NON-EQUIVALENT COMPARISONS OF EXPERIMENTS AND THEIR USE FOR EXPERIMENTS INVOLVING LOCATION PARAMETERS

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1. Introduction and summary. Consider experiments of the following type. Observation is made of a univariate random variable X whose absolutely continuous distribution function $F(x \mid \theta)$ and probability density function $p(x \mid \theta)$ are functions of a real unknown parameter θ. Different experiments of this type with random variables $X_1, X_2, \cdots$ will be denoted $\mathcal{E}_1, \mathcal{E}_2, \cdots$ In the following definitions, Θ represents a subset of θ-values.

(a) Following Blackwell [1], $\mathcal{E}_1$ is sufficient for $\mathcal{E}_2$ with respect to Θ or $\mathcal{E}_1 \succ \mathcal{E}_2(\Theta)$ when there exists a stochastic transformation of X_1 (given by a set of distribution functions $\{G(z \mid x_1) \mid -\infty < x_1 < \infty\}$) to a random variable Z such that, for each $\theta \varepsilon \Theta$, $Z(X_1)$ and X_2 have identical distributions.

(b) Following Lindley [3], $\mathcal{E}_1$ is not less Shannon informative than $\mathcal{E}_2$ with respect to Θ or $\mathcal{E}_1 S \geqq \mathcal{E}_2(\Theta)$ when $\mathcal{G}[\mathcal{E}_1, F(\theta)] \geqq \mathcal{G}[\mathcal{E}_2, F(\theta)]$ for all "prior" distribution functions $F(\theta)$ giving probability one to Θ, where $\mathcal{G}[\mathcal{E}_i, F(\theta)]$ is the mean Shannon information given by $\mathcal{E}_i$ about θ when θ has the prior distribution function $F(\theta)$.

(c) When the Fisher informations

$$I_i(\theta) = \int_{-\infty}^{\infty} p(x_i \mid \theta) \left[\frac{\partial}{\partial\theta} \log p(x_i \mid \theta)\right]^2 dx_i, \qquad i = 1, 2,$$

are definable for $\theta \varepsilon \Theta$, $\mathcal{E}_1$ will be said to be not less Fisher informative than $\mathcal{E}_2$ with respect to Θ, or $\mathcal{E}_1 F \geqq \mathcal{E}_2(\Theta)$, when $I_1(\theta) \geqq I_2(\theta)$ for $\theta \varepsilon \Theta$.

Lindley [3] has shown that $\mathcal{E}_1 \succ \mathcal{E}_2(\Theta) \Rightarrow \mathcal{E}_1 S \geqq \mathcal{E}_2(\Theta)$. In Theorem 1, we show that under certain conditions $\mathcal{E}_1 S \geqq \mathcal{E}_2(\Theta) \Rightarrow \mathcal{E}_1 F \geqq \mathcal{E}_2(\Theta)$. If this implication always held, comparison by $F \geqq$ would be more widely applicable than comparison by $S \geqq$ (and *a fortiori* by $\succ$). However the conditions of Theorem 1 suggest that cases exist where $\mathcal{E}_1 S \geqq \mathcal{E}_2(\Theta)$ but where $I_1(\theta)$ and $I_2(\theta)$ are not even defined for $\theta \varepsilon \Theta$.

When θ is a location parameter, $p(x \mid \theta) = f[x - \theta]$, say. For fixed $f[\cdot]$ consider the class of experiments $\{\mathcal{E}(c) \mid c > 0\}$, where $\mathcal{E}(c)$ is the experiment determined by the probability density function $cf[c(x - \theta)]$. The conditional distribution of $\mathcal{E}(c_1)$ is a contraction of that of $\mathcal{E}(c_2)$ when $c_1 > c_2$. (Example: $\mathcal{E}(c)$ consisting of c^2 observations from the normal distribution $N(\theta, 1)$ and x their mean). In the theorems of Sections 3, 4 and 5, conditions for $\mathcal{E}(c_1) \succ \mathcal{E}(c_2)$, $\mathcal{E}(c_1) S \geqq \mathcal{E}(c_2)$ or $\mathcal{E}(c_1) F \geqq \mathcal{E}(c_2)$ when $c_1 > c_2$ are given. Unless otherwise indicated, integrals will be taken over R^1.

