AN ASSOCIATED POLYNOMIAL FOR LEAST SQUARES APPROXIMATIONS

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Summary. Tabular reconstruction of differences from a curtailed set of moments is ordinarily impossible because of a gap on the integral side of the difference table. This gap can be closed by substituting an associated polynomial for the one that is being operated upon. The associated polynomial has, at the end of a *contracted range*, terminal differences and moments individually proportional to corresponding differences and moments of the polynomial from which it was derived. The function is applied to the problem of matching moments to form least squares approximations.

Introduction. The problem of least squares polynomial approximation to a set of n equally spaced observations reduces eventually to the problem of constructing a polynomial to have a given set of numbers for its moments. A currently favored procedure for this construction is to form a set of linear combinations of the moments which delivers the approximation arranged in terms of Chebychef's orthogonal polynomials. A practical method for applying Chebychef's functions to the case of equidistant intervals was described by Charles Jordan in 1921 [1]. A more convenient variation was given by R. A. Fisher in 1928 in the second edition of his textbook [2], and several variations were described by A. C. Aitken in 1932 [3]. Jordan published a revision of his own method in 1932 [4]. Fisher's method is the one known to most statisticians; rules for applying it are quoted in M. G. Kendall's *Advanced Theory of Statistics* ([5], Vol. 2, p. 164).

All of the procedures that employ Chebychef's functions have the same inconvenient feature; after obtaining the arrangement in orthogonal polynomials it is then necessary to perform another operation of equal magnitude to obtain a single numerical value or to tabulate the approximation over the given range. Another inconvenient detail is that the arrangement in orthogonal polynomials requires extra figures to be carried from the very beginning to absorb arithmetical error introduced by rounded quotients that occur early in the work.

In the present paper we will describe a method for recovering differences from moments *without* the use of Chebychef's functions. This is not a variation of previously known methods, it depends on the use of a new type of function. In application the function makes it possible to obtain the rth degree least squares approximation to a set of numbers in a form immediately useful for the substitution of numerical values or for tabulation. Incidentally, since the arithmetic does

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not require division operations until the last stage, arithmetical error is more easily controlled.

The function that makes this computation possible is designed for use on the difference table. We can assume the reader is familiar with operations on the difference side of this device but the integral side is not as well known and a foreword on the more extended use of the difference table may be helpful.

The Arrangement. The difference table is an instrument for computation. When applied to ordered sets of discrete numbers the use of the table corresponds to the use of a plotting linkage, or mechanical differentiator, or integraph in the study of continuous functions. When both the integral and the difference side are used together the device is capable of delivering, by simple arithmetic, results that would require complicated algebra.

A rectangular difference table is an array of positions connected by a rule of operation. The positions are arranged in horizontal lines and vertical columns and the rule of operation is that numbers are to be placed in the positions subject to the tabular relation

$$a + b = c. \qquad \urcorner\!\downarrow$$

The relation holds uniformly throughout the table for any three positions in the configuration $\begin{matrix} a & b \\ & c \end{matrix}$.

This array of interlocked positions is a computing device. To operate it

1. Place on the table an initial setting, a pattern of starting numbers.
2. Begin with any available position and write in it the number that satisfies the tabular relation. (In Table I, if any one of the numbers a, b, or c is missing, its value can be written in by inspection).
3. Continue in this manner until a region of interest in the table is completely filled in. Completion will proceed along a more or less devious path determined by the pattern of starting numbers.

TABLE I

x	$\Delta^3 u_x$	$\Delta^2 u_x$	Δu_x	u_x	Su_x	$S^2 u_x$	$S^3 u_x$
⋮							
-3							
-2			a	b			
-1				c			
0							
1						a	b
2	a	b					c
3		c		a	b		
4					c		
⋮							

Not all patterns are admissible, since some would lead eventually to an inconsistency with the tabular relation, but the number and variety of admissible patterns is sufficient to make the device a useful and versatile instrument. It has been used for many years and in many different forms. The initial setting depends on the purpose to which the table is applied and the process of filling in the blank spaces in the neighborhood of a given pattern is called differencing, tabulating, summing, or computing moments, according to the region that is filled in.

