SOME ASPECTS OF WEIGHING DESIGNS

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1. Summary. In a previous paper [8] the author proved that the P_N and S_N matrices are the most efficient weighing designs obtainable under Kishen's definition of efficiency [5], when N is odd and $N \equiv 2 \pmod 4$ respectively, subject to the conditions

(i) The variances of the estimated weights are equal;

(ii) The estimated weights are equally correlated.

In this paper, assuming the above conditions, it is proved that the P_N matrices are the best weighing designs under the definitions of Mood [6] and Ehrenfeld [2] when N is odd, while the S_N matrices are the best weighing designs under the definition of Ehrenfeld when $N \equiv 2 \pmod 4$. Under Mood's definition of efficiency, the best weighing design X, when $N \equiv 2 \pmod 4$, is shown to be that for which $X'X = (N - 2)I_N + 2E_{NN}$, where I_N is the Nth order identity matrix and E_{NN} is the Nth order square matrix with positive unit elements everywhere. By applying the Hasse-Minkowski invariant, a necessary condition for the existence of the S_N matrices is obtained, and the impossibilities of the S_N matrices of orders 22, 34, 58 and 78 are shown.

2. Introduction. Suppose we are given N objects to be weighed in N weighings with a chemical balance having no bias. Let

$$\begin{aligned} x_{ij} &= 1, \text{ if the } j\text{th object is placed in the left pan in the } i\text{th weighing;} \\ &= -1, \text{ if the } j\text{th object is placed in the right pan in the } i\text{th weighing;} \\ &= 0, \text{ if the } j\text{th object is not weighed in the } i\text{th weighing.} \end{aligned}$$

The Nth order matrix $X = (x_{ij})$ is known as the design matrix. Also, let y_i be the result recorded in the ith weighing; ϵ_i the error in this result and w_j the true weight of the jth object, so that we have the N equations

$$x_{i1}w_1 + x_{i2}w_2 + \cdots + x_{iN}w_N = y_i + \epsilon_i, \qquad i = 1, 2, \cdots, N. \tag{2.1}$$

If X is non-singular, the method of Least-Squares or theory of Linear Estimation, gives the estimated weights ($\hat{w}_i$) by the equation

$$\hat{w} = S^{-1}X'y, \tag{2.2}$$

where y is the column vector of the observations, $\hat{w}$ is the column vector of the estimated weights and $S = X'X$.

If σ^2 is the variance of each weighing, then

$$\text{Var}(\hat{w}) = S^{-1}\sigma^2 = (C_{ij})\sigma^2, \tag{2.3}$$

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where (C_{ij}) is the inverse matrix of S. Hotelling [3] proved that the minimum minimorum of each of the estimated weights is σ^2/N.

Mood considers as best that weighing design which gives the smallest corresponding joint confidence region for the estimated weights. Consider a set of confidence intervals C_0 for the parameter θ, typified by δ_0, obeying the condition that $P(\delta_0 C\theta \mid \theta) = \alpha$, where we write $\delta_0 C\theta$, that is δ_0 contains θ. Let C_1 be some other confidence intervals for the parameter θ, typified by δ_1, such that $P(\delta_1 C\theta' \mid \theta) = \alpha$. If now for every C_1, we have, for any value θ' other than the true value, $P(\delta_0 C\theta' \mid \theta) \leqq P(\delta_1 C\theta' \mid \theta)$, C_0 is said to be the smallest confidence intervals (cf. Neyman [7]). Hence a design will be called optimum in the sense of Mood if the determinant of the matrix (C_{ij}) is minimum. But we know that the determinant $|C_{ij}|$ is minimum when the determinant $|S|$ is maximum. Thus, the efficiency of a weighing design X can be measured, in the sense of Mood, by

$$\det(S)/\max.\det(S). \tag{2.4}$$

If $\lambda_{\min}$ is the minimum of the distinct characteristic roots of S, then the efficiency of the weighing design X, can be measured, in the sense of Ehrenfeld, by

$$\lambda_{\min}/N. \tag{2.5}$$

The conditions

(i) the variances of the estimated weights are equal;

(ii) the estimated weights are equally correlated are assumed throughout this paper.

