NOTES

DISTRIBUTION OF THE LIKELIHOOD RATIO FOR TESTING MULTIVARIATE LINEAR HYPOTHESES[1]

By S. K. Katti[2]

Iowa State University

1. Introduction. Random orthogonal transformations having elements depending on certain random elements have been used by Wijsman [4] to derive the Wishart distribution and the important statistics such as Hotelling's T^2. The purpose of this paper is to use these transformations in a simple derivation of the result that the likelihood ratio for testing multivariate linear hypotheses is distributed as the product of q independent Beta variables (cf., Anderson [1], Section 8.5.2). Indirect derivations through the use of moments etc., are given in Wilks [5] and Bartlett [2].

2. Notation and results. Let X be a $q \times r$ matrix of $N(0, 1)$ variables and Y a $q \times s$ matrix ($s \geqq q$) of N (0, 1) variables, all variables being independent. Let $A_{q,r} = XX'$, $B_{q,s} = YY'$. In terms of the canonical reduction as given by Hsu [3], it can be shown that the likelihood criterion for testing a general linear hypothesis with r constraints ($r < q$) can be written in the form

$$\Lambda = \frac{|B_{q,s}|}{|A_{q,r} + B_{q,s}|}. \tag{1}$$

If $q = 1$, the problem is trivial. In the following, we shall assume $q > 1$. Denote by x_{ij}, y_{ij} the (i, j)th elements and by $x_{i\cdot}$, $y_{i\cdot}$ the ith rows of the matrices X and Y.

Let c_1 be the column vector $y'_{1\cdot}/(y_{1\cdot}y'_{1\cdot})^{\frac{1}{2}}$, so that $c'_1c_1 = 1$, and complete c_1 with $s - 1$ additional columns to an orthogonal matrix $\|c_1:\Omega_B\|$. Following Wijsman [4] we make a random orthogonal transformation from Y to Z,

$$Z = Y\|c_1:\Omega_B\|. \tag{2}$$

In the first row of Z all elements are 0 except z_{11}, which is equal to

$$z_{11} = (y_{1\cdot}y'_{1\cdot})^{\frac{1}{2}}. \tag{3}$$

If the first row and column of Z are deleted, there results a $(q - 1) \times (s - 1)$ matrix V, whose elements are $N(0, 1)$ variables, independent of each other and

Received June 24, 1959; revised October 9, 1960.

[1] This research was sponsored by the National Science Foundation under Grant NSF G-5248.

[2] Present address: Florida State University, Tallahassee, Florida.

of z_{11} [4]. Furthermore [4],

$$(4)\qquad |B_{q,s}| = |YY'| = |ZZ'| = z_{11}^2|VV'| = z_{11}|B_{q-1,s-1}|,$$

where we have set $B_{q-1,s-1} = VV'$.

Let Ω_A be an $r \times (r-1)$ matrix whose columns are mutually orthogonal, and orthogonal to $x_{1\cdot}'$. Define the following column vectors

$$c_2 = x_{1\cdot}' / (x_{1\cdot}x_{1\cdot}' + y_{1\cdot}y_{1\cdot}')^{\frac{1}{2}}, \qquad c_3 = y_{1\cdot}' / x_{1\cdot}x_{1\cdot}' + y_{1\cdot}y_{1\cdot}')^{\frac{1}{2}},$$

$$c_4 = c_2(c_3'c_3 / c_2'c_2)^{\frac{1}{2}}, \qquad c_5 = -c_3(c_2'c_2 / c_3'c_3)^{\frac{1}{2}},$$

and transform $\|X:Y\|$ to W with the following orthogonal transformation

$$(5)\qquad W = \|X : Y\| \begin{Vmatrix} c_2 : c_4 : \Omega_A : O \\ \text{-- -- -- --} \\ c_3 : c_5 : O : \Omega_B \end{Vmatrix}.$$

Since the vectors c_2 and c_3 are zero with probability zero, that such a random orthogonal transformation can be chosen measurably follows from the arguments of Wijsman [4]. The elements in the first row of W are 0 except w_{11}, which is

$$(6)\qquad w_{11} = (x_{1\cdot}x_{1\cdot}' + y_{1\cdot}y_{1\cdot}')^{\frac{1}{2}}.$$