Like its graphical analogues the difference table delivers its results without requiring the use of algebra; its operation is mechanical. But to design an initial setting for the table, or to explain why the setting will deliver a certain result, it is necessary to use algebra. The indexing arrangement shown, the uniform tabular relation, and the off-set position of the sum adjacent to any column, automatically provide the necessary additive unit in the upper limit of a finite integral. This establishes in the algebra a convenient correspondence with the familiar notation used for definite integrals in the infinitesimal calculus. For a *definite* finite integral

$$\Sigma_a^b u_x = Su_b - Su_a \,, \tag{1}$$

where, by definition,

$$\Sigma_a^b u_x = \sum_{x=a}^{b-1} u_x \,. \tag{2}$$

In the columns on the right side of the table the prefix S is not a mechanical operator, it is an identifying prefix in the compound symbol that represents a set, or any member of a set, which is a *particular* (finite) integral of the set of numbers in the column adjacent on the left. The particular integral represented by the symbol $S^t u_x$ is the set of numbers in the column with that heading. *Any* initial setting consistent with the tabular relation can be used to construct a particular integral and the symbol with prefix S is then used to indicate the result. It is necessary to distinguish between the operation and the result when one constructs a particular integral.

The Operators Δ and Σ. A correspondence between the members of two sets is observed by substituting one set for the other in the attention of the analyst. In the algebra that has been attached to the difference table the attention of the analyst is directed from the numbers in a given column to the corresponding numbers in another column of the same table by the substitution operators Δ and Σ. When prefixed to a symbol representing the set of numbers in a given column Δ directs the attention of the analyst to the adjacent column on the left, Σ directs the attention to the adjacent column on the right. These operators can also be applied to a literal expression conditioned to describe the numbers in a given set. The operators are distributive with respect to algebraic addition and they commute with constant multipliers other than zero. Repetitions of the operations are indicated by positive integer superscripts that obey the law of addition for exponents.

There is no symbolic operator that deletes numbers from the difference table. Application of the operator Δ to an expression u_x, conditioned to describe a set of numbers on the table, will remove a constant term from the attention of the analyst in that expression but it does not remove it from the position it occupies on the table. Application of the operator Σ to an expression for the set Δu_x will restore the constant in the attention of the analyst, either (1) as an additive integration constant, or (2) as a term in a conditioning equation that accompanies the algebraic description of the result of the substitution.

The substitution operators Δ and Σ, when applied to a specific table, commute with each other. But when the operators are applied to an abstract algebraic expression not conditioned to relate to a particular table the commutative property is lost.

It is important to observe that the result of the operation Σ applied to a set u_x is Su_x, *not* $Su_x + C$. The set represented by the symbol Su_x contains at least one member which was originally a part of the initial setting and was selected without reference to the set u_x. The addition of another integration constant would be incorrect. To identify the particular integral represented by the symbol it is necessary to look at the difference table the algebra is intended to describe or to refer to a conditioning equation that identifies the set. The necessary integration constant must, of course, be written in a detailed algebraic description of the set, but it does not appear in the compact symbol.