3. Most efficient designs when N is odd and $N \equiv 2 \pmod 4$ under the definitions of efficiency of Ehrenfeld and Mood. With the conditions assumed in Section 2, the matrix S takes the form

$$(r - \lambda)I_N + \lambda E_{NN}. \tag{3.1}$$

Now

$$\det(S) = (r - \lambda)^{N-1}\{r + \lambda(N - 1)\}. \tag{3.2}$$

Since $\det(X)$ is real and non zero, we have

$$r > \lambda \geqq 0, \text{ or } r = N, \lambda = -1. \tag{3.3}$$

Therefore, in this paper we consider only those values of r and λ satisfying (3.3).

Replacing r in (3.1) by $(r - z)$ and equating the value of $\det(S)$ to zero, we get $(r - \lambda)$ and $\{r + \lambda(N - 1)\}$ as the distinct characteristic roots of S with multiplicities $(N - 1)$ and 1 respectively when $\lambda \neq 0$. If $\lambda = 0$, r is the only distinct characteristic root and it has multiplicity N. In either case, among the distinct characteristic roots, $(r - \lambda)$ is always minimum except when $r = N$, $\lambda = -1$, in which case 1 is the minimum characteristic root. Hence from (2.5), we measure the efficiency of a weighing design X, satisfying (3.1) under the definition of efficiency of Ehrenfeld, by

$$f_1(r, \lambda) = \begin{cases} (r - \lambda)/N, & r > \lambda \geqq 0; \\ 1/N, & r = N, \quad \lambda = -1. \end{cases}$$

Using the method and Lemma 2.1 of [8] we can easily prove the following two theorems:

THEOREM 3.1. *For Ehrenfeld's definition of efficiency the best weighing design X, when N is odd, is that for which*

$$S = (N - 1)I_N + E_{NN}. \tag{3.5}$$

THEOREM 3.2. *For Ehrenfeld's definition of efficiency, the best weighing design X, when $N \equiv 2 \pmod 4$ and $N \neq 2$, is that for which*

$$S = (N - 1)I_N. \tag{3.6}$$

If we let $f_2(r, \lambda)$ be the value of $\det(S)$, we have the following Lemma.

LEMMA 3.1. *For $r > \lambda \geqq 0$,*

(i) *$f_2(r, \lambda)$ is a monotonic increasing function in r for a fixed λ, and*

(ii) *$f_2(r, \lambda)$ is a monotonic decreasing function in λ for fixed r.*

The Lemma can be easily proved by partially differentiating $f_2(r, \lambda)$ with respect to r and λ, and examining the signs of the derivatives.

We now prove

THEOREM 3.3. *For Mood's definition of efficiency, the best weighing design X, when N is odd, is that whose S is* (3.5).

PROOF. Since max. $\det(S)$ is not known, we prove that $\det(S)$, where S is given by (3.5), is greater than $\det(S)$ for all other possible S. Now,

$$f_2(N, 1) - f_2(N - 1, 0) = N(N - 1)^{N-1} > 0. \tag{3.7}$$

Again

$$\begin{aligned} f_2&(N, 1) - f_2(N, -1) \\ &= (N - 1)^{N-1}(2N - 1) - (N + 1)^{N-1} \\ &= 2N(N - 1)^{N-1} - 2\left\{N^{N-1} + \binom{N-1}{2}N^{N-3} + \cdots + 1\right\} \\ &= 2\Bigg[(N - 1)\left\{N^{N-1} + \binom{N-1}{2}N^{N-3} + \cdots + 1\right\} \\ &\qquad - N\left\{\binom{N-1}{1}N^{N-2} + \binom{N-1}{3}N^{N-4} + \cdots + \binom{N-1}{N-2}N\right\}\Bigg] \\ &= 2\Bigg[N^{N-3}\left\{(N - 1)\binom{N-1}{2} - \binom{N-1}{3}\right\} \\ &\qquad + N^{N-5}\left\{(N - 1)\binom{N-1}{4} - \binom{N-1}{5}\right\} \\ &\qquad + \cdots + N^2\left\{(N - 1)\binom{N-1}{N-3} - \binom{N-1}{N-2}\right\} + (N - 1)\Bigg]. \end{aligned} \tag{3.8}$$

