(n-k)!}, \tag{4.2}$$

$$P(l = j) = \sum_{k=j}^{n} \frac{(n-2)!}{(n-1)^{k-1}(n-k)!}, \qquad 2 \leqq j \leqq n. \tag{4.3}$$

Comparing these results with (3.1), (3.2) and (3.3), we have

$$\begin{aligned} &P(s = k \mid I, n) = P(s = k + 1 \mid II, n + 1) \\ &P(s = k, l = j \mid I, n) = P(s = k + 1, l = j + 1 \mid II, n + 1), \end{aligned} \tag{4.4}$$

and

$$P(l = j \mid I, n) = P(l = j + 1 \mid II, n + 1).$$

Hence

$$E(l \mid II, n + 1) = E(l \mid I, n) + 1 \tag{4.5}$$

and

$$E(s \mid II, n + 1) = E(s \mid I, n) + 1. \tag{4.6}$$

From (3.4) we have

$$\tfrac{1}{2}E(s \mid II, n) + 1 = E(l \mid II, n). \tag{4.7}$$

Then, by analogy with (3.6), we note that the asymptotic density of $(s/\sqrt{n-1}, l/\sqrt{n-1})$ is

$$f(x, y) = e^{-\frac{1}{2}x^2}, \qquad 0 \leqq y \leqq x < \infty, \tag{4.8}$$

giving the same marginal density functions as (3.7) and (3.8).

Now consider the probability distribution of the number of elements of X cyclical under T. We show that

$$P(q = j) = n^{n-j} D_j \binom{n-1}{j-1} \Big/ (n-1)^n, \tag{4.9}$$

where D_j is the jth derangement number, i.e., D_j is the nearest integer to $j!/e$, $j \neq 0$, and $D_0 = 1$.

The proof is identical with the proof of (3.12) except that the $j!$ in the numerator of (3.13) is replaced by D_j. Hence an application of Katz's lemma gives

$$(4.10)\quad P(q = j) = \frac{1}{(n-1)^n} \sum \frac{n!}{j!\,n_1!\,n_2! \cdots n_m!} D_j\, j^{n_1} n_1^{n_2} \cdots n_{m-1}^{n_m}, \qquad j \neq n,$$

the sum being taken over all non-empty m-part partitions of $n - j$. Hence

$$(4.11)\qquad P(q = j) = \frac{1}{(n-1)^n} D_j \frac{n!\,n^{n-j-1}}{(j-1)!(n-j)!}, \qquad j \neq n,$$

and thus $P(q = j)$ is given by (4.9). The case $j = n$, is given trivially by (4.9).

The asymptotic distribution is obtained by replacing D_j by $j!/e$, and replacing factorials by Stirling's approximation. Then, letting $j = \sqrt{n}y$, we get

$$(4.12)\qquad f(y) = y^{-\frac{1}{2}y^2}, \qquad 0 < y < \infty,$$

for the asymptotic density of $qn^{-\frac{1}{2}}$. The agreement of (4.12) with (3.7) can hardly be surprising in view of the agreement of (3.2) and (3.12).

We now obtain the distribution of p,

$$(4.13)\qquad \begin{aligned} P(p = q) &= \frac{(n-1)!(n-j-1)^{n-j}j^{j-2}}{(j-1)!(n-j)!(n-1)^{n-1}}, \qquad j = 1, 2, \cdots, n-2, \\ P(p = n-1) &= 0, \\ P(p = n) &= 1 - \sum_{j=1}^{n-2} P(p = j) = \frac{n!}{(n-1)^n} \sum_{j=2}^{n} \frac{n^{n-j-2}}{(n-j)!}. \end{aligned}$$