Repeated substitutions of sets on the difference table may replace a set having prefix S with a set having prefix Δ, or a different substitution may have the reverse effect. The possible changes are exhibited well enough by the arrangement of the columns in the table, but in detail they are

$$\begin{array}{rclll} & & p < q & p = q & p > q \\ \Delta^p(\Delta^q u_x) & = & \Delta^{p+q}u_x & \Delta^{p+q}u_x & \Delta^{p+q}u_x \\ \Delta^p(S^q u_x) & = & S^{q-p}u_x & u_x & \Delta^{p-q}u_x \\ \Sigma^p(\Delta^q u_x) & = & \Delta^{q-p}u_x & u_x & S^{p-q}u_x \\ \Sigma^p(S^q u_x) & = & S^{p+q}u_x & S^{p+q}u_x & S^{p+q}u_x . \end{array} \tag{3}$$

Reduced Factorial Powers. Any ordered set of n numbers can be represented exactly by a conditioned polynomial of some degree less than n, and for some purposes the polynomial expression may be more convenient than the direct representation of the set on the difference table. A polynomial to be used as a description of a set of numbers on the difference table is arranged, for convenience, in factorial powers or reduced factorial powers. The nomenclature *reduced* is due to A. C. Aitken ([3], p 56). In the notation of Whittaker and Robinson ([6], Ch. 1, Sec. 6; Ch. 3, Sec. 28) a descending factorial power is written with square brackets and an exponent: $[x + a]^p$ means the product

$$(x + a)(x + a - 1)(x + a - 2)\cdots$$

to p factors, ending in $(x + a - p + 1)$. A reduced factorial power is written as $(x + a)_p$ meaning $[x + a]^p/p!$.

We assume the reader is acquainted with the effects of the operators Δ and Σ on the sets represented by polynomials arranged in terms of these functions.

Some Properties of the Table. The difference table of a polynomial u_x, tabulated for integer values of the independent variable x, is distinguished by a column of zeros for all of the differences of order $r + 1$, where r is the degree of the polynomial. In practical applications this column of zeros may be part of the starting pattern written explicitly in the table. Some of the useful properties of the difference table of a polynomial will be applied in the present work. We consider a completed table that contains related sets of differences, values, and particular iterated sums of a polynomial, tabulated for integer values of x, positive, negative, and zero. The lines of the table are indexed by the value of the independent variable and we refer to the numbers on a given line of the table as the numbers on line a, meaning the numbers on the line for which $x = a$. We select a certain column in the table and refer to the numbers in that column as the function of interest. The function of interest might be, at one time the set $\Delta^3 u_x$, at another time it might be the set u_x itself, or at another time it might be the set $S^4 u_x$, the particular iterated sum of that order which is exhibited on this table. The property is as follows:

> The numbers on any line a extending from the left, to and including the number in the column of interest, are the coefficients of the function of interest when it is arranged in reduced factorial powers $(x - a)_p$. The terms are in descending order from left to right, ending in a constant for the last term.

Another property applied in the present work relates specifically to the integral side of the table. This second property is not restricted to the table of a polynomial but applies to any ordered set of numbers.

> Given, as the initial setting on the table, a column of n numbers v_x, $x = 0$ to $n - 1$ inclusive, and a line of zeros on line zero in the positions $S^t v_0$ so that $S^t v_0 = 0$ for $t = 1, 2, 3, \cdots$ as far as desired. When the right side of this table is filled in the sums on line n will be reduced factorial moments of the form

$$S^t v_n = (-1)^{t-1} \sum_{x=0}^{n-1} (x - n + t - 1)_{t-1} v_x \qquad (t > 0). \tag{4}$$

The first property is the basis of Newton's formula for interpolation with forward differences ([6], Ch. 1, Sec. 8); the second is an example of the computation of moments by summation, an operation introduced by G. F. Hardy in the late 1800's in the work of graduating the British mortality tables ([7], Ch. 3, Sec. 9). Ordinarily these two properties are used separately; our present object is to establish a simple arithmetical link between them and compute differences from moments in the same way we compute moments from differences.

Matching Moments. In the normal equations which impose the least squares condition on an rth degree polynomial u_x to approximate a set of n observations v_x the numerical coefficients are moments, sums of products. The set of normal equations reduces to the statement that a set of $r + 1$ moments of the polynomial must equal the same set of moments of the observations.