It can be easily checked from (5) and (2) that the $(q-1) \times (r+s-1)$ matrix T, which results after deleting the first row and column of W, can be written as

$$(7)\qquad T = \|U:V\|,$$

where U is the $(q-1) \times r$ matrix

$$(8)\qquad U = \begin{Vmatrix} x_{2\cdot} & \vdots & y_{2\cdot} \\ \vdots & \vdots & \vdots \\ x_{q\cdot} & \vdots & y_{q\cdot} \end{Vmatrix} \begin{Vmatrix} c_4 & \Omega_A \\ & \\ c_5 & O \end{Vmatrix},$$

and V is as defined before. Moreover, the elements of U are $N(0, 1)$, independent of each other, of V, of w_{11} and of z_{11}. Setting $UU' = A_{q-1,r}$, we can write, analogously to (4),

$$(9)\qquad \begin{aligned} |A_{q,r} + B_{q,s}| &= |\|X:Y\|\|X:Y\|'| = |WW'| = w_{11}^2|TT'| \\ &= w_{11}^2|A|_{q-1,r} + B_{q-1,s-1}|. \end{aligned}$$

Substitution of (4) and (9) into (1) gives

$$(10)\qquad \Lambda = \frac{z_{11}^2}{w_{11}^2} \frac{|B_{q-1,s-1}|}{|A_{q-1,r}+ B_{q-1,s-1}|}$$

Using (3) and (6), the first factor, z_{11}^2/w_{11}^2, on the right-hand side in (10), which we will denote by $\beta_{r/2,s/2}$, is a β-variable with degrees of freedom $r/2$ and $s/2$.

Moreover, the second term is independent of the first. By repeated application of the above procedure, we obtain Λ as the product of q independent β-variables $\beta_{r/2,s}$, $\beta_{r/2,(s-1)/2}, \cdots, \beta_{r/2,(s-q+1)/2}$.

If $r = 1$, i.e., if x is a column vector, we have

$$\Lambda^{-1} = 1 + X'(YY')^{-1}X. \tag{11}$$

Since $X'(YY')^{-1}X$ is Hotelling's T^2 times a constant, equation (11) implies that the product of the q independent Beta variables $\beta_{1/2,s/2}$, $\beta_{1/2,(s-1)/2}$, $\cdots$, $\beta_{1/2,(s-q+1)/2}$ is distributed as the reciprocal of one plus a constant times an F variable.

If the null hypothesis is not true, let $E(x_{ij}) = \mu_{ij}$. If the matrix (μ_{ij}) is of rank 1, which really means $r = 1$ (since the multivariate hypothesis is assumed to be in canonical form, we transform X and Y to ξ and η respectively through the relations

$$\xi = MX \text{ and } \eta = MY,$$

where M is a $q \times q$ orthogonl matrix with $(\mu_{11}, \mu_{21}, \cdots, \mu_{q1})/(\sum_{i=1}^{q} \mu_{i1}^2)^{\frac{1}{2}}$ for the first row. Obviously $E(\xi_{11}) = (\sum_{i=1}^{q} \mu_{i1}^2)^{\frac{1}{2}}$, $E(\xi_{i1}) = 0$ for $i \neq 1$ and $E(\eta) = 0$. By treating ξ and η along the same lines as the matrices X and Y in the above discussion, we obtain equation (10) in which, now, $z_{11}^2 = \eta_{1\cdot}\eta_{1\cdot}'$,

$$w_{11}^2 = \xi_{11}^2 + \eta_{1\cdot}\eta_{1\cdot}',$$

and the components U and V of $A_{q-1,r}$ and $B_{q-1,s-1}$ are matrices of independent variables, distributed as $N(0, 1)$. Since ξ_{11} has a non zero mean, we refer to z_{11}^2/w_{11}^2 as a noncentral Beta variable and conclude that Λ is distributed as the product of one noncentral and $q - 1$ central independent Beta variables.

REFERENCES

[1] T. W. Anderson, *Introduction to Multivariate Statistical Analysis*, John Wiley and Sons, New York, 1957.

[2] M. S. Bartlett, "The vector representation of a sample," *Proc. Camb. Phil. Soc. Edinburgh*, Vol. 30 (1934), pp. 327–340.

[3] P. L. Hsu, "Canonical reduction of the general regression problem," *Ann. Eug.*, Vol. 11 (1941–2), pp. 42–46.

[4] Robert A. Wijsman, "Random orthogonal transformations and their use in some classical distribution problems in multivariate analysis," *Ann. Math. Stat.*, Vol. 28 (1957), pp. 415–422.

[5] S. S. Wilks, "Certain generalizations in the analysis of variance," *Biometrika*, Vol. 24 (1932), pp. 471–494.