This is established as follows. Define X_{j-1}, $\mathfrak{J}_1$, $\mathfrak{J}_2$, and $\mathfrak{J}^*$, as in case I. Let x be a distinguished element of X. Then, as before,

$$(4.14)\qquad P(p = j) = \binom{n-1}{j-1} P(T \varepsilon \mathfrak{J}^*),$$

and

$$(4.15)\qquad P(T \varepsilon \mathfrak{J}^*) = P(T \varepsilon \mathfrak{J}_1) P(T \varepsilon \mathfrak{J}_2).$$

Then

$$(4.16)\qquad P(T \varepsilon \mathfrak{J}_1) = [(n-j-1)/(n-1)]^{n-j}.$$

Exactly as in (3.22), we can employ Katz's Lemma to obtain

$$(4.17)\qquad P(T \varepsilon \mathfrak{J}_2) = j^{j-2}/(n-1)^{j-1}.$$

Combining these we have

$$P(p = j) = \binom{n-1}{j-1} \left(\frac{n-j-1}{n-1}\right)^{n-j} \frac{j^{j-2}}{(n-1)^{j-1}}.$$

The condition $Tx \neq x$ for all $x \varepsilon X$, precludes the possibility of $p = n - 1$. There remains the case $p = n$.

$$(4.18) \quad \begin{aligned} P(p = n \mid x) &= \sum_{j=2}^{n} P(q = j, K_T(x) = X, C_T(x) \neq 0) \\ &= \sum_{j=2}^{n} \frac{n^{n-j-1} D_j n!}{(n-1)^n (j-1)!(n-j)!} \cdot \frac{(j-1)!}{D_j} \cdot \frac{j}{n} \end{aligned}$$

Inasmuch as (3.23) depends only on invariance under the symmetric group operating on X and (3.26) is a property of the directed graphs in general, both of these apply in II.

The distribution of $p^{(1)}$ is obtained trivially,

$$(4.19) \quad P(p^{(1)} = j) = \binom{n-1}{j} \left(\frac{1}{n-1}\right)^j \left(\frac{n-2}{n-1}\right)^{n-j}$$

The distribution of r, the number of components, has been computed by Folkert [3], and shown to be

$$(4.20) \quad \begin{aligned} P(r = k) &= \frac{1}{(n-1)^n} \sum_{\mu=k}^{[n/2]} \frac{S_\mu^k}{\mu!} \sum_{k_1, k_2, \cdots, k_\mu} \\ &\cdot \frac{n!}{k_1! k_2! \cdots k_\mu!} (k_1 - 1)^{k_1} (k_2 - 1)^{k_2} \cdots (k_\mu - 1)^{k_\mu}, \end{aligned}$$

where the sum over $k_1, k_2, \cdots, k_\mu$ is over all μ-tuples with $k_i > 1$ and $\sum_1^\mu k_i = n$.

We will now develop a probability generating function for the number of components, and obtain an alternate derivation of (4.20). The argument parallels the same discussion in Case I and hence will only be sketched briefly.

As in case I,

$$(4.21) \quad P(T \varepsilon \mathfrak{Z}_{k_1, k_2, \cdots, k_n}) = \frac{n! I_2^{k_2} I_3^{k_3} \cdots I_n^{k_n}}{2!^{k_2} 3!^{k_3} \cdots n!^{k_n} k_2! k_3! \cdots k_n! (n-1)^n},$$

where $I_j/(j-1)^j$ is the probability that a transformation T_j on j elements X_j is indecomposable, i.e., $K_{T_j}(x) = X_j$ for all $x \varepsilon X_j$, $0 \leqq k_i \leqq n$ and $\sum_{i=2}^n ik_i = n$.

$$(4.22) \quad \begin{aligned} \frac{I_j}{(j-1)^j} &= \sum_{i=2}^{j} P(q = i, K_{T_j}(x) = X_j) = \sum_{i=2}^{j} \frac{j^{j-i-1} j!}{(j-1)^j (j-i)!} \\ &= \sum_{i=0}^{j-2} \frac{(j-1)! j^i}{(j-1)^j i!}. \end{aligned}$$

(4.22) has previously been established by Katz [7] using a somewhat different argument. Then

$$(4.23) \quad G(x_2, x_3, \cdots, x_n) = \sum_{k_2, k_3, \cdots, k_n} \frac{n!(I_2 x_2)^{k_2} (I_3 x_3)^{k_3} \cdots (I_n x_n)^{k_n}}{2!^{k_2} 3!^{k_3} \cdots n!^{k_n} k_2! k_3! \cdots k_n! (n-1)^n}$$

is the generating function of $k_2, k_3, \cdots, k_n$ in the same manner as (3.32).