We have freedom of choice in selecting the type of moment to be matched. The only requirement is that the moment multipliers be a linearly independent set of polynomials derivable by a non-singular transformation from a set of polynomials of degrees 0 to r in x. This set can be chosen to suit the convenience of the analyst.

When $r \leqq n - 1$ the normal equations have a unique solution and the different types of moments that can be matched all deliver the same approximating polynomial but, depending on the method of solution, the terms may be arranged in different ways. The usual choice for matching is a set of reduced factorial moments since these are so easily computed by iterated summation of the set of observations.

In practice the degree r of the approximation is limited to values less than $n - 1$. This is a practical, rather than a mathematical, limitation. One can easily construct a least squares polynomial approximation for a degree r equal to or greater than $n - 1$ by direct differencing. The polynomial will coincide precisely with the set and the coefficients of the unnecessary terms of higher degree (e.g., the differences on line 0 of order greater than $n - 1$) may be chosen arbitrarily. A set of n numbers does not have more than n linearly independent power moments (and it may have less than n). Moments of order higher than $n - 1$ are linear combinations of the lower order moments and contain no new information.

We will use a type of factorial moment which is zero for all orders greater than $n - 1$. (Equation 4).

Rearranging the Solution. So far as algebra is concerned any computation of the first $r + 1$ moments of the set v_x completes the construction of the approximating polynomial u_x. The numbers are taken as the first $r + 1$ moments of the polynomial. They are also the coefficients of the polynomial when it is arranged in terms of a complementary family of polynomials orthogonal to the moment multipliers in the given range. For the commonly used types of moments (those that are easiest to compute) the complementary polynomials are not well suited for the substitution of numerical values or for tabulation. For these purposes it is more convenient to have the polynomial arranged in reduced factorial powers and rearrangements of this kind are most easily performed by operating directly on the difference table.

When $n = r + 1$ it is evident that the difference table of a polynomial given in terms of its moments can be reconstructed in reverse order, but when n is greater than $r + 1$ there is a gap on the integral side that cannot be filled in directly. This gap can be closed by substituting, for the polynomial u_x which is to be rearranged, an associated polynomial u'_x of the same degree and operating on the associated polynomial in the contracted range $x = 0$ to $x = r + 1$. In this range differences can be recovered from moments simply by reversing the summation process ordinarily used to compute the moments.

The Participating Functions. The complete process which we are now in a position to describe deals with differences and particular finite integrals of three distinct functions: v_x, u_x and u'_x.

The *set of observations* v_x is a set of n discrete numbers, known only for $x = 0, 1, 2, \cdots (n - 1)$. Nothing is known or assumed about possible values of v_x at values of x other than these. The only part played by the set v_x is to furnish the moments that determine the approximating polynomial.

The *approximation* u_x is a polynomial of degree r. This polynomial is simply an algebraic form in which numbers can be substituted. The form is pinned to the set v_x at the n values $x = 0, 1, 2, \cdots (n - 1)$ by the least squares condition but it has no assigned connection with possible values of v_x at points other than these. It should be clear that extrapolation with a function chosen to suit arithmetical convenience has no mathematical justification. Although it is true that, when $r \leqq n - 1$ the rth degree least squares polynomial approximation to a given set is unique, there is an unlimited number of different least squares approximations having the same number of parameters based on functions other than polynomials.

We assume the approximating polynomial will be used to summarize experimental information, such as a frequency distribution. In practical applications the degree r will be some integer less than $n - 1$. If the analyst has observed that a set of differences $\Delta^r v_x$ is approximately constant and the variations are assumed to be due to chance the choice of a polynomial to describe the distribution function is reasonable. The observed distribution is itself empirical and the use of an empirical formula to summarize the distribution has the advantage of conveying the useful information in a smaller number of terms.

The particular integral $S^t u_x$ is a polynomial of degree $r + t$. In the present application it will be the particular finite integral of the polynomial u_x characterized by the values $S^t u_0 = 0$ for $t = 1, 2, 3, \cdots$ as far as desired.