Received July 14, 1960; revised October 7, 1960.

2. Theorem 1. *If $p(x_1 \mid \theta)$ and $p(x_2 \mid \theta)$ are twice-differentiable with respect to θ and well-behaved enough to justify double differentiation of expression (2.1) under the integral sign with respect to θ at $\theta = \theta^*$ for all $\theta^* \varepsilon \Theta$ and if every point of Θ is a limit point, then $\mathcal{E}_1 S \geqq \mathcal{E}_2(\Theta) \Rightarrow \mathcal{E}_1 F \geqq \mathcal{E}_2(\Theta)$.*

PROOF. Choose $\theta^* \varepsilon \Theta$. Since θ^* is a limit point of Θ, there exists a sequence $\{\theta_n\}$ in Θ such that $|\theta_n - \theta^*| \to 0$ as $n \to \infty$. Let $F^{(n)}$ be the prior distribution assigning probability $\frac{1}{2}$ to each of θ^* and θ_n. Then, for $i = 1, 2$,

$$\begin{aligned} \mathcal{J}[\mathcal{E}_i, F^{(n)}] &= \int\int p(x_i \mid \theta) \log [p(x_i \mid \theta)/p(x_i)]\, dx_i\, dF(\theta) \\ &= \tfrac{1}{2} \int \{p(x_i \mid \theta^*) \log p(x_i \mid \theta^*) + p(x_i \mid \theta) \log p(x_i \mid \theta) \\ &\quad - [p(x_i \mid \theta^*) + p(x_i \mid \theta)] \log [\tfrac{1}{2} p(x_i \mid \theta^*) + \tfrac{1}{2} p(x_i \mid \theta)]\}\, dx_i \end{aligned} \tag{2.1}$$

at $\theta = \theta_n$, where $p(x_i) = \int p(x_i \mid \theta)\, dF(\theta)$. Differentiating (2.1) twice with respect to θ under the integral sign, it is readily verified that

$$\lim_{n\to\infty} \{8\mathcal{J}[\mathcal{E}_i, F^{(n)}]/(\theta_n - \theta^*)^2\} = I_i(\theta^*), \qquad i = 1, 2.$$

But $\mathcal{J}[\mathcal{E}_1, F^{(n)}] \geqq \mathcal{J}[\mathcal{E}_2, F^{(n)}]$ for all n. Therefore $I_1(\theta^*) \geqq I_2(\theta^*)$ for $\theta^* \varepsilon \Theta$ and $\mathcal{E}_1 F \geqq \mathcal{E}_2(\Theta)$.

3. Theorems 2, 3, and 4. For this section, $\Theta = R^1$. For any $\Theta^* \subset R^1$, $\mathcal{E}_1 \succ \mathcal{E}_2(R^1) \Rightarrow \mathcal{E}_1 \succ \mathcal{E}_2(\Theta^*)$.

THEOREM 2. *If $f[\cdot]$ is bounded, $\phi(t) = \int \exp(itu) f[u]\, du$ and $c_1 > c_2 > 0$, a sufficient condition that $\mathcal{E}(c_1) \succ \mathcal{E}(c_2)(R^1)$ is that $\phi(t/c_2)/\phi(t/c_1)$ be a characteristic function.*