Since $r = \sum_{i=1}^{n} k_i$, we obtain, after extending the definition to $G(x_2, x_3, \cdots)$,

$$G(x_2, x_3, \cdots) = \frac{n!}{(n-1)^n} \exp \sum_{i=2}^{\infty} \frac{I_i x_i}{i!} \tag{4.24}$$

If, in (4.24), we replace x_i by x^i, we obtain

$$\frac{n!}{(n-1)^n} \exp \sum_{i=2}^{\infty} \frac{I_i x^i}{i!} = \frac{n!}{(n-1)^n} \sum_{j=0}^{\infty} \frac{(j-1)^j x^j}{j!}. \tag{4.25}$$

Thus

$$\sum_{i=2}^{\infty} \frac{I_i x^i}{i!} = \log \sum_{j=0}^{\infty} \frac{(j-1)^j x^j}{j!}. \tag{4.26}$$

Replacing x_i by $t^i x_i$ in (4.24), we obtain:

$$G(t^2 x_2, t^3 x_3, \cdots) = \frac{n!}{(n-1)^n} \exp \sum_{i=2}^{\infty} \frac{I_i t^i x_i}{i!}. \tag{4.27}$$

Then the coefficient of t^n in (4.27) gives the probability of every possible decomposition of X into components, in the same manner as (3.37). If we replace x_i by tx^i, we get

$$G(tx^2, tx^3, \cdots) = \frac{n!}{(n-1)^n} \exp \left[t \sum_{i=2}^{\infty} \frac{I_i x^i}{i!} \right] \tag{4.28}$$

or

$$G(tx^2, tx^3, \cdots) = \frac{n!}{(n-1)^n} \left[\sum_{i=0}^{\infty} \frac{(i-1)^i x^i}{i!} \right]^t, \tag{4.29}$$

giving

$$\text{coefficient of } t^k x^n \text{ in } G(tx^2, tx^3, \cdots) = P\{r = k\}.$$

We now employ the generating function to obtain an alternate form of Folkert's formula (4.20). From (4.28) we have

$$\text{coefficient of } t^k = \frac{n!}{(n-1)^n k!} \left[\sum_{i=2}^{\infty} \frac{I_i x^i}{i!} \right]^k, \tag{4.30}$$

and from (4.26) we have

$$\text{coefficient of } t^k = \frac{n!}{(n-1)^n k!} \left[\log \left(1 + \sum_{i=1}^{\infty} \frac{(i-1)^i x^i}{i!} \right) \right]^k. \tag{4.31}$$

Hence

$$\text{coefficient of } t^k = \frac{n!}{(n-1)^n k!} \sum_{\mu=k}^{\infty} \frac{k!}{\mu!} S_\mu^k \left[\sum_{i=1}^{\infty} \frac{(i-1)^i}{i!} x^i \right]^\mu,$$

and, as in (3.43), we get

$$\text{coefficient of } t^k = \frac{n!}{(n-1)^n k!} \sum_{\mu=k}^{\infty} \frac{k!}{\mu!} S_\mu^k \sum_{\substack{k_1,k_2,\cdots,k_n \geqq 0 \\ \sum_1^n k_i = \mu}} \cdot \frac{\mu!}{k_1!k_2!\cdots k_n!} \left[\frac{(1-1)x^1}{1!}\right]^{k_1} \left[\frac{(2-1)^2x^2}{2!}\right]^{k_2} \cdots \left[\frac{(n-1)^n x^n}{n!}\right]^{k_n}. \tag{4.32}$$