But

$$(N-1)\binom{N-1}{i} > \binom{N-1}{i+1}. \tag{3.9}$$

Hence the last expression of (3.8) is greater than zero and we have

$$f_2(N, 1) > f_2(N, -1). \tag{3.10}$$

Also, we know from Lemma 2.1 of [8] that λ cannot be zero, since N is odd. Therefore from the inequalities (3.7), (3.10) and Lemma 3.1, we see that $\det(S)$ is maximum when S is given by (3.5). This completes the proof.

THEOREM 3.4 *For Mood's definition of efficiency, the best weighing design X, when $N \equiv 2 (\text{mod } 4)$ and $N \not\equiv 2$ is that for which*

$$S = (N-2)I_N + 2E_{NN}. \tag{3.11}$$

PROOF.

$$\begin{aligned} f_2(N, 2) &- f_2(N-1, 0) \\ &= \{(N-2)^{N-1}(3N-2) - (N-1)^N\} \\ &= (N-1)^N[\{1 - 1/(N-1)\}^{N-1}\{3 + 1/(N-1)\} - 1]. \end{aligned} \tag{3.12}$$

Considering the inequality

$$t < -\log(1-t) < t/(1-t), \qquad 0 < t < 1, \tag{3.13}$$

and substituting $t = 1/(N-1)$, we can easily show that

$$\{1 - 1/(N-1)\}^{N-1} > \text{Exp}\{-(N-1)/(N-2)\}. \tag{3.14}$$

Making use of (3.14), (3.12) is greater than zero, if

$$3\,\text{Exp}\{-(N-1)/(N-2)\} - 1 > 0. \tag{3.15}$$

We easily see that (3.15) is true for $N > 11$. We also see that $f_2(N, 2) > f_2(N-1, 0)$ for $N = 5, 6, 7, 8, 9, 10, 11$ by actual substitution. Thus, we have

$$f_2(N, 2) > f_2(N-1, 0) \quad \text{for} \quad N \geqq 5. \tag{3.16}$$

The only value of $N \leqq 4$ and $\equiv 2 (\text{mod } 4)$ is 2, and in this case we know that the Hadamard matrix provides the optimum weighing design. Hence, if we delete this case we see that $f_2(N, 2) > f_2(N-1, 0)$ when $N \equiv 2 (\text{mod } 4)$.

We know from Lemma 2.1 of [8] that λ cannot be equal to ± 1 when $r = N \equiv 2$ (mod 4). Also, as no Hadamard matrix exists in this case, λ cannot be equal to zero when $r = N$. Thus, from Lemma 3.1 and the inequality (3.16) we see that the $\det(S)$ is maximum when S is given by (3.11). Thus the theorem is proved.

The proof of the following theorem is similar to that of Theorem 3.1 of [8].

THEOREM 3.5. *A necessary condition for the existence of a weighing design X satisfying* (3.11) *is that*

$$N = \{4 + (3f^2 + 4)^{\frac{1}{2}}\}/3, \tag{3.17}$$

where f is an integer.

An application of the above theorem shows that the weighing design X satisfying (3.11) exists only for $N = 6$ and $N = 66$ out of all $N < 200$ and $\equiv 2 \pmod 4$.

