A THIRD ORDER ROTATABLE DESIGN IN FOUR DIMENSIONS[1]

By Norman R. Draper

Mathematics Research Center, United States Army, Madison, Wisconsin

1. Introduction. Only one third order rotatable design in four dimensions is known, namely the 128 point design presented by Gardiner, Grandage and Hader [2]. This is formed by combining the 96 points

$$(\pm c, \pm c, \pm d, \pm d), \quad (\pm c, \pm d, \pm c, \pm d), \quad (\pm c, \pm d, \pm d, \pm c),$$
$$(\pm d, \pm c, \pm c, \pm d), \quad (\pm d, \pm c, \pm d, \pm c), \quad (\pm d, \pm d, \pm c, \pm c),$$

of the "truncated cube (2)" with the 16 points arising from two cross-polytopes

$$(\pm c_i, 0, 0, 0), \quad (0, \pm c_i, 0, 0), \quad (0, 0, \pm c_i, 0), \quad (0, 0, 0, \pm c_i),$$

$i = 1, 2$, and the 16 points of a measure polytope

$$(\pm a, \pm a, \pm a, \pm a),$$

where a, c_i, c and d take appropriate values obtained after considerable computation.

We now give a very simple design which requires 96 points (32 less) and is a combination of four second order rotatable arrangements, each containing 24 points. This permits the use of four convenient blocks of equal size or, alternatively, sequential performance in several ways. The notation and definitions of reference [1] are used in obtaining this design.

2. The construction of the design. Consider in four dimensions (a) the 24 points

$$(\pm p, \pm p, 0, 0), \quad (\pm p, 0, \pm p, 0), \quad (\pm p, 0, 0, \pm p),$$
$$(0, \pm p, \pm p, 0), \quad (0, \pm p, 0, \pm p), \quad (0, 0, \pm p, \pm p),$$

which we shall denote by $S(p, p, 0, 0)$; (b) the 8 points

$$(\pm c, 0, 0, 0), \quad (0, \pm c, 0, 0), \quad (0, 0, \pm c, 0), \quad (0, 0, 0, \pm c),$$

which we shall denote by $S(c, 0, 0, 0)$; (c) the 16 points

$$(\pm a, \pm a, \pm a, \pm a),$$

which we shall denote by $S(a, a, a, a)$.

Applying the formulae for the various excess functions (as defined in [1]) for these sets, we find the results shown in Table I.

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TABLE I

The Excess Functions for Three Chosen Point Sets

	$S(p, p, 0, 0)$	$S(c, 0, 0, 0)$	$S(a, a, a, a)$
Number of points	24	8	16
Ax	$12p^2$	$2c^2$	$16a^2$
Ex	0	$2c^4$	$-32a^4$
Fx, Gx	0	0	0
Hx	$12p^6$	$2c^6$	$-224a^6$
Ix	$4p^6$	0	$-32a^6$

As is shown in [1], the combined set of points

$$S = S(p, p, 0, 0) + S(c, 0, 0, 0) + S(a, a, a, a)$$

will form a third order rotatable arrangement if

$$Ex(S) = Fx(S) = Gx(S) = Hx(S) = Ix(S) = 0.$$

This will happen when, from Table I, $2c^4 - 32a^4 = 0$, $12p^6 + 2c^6 - 224a^6 = 0$, $4p^6 - 32a^6 = 0$, i.e., when $c^2 = 4a^2$, $p^2 = 2a^2$. The radii of the three separate sets which comprise the arrangement are $(2)^{\frac{1}{2}}p$, c, and $2a$, all of which equal $2a$. It follows [1] that the 48 points given by

$$S(a\sqrt{2}, a\sqrt{2}, 0, 0) + S(2a, 0, 0, 0) + S(a, a, a, a),$$

which we shall denote, as a whole, by $S(a)$, form a *singular* third order arrangement for which $\lambda_6\lambda_2/\lambda_4^2 = (k + 2)/(k + 4) = \frac{3}{4}$. Thus $S(a)$ cannot, by itself, be used as a design. Now consider two such arrangements characterized by $S(a_1)$ and $S(a_2)$. As has been shown [1], the combination of points $S(a_1) + S(a_2)$ will, if $a_1 \neq a_2$, form a third order rotatable arrangement which is non-singular and so *may* be used as a design. For the whole set, $\lambda_2 N = 48(a_1^2 + a_2^2)$, where $N = 96 + n_0$ and n_0 is the number of center points (if any) added. The condition $\lambda_2 = 1$ then implies that $a_1^2 + a_2^2 = 2 + n_0/48$. Thus, given a_1^2,

$$0 < a_1^2 \leqq 1 + n_0/96,$$

we have a rotatable design of third order in four dimensions given by the $(96 + n_0)$ points of $S(a_1) + S(a_2)$, where a_1 is chosen from the range given above and $a_2 = [2 + n_0/48 - a_1^2]^{\frac{1}{2}}$.

Each of the four sets of 24 points given by $S(a_i(2)^{\frac{1}{2}}, a_i(2)^{\frac{1}{2}}, 0, 0)$ and $S(2a_i, 0, 0, 0) + S(a_i, a_i, a_i, a_i)$, $(i = 1, 2)$, is itself a rotatable arrangement of second order in four dimensions. These are, of course, the second order arrangements most frequently used by experimenters for four factors. Hence the completion of the 96 point design provides an excellent way of proceeding from a second order investigation when the second order polynomial is found to be inadequate. More generally, any one, two, or three of the sets may be performed (with center points if necessary to satisfy the condition $\lambda_4/\lambda_2^2 > k/(k + 2) = \frac{2}{3}$)

as the first part of a sequential third order design. Alternatively, the four sets of 24 points may be performed together as four separate blocks of a third order design. The complete design has parameters

$$\lambda_2 = 48(a_1^2 + a_2^2)/N = 1, \qquad \lambda_4 = 32(a_1^4 + a_2^4)/N, \qquad \lambda_6 = 16(a_1^6 + a_2^6)/N.$$

REFERENCES

[1] NORMAN R. DRAPER, "Third order rotatable designs in three dimensions," *Ann. Math. Stat.*, Vol. 31 (1960), pp. 865–874.

[2] D. A. GARDINER, A. H. E. GRANDAGE, AND R. J. HADER, "Third order rotatable designs for exploring response surfaces," *Ann. Math. Stat.*, Vol. 30 (1959), pp. 1082–1096.