To find the coefficient of x^n in (4.32), it is sufficient to restrict the second sum to non-negative n-tupules $(k_1, k_2, \cdots, k_r)$ with $\sum_{i=1}^n k_i = \mu$ and $\sum_{i=1}^n ik_i = n$. Thus, we have

$$P(r=k) = \frac{n!}{(n-1)^n} \sum_{\mu=k}^{[n/2]} S_\mu^k \sum_{k_2,k_3,\cdots,k_n} \frac{1}{k_2!\,k_3!\cdots k_n!} \left(\frac{1^2}{2!}\right)^{k_2} \left(\frac{2^3}{3!}\right)^{k_3} \cdots \left(\frac{(n-1)^n}{n!}\right)^{k_n}, \tag{4.33}$$

with the second sum restricted to $k_2, k_3, \cdots, k_n \geqq 0$, $\sum_{i=2}^n k_i = \mu$, and $\sum_{i=2}^n ik_i = n$ by the deletion of zero terms, and thus we have an alternate form of Folkert's formula.

5. Probability Distributions for Cases III and IV. In case III, we have $P(T) = n!^{-1}$, and in case IV, we have $P(T) = D_n^{-1}$. In both cases, every $x \varepsilon X$ is cyclic, since $\mathfrak{J}$ is a collection of mappings which are one-to-one and onto in each case. As a consequence

$$S_T(x) = P_T(x) = C_T(x) = K_T(x). \tag{5.1}$$

Therefore, many of the probability distributions considered for cases I and II coincide in cases III and IV. Hence, we consider only the distributions of l and r. Then, in case III, we have

$$P(l=j) = \left[\binom{n}{j}(j-1)!\,j(n-j)!\right] \Big/ nn! = 1/n. \tag{5.2}$$

Gontcharoff [4] has shown that the probability that the number of components r of T is k is given by

$$P(r=k) = \text{coefficient of } t^k \text{ in } \frac{t(t+1)\cdots(t+n-1)}{n!}, \tag{5.3}$$

and therefore is given by the well-known result,

$$P(r=k) = |S_n^k|/n!, \tag{5.4}$$

which may be found in Riordan [8]. Gontcharoff [4] has also shown that the distribution of $(r - Er)/\sigma_r$ is asymptotically normally distributed with mean 0 and variance 1. Feller [2] and Greenwood [5] have also computed Er and σ_r^2. We show an alternative computation using (5.2).

Let m_j be the number of components of T with exactly j elements. Then

$$Er = E\sum_{j=1}^{n} m_j = \sum_{j=1}^{n} Em_j . \tag{5.5}$$

From (5.2) we note that $Em_j = 1/j$, and hence

$$Er = \sum_{j=1}^{n} \frac{1}{j} \backsim \log n + \gamma, \tag{5.6}$$

where γ is Euler's constant.

The variance has been shown to be

$$\sigma_r^2 = \sum_{j=1}^{n} \frac{1}{j} - \left(\sum_{j=1}^{n} \frac{1}{j^2}\right) \backsim \log n + \gamma - \frac{\pi^2}{6} . \tag{5.7}$$

In case IV, we have

$$\begin{aligned} P(l = j) &= \left[\binom{n}{j} (j-1)!\, j\, D_{n-j}\right] \Big/ [n\, D_n] \\ &= \frac{(n-1)!\, D_{n-j}}{(n-j)!\, D_n} . \end{aligned} \tag{5.8}$$

For large n; and $j \geqq 2$ and sufficiently small compared to n,

$$P(l = j) \backsim 1/n, \qquad 2 \leqq j. \tag{5.9}$$

Furthermore

$$\begin{aligned} P(l = n) &\backsim e/n \\ P(l = n - 1) &= 0 \\ P(l = n - 2) &\backsim e/2n \\ P(l = n - 3) &\backsim e/3n. \end{aligned}$$