For $N = 6$, the best weighing design X satisfying (3.11) is

$$\begin{bmatrix} 1 & -1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & -1 \end{bmatrix}. \tag{3.18}$$

If we adopt the above design to weigh 6 objects,

$$\begin{aligned} &\text{Variance of each estimated weight} = 7\sigma^2/32, \quad \text{and} \\ &\text{Covariance of each pair of estimated weights} = -\sigma^2/32. \end{aligned} \tag{3.19}$$

4. Some known results about the Legendre symbol, the Hilbert norm residue symbol and the Hasse-Minkowski invariant. The Legendre symbol (a/p) is defined for odd primes p as

$$(a/p) = \begin{cases} +1, & \text{if } a \text{ is a quadratic residue of } p; \\ -1, & \text{if } a \text{ is a non quadratic residue of } p. \end{cases} \tag{4.1}$$

A slight generalisation of the Legendre symbol is the Hilbert norm residue symbol $(a, b)_p$. If a and b are non zero rational numbers, we define $(a, b)_p$ to have the value $+1$ or -1 according as the congruence,

$$ax^2 + by^2 \equiv 1 \pmod{p^r}, \tag{4.2}$$

has or has not for every value of r, rational solutions x_r and y_r. Here p is any prime, including the conventional prime $p_\infty = \infty$. Many properties of $(a, b)_p$ are given by Jones [4] and Shrikhande [9].

Let $A = (a_{ij})$ be any $n \times n$ symmetrical matrix with rational elements. The matrix B is said to be rationally congruent to A, written $A \sim B$, provided there exists a non singular matrix C with rational elements such that $A = CBC'$, where C' is the transpose of C. If D_i $(i = 1, 2, \cdots, n)$ denotes the leading principal minor determinant of order i in the matrix A, then, if none of the D_i vanish, the quantity

$$C_p = C_p(A) = (-1, -D_n)_p \prod_{i=1}^{n-1} (D_i, -D_{i+1})_p \tag{4.3}$$

is invariant for all matrices rationally congruent to A. C_p is known as the Hasse-Minkowski invariant.

The following Lemma, given by Bose and Connor [1], will be of use for the next section.

LEMMA 4.1. *If t is a rational number and $\Delta_m = tI_m$, then,*

$$C_p(\Delta_m) = (-1, -1)_p(t, -1)_p^{m(m+1)/2}. \tag{4.4}$$

5. On the impossibilities of the S_N matrices. Since the S_N matrix is a square matrix with rational elements and $\text{Det}(S_N) \neq 0$, its inverse exists and is also a matrix with rational elements. Thus, $I_N = (S_N^{-1\prime})(S_N' S_N)(S_N^{-1})$. From the last section, we see that I_N and $S_N' S_N$ are rationally congruent and they can be written $S_N' S_N \sim I_N$. Hence

$$(5.1) \qquad C_p(S_N' S_N) = C_p(I_N) = (-1, -1)_p.$$

But

$$(5.2) \qquad S_N' S_N = (N - 1) I_N .$$

From Lemma 4.1, we see that

$$(5.3) \qquad C_p(S_N' S_N) = (-1, -1)_p (N - 1, -1)_p^{N(N+1)/2}.$$

But, as $N \equiv 2 \pmod 4$, $N(N + 1)/2$ is odd and (5.3) reduces to

$$(5.4) \qquad C_p(S_N' S_N) = (-1, -1)_p (N - 1, -1)_p.$$

Equating the right hand sides of (5.1) and (5.4), we have for all primes p,

$$(5.5) \qquad (N - 1, -1)_p = +1.$$

This result can be stated in the form of the following theorem.

THEOREM 5.1. *A necessary condition for the existence of the S_N matrix where $N \equiv 2 \pmod 4$ is that $(N - 1, -1)_p = +1$, for all primes p.*

ILLUSTRATION 5.1.1. When $N = 22$,

$$(N - 1, -1)_p = (21, -1)_p = (3, -1)_p (7, -1)_p$$
$$= -1, \text{ for } p = 3.$$

The Theorem 5.1 is violated and S_{22} does not exist.

The non existence of S_{34}, S_{58} and S_{78} can also be easily shown by applying Theorem 5.1.

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