Similarly, $S^t u'_x$ will be the particular finite integral of u'_x characterized by the same set of initial values $S^t u'_0 = 0$ for $t = 1, 2, 3, \cdots$ as far as desired.

Both polynomials u_x and u'_x will be tabulated for integer values of x but the range of tabulation for u_x is $x = 0$ to n. The range of tabulation for u'_x is $x = 0$ to $r + 1$.

The *associated polynomial* u'_x is a polynomial of degree r derived from the polynomial u_x. The terminal differences and the sums of u'_x on line $r + 1$ are individually proportional to those of u_x on line n. After reconstructing the differences from the sums of the associated polynomial the proportionality factors can be removed, leaving the terminal differences of the desired polynomial. We construct the associated polynomial from a special set of particular integrals of the original polynomial. This is the set obtained by choosing zero for the initial values of all of the sums on line zero.

For a given polynomial u_x the associated polynomial u'_x is defined by the following equation:

$$u'_x = \Delta^n\{(x - r - 1)_{n-(r+1)}S^{r+1}u_x\}. \tag{5}$$

Then the proportionality factors are given by

$$S^t u'_{r+1} = (n - t)_{n-(r+1)}S^t u_n \qquad (t > 0) \tag{6}$$

$$\Delta^t u'_{r+1} = (n + t)_{n-(r+1)}\,\Delta^t u_n \qquad (t \geqq 0). \tag{7}$$

Here $S^t u_x$ is the tth iterated sum of u_x formed from the particular set of initial values $S^t u_0 = 0$ for $t = 1, 2, 3, \cdots$ as far as desired, and $S^t u'_x$ is the tth iterated sum of u'_x formed from the particular set of initial values $S^t u'_x = 0$ for $t = 1, 2, 3, \cdots$ as far as desired. The proportionality factors are reduced factorial powers of the descending kind and can be obtained from a table of binomial coefficients.

Algebraic Demonstration. In numerical computation the construction of the associated polynomial consists merely in applying the proportionality factors to the set of moments. The only need for algebra is to identify the proportionality factors. To explain why the process delivers its intended result by such simple means we will use one of the difference calculus analogues of Leibniz's theorem, but this algebra is not used in the performance of the computation.

Referring to equations (3) it will be observed that repeated application of the operator Σ to the defining equation for u'_x reduces the exponent of the operator Δ on the right, and that differencing increases it. We apply the operator Σ^t, or the operator Δ^t, to the defining equation (5) and expand the right side by using one of the difference calculus forms ([8], Ch. 2, Art. 10, Ex. 3) for the difference of a product

$$\Delta^p(w_x y_x) = \sum_{j=0}^{p} (p)_j \Delta^{p-j} w_x \Delta^j y_{x+p-j}\,. \tag{8}$$

This gives, for the particular sums of the associated polynomial, with initial values $S^t u'_0 = 0$ for all $t > 0$,

$$S^t u'_x = \sum_{j=0}^{n-t} (n - t)_j \Delta^{n-t-j}(x - r - 1)_{n-(r+1)} S^{r+1-j} u_{x+n-t-j}\,. \tag{9}$$

In this expanded form all but one of the terms will contain the factor $(x - r - 1)$ and will vanish at $x = r + 1$. The single non-vanishing term is the one in which that factor has been reduced to unity by repeated differencing; this is the term for which $j = r + 1 - t$. Writing out that term for $x = r + 1$,

$$S^t u'_{r+1} = (n - t)_{r+1-t} S^t u_n\,. \tag{10}$$

Using the relation $(a)_b = (a)_{a-b}$, the proportionality factors for the sums can be written more conveniently as $(n - t)_{n-(r+1)}$.