To get the probability distribution of the number of components, we employ the same type of generating function used earlier. First we note that

$$P(K_T(x) = X) = (n-1)!/D_n \tag{5.10}$$

since all $(n - 1)!$ n-cycles belong to $\mathfrak{J}$. From this, we obtain:

$$\begin{aligned} P(T \,\varepsilon\, \mathfrak{J}_{k_1,k_2,\cdots,k_n}) &= \frac{n!\, 1!^{k_2}\, 2!^{k_3} \cdots (n-1)!^{k_n}}{D_n\, 2!^{k_2}\, 3!^{k_3} \cdots n!^{k_n}\, k_2!\, k_3! \cdots k_n!} \\ &= \frac{n!}{D_n\, 2^{k_2}\, 3^{k_3} \cdots n^{k_n}\, k_2! \cdots k_n!} \end{aligned} \tag{5.11}$$

where $0 \leqq k_i \leqq n$ and $\sum_{i=2}^{n} ik_i = n$. Then

$$G(x_2, x_3, \cdots, x_n) = \sum_{k_2, k_3, \cdots, k_n} \frac{n!}{D_n} \left(\frac{x_2}{2}\right)^{k_2} \left(\frac{x_3}{3}\right)^{k_3} \cdots \left(\frac{x_n}{n}\right)^{k_n} \frac{1}{k_2!\, k_3! \cdots k_n!} \tag{5.12}$$
$$= \sum_r \frac{n!}{D_n} \left(\frac{x_2}{2} + \frac{x_3}{3} + \cdots + \frac{x_n}{n}\right)^r \Big/ r!,$$

where $r = \sum_{i=2}^{n} k_i$. Proceeding as before, we have

$$G(x_2, x_3, \cdots) = \frac{n!}{D_n} \exp \sum_{i=2}^{\infty} \frac{x_i}{i}. \tag{5.13}$$

Replacing x_i by x^i, we have

$$\frac{n!}{D_n} \exp \sum_{i=2}^{\infty} \frac{x_i}{i} = \frac{n!}{D_n} \sum_{j=0}^{\infty} \frac{D_j}{j!} x^j. \tag{5.14}$$

If we replace x_i by tx^i in (5.13), we obtain

$$G(tx^2, tx^3, \cdots) = \frac{n!}{D_n} \exp t \sum_{i=2}^{\infty} \frac{x^i}{i} = \frac{n!}{D_n} \left[\sum_{j=0}^{\infty} \frac{D_j}{j!} x^j\right]^t. \tag{5.15}$$

Since

$$\sum_{i=2}^{\infty} \frac{x^i}{i} = -\log(1-x) - x,$$

we have

$$G(tx^2, tx^3, \cdots) = \frac{n!}{D_n} \exp^{-t[\log(1-x)+x]} \tag{5.16}$$
$$= \frac{n!}{D_n} \frac{e^{-tx}}{(1-x)^t}.$$

Then

$$\text{coefficient of } t^k x^n \text{ in } G(tx^2, tx^3, \cdots) = P(r = k). \tag{5.17}$$

From (5.15) we have

$$\text{coefficient of } t^k = \frac{n!}{D_n k!} \left(\sum_{i=2}^{\infty} \frac{x^i}{i}\right)^k, \tag{5.18}$$

and, expanding by the multinomial theorem, we get

coefficient of t^k

$$= n!/D_n k! \sum_{\substack{k_2, k_3, \cdots, k_n \geqq 0 \\ \sum_1^n k_i = k}} \frac{k!}{k_2!\; k_3! \;\cdots\; k_n!} \left(\frac{x^2}{2}\right)^{k_2} \left(\frac{x^3}{3}\right)^{k_3} \cdots \left(\frac{x^n}{x}\right)^{k_n}, \tag{5.19}$$

and thus

$$(5.20)\qquad P(r = k) = \frac{n!}{D_n} \sum_{k_2, k_3, \cdots, k_n} [k_2!\, k_3! \cdots k_n!\, 2^{k_2}\, 3^{k_3} \cdots n^{k_n}]^{-1},$$

the sum over all non negative $n - 1$ tuples $k_2, k_3, \cdots, k_n$ with $\sum_{i=2}^n k_i = k$ and $\sum_{i=2}^n ik_i = n$. Another form of the same result is obtained from (5.16). Here we have

$$(5.21)\quad \text{coefficient of } x^n = n!/D_n \sum_{j=0}^{n} \frac{(-1)^{n-j} t^{n-j}}{(n-j)!} \frac{(-1)^{2j} t(t+1) \cdots (t+j-1)}{j!}.$$

Hence, we find that the coefficient of $t^k x^n$ is

$$(5.22)\qquad P(r = k) = n!/D_n \sum_{j=0}^{n} \frac{(-1)^{n+j} \mid S_j^{k-n+j} \mid}{(n-j)!\, j!},$$

or

$$(5.23)\qquad P(r = k) = n!/D_n \sum_{j=0}^{n} \frac{(-1)^{j} \mid S_{n-j}^{k-j} \mid}{(n-j)!\, j!}.$$

Since

$$(5.24)\qquad Er = \text{coefficient of } x^n \text{ in } (d/dt) G(tx^2, tx^3, \cdots)\mid_{t=1}$$

and

$$(5.25)\quad Er(r-1) = \text{coefficient of } x^n \text{ in } (d^2/dt^2) G(tx^2, tx^3, \cdots)\mid_{t=1},$$

then

$$(5.26)\quad Er = \text{coefficient of } x^n \text{ in } (n!/D_n)(e^{-x}/(1-x))(-\log(1-x) - x),$$

$$(5.27)\ Er(r-1) = \text{coefficient of } x^n \text{ in } (n!/D_n)(e^{-x}/(1-x))(\log(1-x) + x)^2.$$

Expanding (5.26) and (5.27) in a power series we obtain

$$(5.28)\qquad Er = \frac{n!}{D_n} \sum_{s=2}^{n} \frac{D_{n-s}}{s(n-s)!},$$

$$(5.29)\qquad \begin{aligned} Er(r-1) &= \frac{n!}{D_n} \sum_{s=4}^{n} \frac{D_{n-s}}{(n-s)!} \sum_{\substack{j,k \geq 2 \\ j+k=s}} \frac{1}{jk} \\ &= \frac{n!}{D_n} \sum_{s=4}^{n} \frac{D_{n-s}}{(n-s)!} \sum_{j=2}^{s-2} \frac{1}{j(s-j)}. \end{aligned}$$

Since $D_j = (j!/e) + O(1)$,

$$Er \sim e \sum_{s=2}^{n} \frac{\frac{(n-s)!}{e} + O(1)}{s(n-s)!} \sim \sum_{s=2}^{n} \frac{1}{s} + O(1).$$

Hence

$$Er \backsim \log n + O(1). \tag{5.30}$$

Similarly

$$\begin{aligned} Er(r-1) &= \frac{n!}{D_n} \sum_{s=4}^{n} \frac{D_{n-s}}{(n-s)!} \frac{1}{s} \sum_{j=2}^{s-2} \left[\frac{1}{j} + \frac{1}{s-j}\right] \\ &\backsim e \sum_{s=4}^{n} \left[\frac{\frac{(n-s)!}{e} + O(1)}{(n-s)!}\right] \frac{2}{s}\left[\log s - 1 - \frac{1}{s-1} - \frac{1}{s} + \gamma + O\left(\frac{1}{s}\right)\right] \end{aligned} \tag{5.31}$$