The proportionality factors for the differences of the associated polynomial at $x = r + 1$ may be verified similarly. The factor for $\Delta^0 u'_x$ is obtained by operating zero times on equation (5) and expanding the right side. For any integer $t \geqq 0$ the expansion is

$$\Delta^t u'_x = \sum_{j=0}^{n+t} (n + t)_j \Delta^{n+t-j}(x - r - 1)_{n-(r+1)} \Delta^j S^{r+1} u_{x+n+t-j} . \tag{11}$$

Repeated differencing of the last factor on the right reduces the order of the S to zero and for $j \geqq r + 1$ the factor is a difference: $\Delta^{j-(r+1)}$ of the object function u. At $x = r + 1$ the single non-vanishing term is the one for which $j = r + 1 + t$ and the result is

$$\Delta^t u'_{r+1} = (n + t)_{n-(r+1)} \Delta^t u_n . \tag{12}$$

So the differences and sums of u' on line $r + 1$ of its difference table are individually proportional to the corresponding differences and sums of u on line n of the difference table of u. It is only these terminal values that are proportional, the others are not.

Integration constants are an important part of the process and the associated polynomial is designed for use *only* with the particular set $S^t u_0 = 0$. In practice the index t will not be greater than $r + 1$. This is a practical rather than a mathematical restriction. The index t can be greater than $r + 1$ but equations (6) and (7), though still true, become vacant identities. In equation (7) the index t can be zero with the usual interpretation of $\Delta^0 u_x$ as identical with u_x.

We avoid the use of negative exponents with Δ and Σ because, when these operators are expressed as matrices, they are singular and have no reciprocals.

Numerical Demonstration. The application of the associated polynomial to least squares approximation can be more easily understood by following the steps in a numerical example.

To construct a second degree least squares polynomial to represent the set of seven "observations"

x	0	1	2	3	4	5	6
v_x	6	8	14	27	50	86	138

The process requires the following 5 steps:

1. Compute the moments to be matched.
2. Apply the proportionality factors to the moments.
3. Reconstruct the terminal differences of the associated polynomial.
4. Remove the proportionality factors from the differences.
5. Tabulate the result.

Three of the steps are tabular computations. In each of these the arithmetic consists in (1) writing down the pattern of starting numbers, (2) filling in the blank spaces with the numbers necessary to preserve the tabular relation.

The example printed here is shown as it appears when completed. The starting numbers are shown in bold face.

STEP 1

Computation of the Necessary Moments

x	Δ^3	Δ^2	Δ	v_x	S	S^2	S^3
0				**6**	0	0	0
1				**8**	6	0	0
2				**14**	14	6	0
3				**27**	28	20	6
4				**50**	55	48	26
5				**86**	105	103	74
6				**138**	191	208	177
$n = 7$					329	399	385

STEP 2 Multiplying by:	15	5	1
Gives the sums:	4935	1995	385

of the associated polynomial.

STEP 3

Reconstruction of the Differences from the Sums of the Associated Polynomial

x				u'_x			
0	0	1260	840	385	0	0	0
1	0	1260	2100	1225	385	0	0
2	0	1260	3360	3325	1610	385	0
$r + 1 = 3$	0	1260	4620	6685	**4935**	**1995**	**385**

STEP 4 Dividing by:	126	70	35
Gives the differences:	10	66	191

of the approximating polynomial.

STEP 5

Tabulation of the Values of the Approximating Polynomial

x				u_x	
0	0	10	−4	9	
1	0	10	6	5	
2	0	10	16	11	
3	0	10	26	27	
4	0	10	36	53	
5	0	10	46	89	
6	0	10	56	135	
$n = 7$	**0**	**10**	**66**	**191**	

CHECK

Apply Step 1 to the Result of Step 5

The sum of the squared differences between the observations and their approximations can be obtained by taking advantage of the least squares relationship.

$$\Sigma_0^n (v_x - u_x)^2 = \Sigma_0^n v_x^2 - \Sigma_0^n u_x^2 \tag{13}$$

when u_x is a least squares polynomial approximation to v_x in the range of summation. The absence of a cross product term is the advantage here.

If one requires only an algebraic expression for the approximating polynomial Step 5 can be omitted. Step 4 delivers the coefficients of the polynomial in a form convenient for the substitution of any numerical value that may be of interest. In the example illustrated the polynomial approximation is delivered by Step 4 as

$$u_x = 10(x - 7)_2 + 66(x - 7) + 191. \tag{14}$$

If Step 5 is completed the difference table presents the coefficients of the approximating polynomial in all of the different arrangements that are described graphically in a Fraser diagram. For example, taking the numbers on line 0 as the coefficients, another arrangement of the solution is

$$u_x = 10(x)_2 - 4x + 9. \tag{15}$$

The arrangement of a polynomial in continuous powers x^p is familiar through habit and custom but it has no mathematical pre-eminence. It is convenient for multiplication and division but for other operations there are more convenient arrangements. The arrangement in reduced continuous powers $x_p = x^p/p!$ has certain advantages in connection with differentiation and integration. An arrangement in reduced factorial powers is convenient for operations connected with the difference table and is equally convenient for the substitution of numerical values.

An arrangement in continuous powers is seldom necessary in the calculus of finite differences except in problems connected with interpolation or sub-tabulation. If it is needed the expansion of the factorial powers and the collection of terms is a minor arithmetical detail. In the illustration, from the arrangement in equation (15), it can be performed by inspection.

$$u_x = 5x^2 - 9x + 9. \tag{16}$$

In more complicated examples a collection in continuous powers would be performed by writing out the expansions of the (un-reduced) factorial powers or, what amounts to the same thing, by using a table of Stirling's numbers.

It is interesting to observe in numerical examples that the degree r of the two polynomials u_x and u_x' may be redundant (the coefficient of the highest power can be zero). Equations (6) and (7) will still be true and reconstruction of the difference table will automatically deliver the zero coefficients along with the others. Of course both polynomials will be treated as being of the same degree and both will be found to have the same number of terms with zero coefficients.

As an example for practice the reader may construct the fourth degree least squares approximation to the set of seven numbers used in the numerical demonstration.

Arithmetical Error. The numbers in the example illustrated were chosen to make the demonstration easy to follow; in practice the coefficients of the approximating polynomial (the differences on line n) will have recurring decimal parts that must be rounded off. Rounding the coefficients introduces arithmetical error in the values of the polynomial computed from them and for values of x far from n the error may be greater than the value of the polynomial itself. To absorb the arithmetical error one can carry extra figures in the results of the division operations that remove the proportionality factors and then discard the extra figures after tabulating the approximation. It can be shown by finite integration that the maximum possible error in the last decimal place of u_0, when computed from an rth degree polynomial arranged in terms of $(x - n)_p$ with rounded coefficients, is $\pm 0.5(n + r)_r$. To determine how many extra figures should be carried in the differences of u_n, write out the numerical value of this maximum and count the effective number of figures in it to the left of the decimal point. For example when $n = 12$, $r = 4$, the error in the last place of any value in the range is not greater than ± 910. The effective number of figures would be counted as four to permit the usual rounding process when discarding the extra figures.

The control of arithmetical error by extra figures is satisfactory enough but in the present computation there is another method available. We can postpone all approximate divisions to the very end of the work by tabulating a multiple ku_x instead of u_x itself. The multiplier k can be any common multiple of the proportionality factors to be removed from the differences. The least common multiple is convenient but any common multiple will do.

To tabulate ku_x multiply all the terminal differences $\Delta^t u'_{r+1}$, including $\Delta^0 u'_{r+1}$, by k before removing the proportionality factors. Then remove the proportionality factors, which of course will divide out exactly since k is a common multiple of them. Tabulate the result and finally, as a last step, divide the individual values by k, carrying out the division in each case to as many decimal places as desired. All of the figures will be free of arithmetical error. This provides complete control over the arithmetical accuracy of the computation.

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