where γ is Euler's constant. Hence

$$Er(r-1) \backsim \sum_{s=4}^{n} \frac{2}{s}\left[\log s - 1 - \frac{1}{s-1} - \frac{1}{s} + \gamma + O\left(\frac{1}{s}\right)\right] + O(1),$$

and

$$Er(r-1) \backsim \log^2 n + 2(\gamma - 1)\log n + O(1). \tag{5.32}$$

Thus

$$\sigma_r^2 \backsim (2\gamma - 1)\log n + O(1). \tag{5.33}$$

6. Miscellaneous Remarks. The problem of random mappings is of interest in various studies of human behaviors. We produce one such example. If we ask each of n individuals in a group to name his best friend from among the members of the group, the individual asked is the element x, and his choice Tx. In this case we have $Tx \neq x$, and the hypothesis of "randomness" leads to case II.

APPENDIX

Index of Notations Having a Fixed Meaning

$(X, \mathfrak{J}, P)$–random mapping space
X–a finite set of n elements
$\mathfrak{J}$–a set of transformations T of X into X
P–a probability measure over $\mathfrak{J}$
Case I–is set of all transformations of X into X, $P(T) = n^{-n}$ for each $T \varepsilon \mathfrak{J}$
Case II–is set of all transformations of X into X with $Tx \neq x$ for each $x \varepsilon X$, $P(T) = (n-1)^{-n}$ for each $T \varepsilon \mathfrak{J}$
Case III–is set of all one-to-one mappings of X onto X, $P(T) = n!^{-1}$ for each $T \varepsilon \mathfrak{J}$
Case IV–is set of all one-to-one mappings of X onto X with $Tx \neq x$ for each $x \varepsilon X$, $P(T) = D_n^{-1}$, D_n is the nth derangement number for each $T \varepsilon \mathfrak{J}$
$S_T(x)$–the set of all images of x in T.
$P_T(x)$–the set of predecessors of X in T.
$C_T(x)$–the cycle containing x.
$K_T(x)$–the component containing x.

$s_T(x)$, s–the number of elements in $S_T(x)$.
$p_T(x)$, p–the number of elements in $P_T(x)$.
$l_T(x)$, l–the number of elements in cycle contained in $K_T(x)$.
$q_T(x)$, q–the number of cyclical elements of X in T.
r_T, r–the number of components of T.
S_n^k–Stirling's Numbers of the First Kind.
$\mathfrak{Z}_{k_1,k_2,\cdots,k_n}$ the subset of $\mathfrak{Z}$ with k_i components with exactly i elements, $i = 1, 2, \cdots, n$.

REFERENCES

[1] E. T. Bell, "Exponential polynomials," *Ann. Math.*, Vol. 35 (1934), pp. 258–277.

[2] William Feller, *An Introduction to Probability Theory and its Applications*, 1st ed., John Wiley and Sons, New York, 1950.

[3] Jay E. Folkert, "The distribution of the number of components of a random mapping function," Unpublished Ph.D. Dissertation, Michigan State University, 1955.

[4] W. Gontcharoff, "On the field of combinatory analysis," *Bull. de l'Academie des Sciences de U.R.S.S.*, (in Russian, with a French Summary), Serie Mathematique, Vol. 8 (1944), pp. 1–48.

[5] R. E. Greenwood, "The number of cycles associated with the elements of a permutation group," *Amer. Math. Monthly*, Vol. 60 (1953), pp. 407–409.

[6] Charles Jordan, *Calculus of Finite Differences*, Chelsea Publishing Company, New York, 1950.

[7] Leo Katz, "Probability of indecomposability of a random mapping function," *Ann. Math. Stat.*, Vol. 26 (1955), pp. 512–517.

[8] John Riordan, *An Introduction to Combinatorial Analysis*, John Wiley and Sons, New York, 1958.

[9] H. Rubin and R. Sitgreaves, "Probability distributions related to random transformations on a finite set," Tech. Rept. No. 19A, Applied Mathematics and Statistics Laboratory, Stanford University, 1954.