PROOF. There exists a distribution function $G^*(u)$ such that

$$\phi(t/c_2) = \phi(t/c_1) \int \exp(itu)\, dG^*(u)$$

or

$$\begin{aligned} \int \exp(itw) c_2 f[c_2 w]\, dw &= \int \exp(itv) c_1 f[c_1 v]\, dv. \int \exp(itu)\, dG^*(u) \\ &= \int \exp(itw) \{ \int c_1 f[c_1(w - u)]\, dG^*(u) \} dw \end{aligned}$$

with a change of variables. The final expression exists when $f[\cdot]$ is bounded, for $\int c_1 f[c_1(w - u)]\, dG^*(u)$ is uniformly convergent in $-\infty < w < \infty$ and $\int\int c_1 f[c_1(w - u)]\, dG^*(u) dw$ exists. Hence, by Fourier's uniqueness theorem,

$$c_2 f[c_2 w] = \int c_1 f[c_1(w - u)]\, dG^*(u),$$

which gives $F[c_2 w] = \int F[c_1(w - u)]\, dG^*(u) = \int G^*(w - v)\, d_v F[c_1 v]$, where $F[X] = \int_{-\infty}^{X} f[u]\, du$. Putting $w = z - \theta$ and $v = x_1 - \theta$, $F[c_2(z - \theta)] = \int G^*(z - x_1)\, d_{x_1} F[c_1(x_1 - \theta)]$. If X_1, X_2 are the random variables of $\mathcal{E}(c_1)$, $\mathcal{E}(c_2)$, the set of distribution functions $\{G^*(z - x_1) \mid -\infty < x_1 < \infty\}$ for Z therefore determines a stochastic transformation of X_1 such that Z and X_2 are identically distributed for each $\theta \varepsilon R^1$. Hence $\mathcal{E}(c_1) \succ \mathcal{E}(c_2)(R^1)$.

THEOREM 3. *If* (i) *$f[\cdot]$ is bounded* (ii) *the class of functions*

$$\{f[u - \psi] \mid -\infty < \psi < \infty\}$$

is closed with respect to bounded convolands (that is, if $\int H(u)f[u - \psi]\, du = 0$, $-\infty < \psi < \infty$, *and* $H(u)$ *bounded in* $-\infty < u < \infty$ *implies* $H(u) = 0$ a.e.) *and* (iii) $c_1 > c_2 > 0$, *a necessary condition that* $\mathcal{E}(c_1) \succ \mathcal{E}(c_2)(R^1)$ *is that* $\phi(t/c_2)/\phi(t/c_1)$ *be a characteristic function.*

PROOF. $\mathcal{E}(c_1) \succ \mathcal{E}(c_2)(R^1)$ implies (see (a), Section 1)

$$\int G(z \mid x_1)c_1 f[c_1(x_1 - \theta)]\, dx_1 = F[c_2(z - \theta)], \quad -\infty < z < \infty, -\infty < \theta < \infty. \tag{3.1}$$

In (3.1), put $c_1x_1 = u + c_1z$ and $\theta = \phi + z$; then

$$\int G(z \mid c_1^{-1}u + z)f[u - c_1\phi]\, du = F[-c_2\phi], \quad -\infty < z < \infty, -\infty < \phi < \infty.$$

Choosing any z_1 and z_2 and writing $H(u)$ for $G(z_1 \mid c_1^{-1}u + z_1) - G(z_2 \mid c_1^{-1}u + z_2)$, we have (i) $|H(u)| \leqq 1$, $-\infty < u < \infty$, and (ii) $\int H(u)f[u - c_1\phi]\, du = 0$, $-\infty < \phi < \infty$. Hence $H(u) = 0$ a.e. and therefore $G(z \mid c_1^{-1}u + z)$ is a.e. a function of u, $G^*(-c_1^{-1}u)$ say; or $G(z \mid x_1) = G^*(z - x_1)$ a.e. The function $G^*(\cdot)$ will be a distribution function on R^1. Substituting in (3.1),

$$\int G^*(z - x_1)c_1f[c_1(x_1 - \theta)]dx_1 = F[c_2(z - \theta)], \quad -\infty < z < \infty, -\infty < \theta < \infty,$$

or, reversing some steps in the last proof,

$$\int F[c_1(w - u)]\, dG^*(u) = F[c_2w], \qquad -\infty < w < \infty,$$

$$\int c_1f[c_1(w - u)]\, dG^*(u) = c_2f[c_2w], \qquad -\infty < w < \infty,$$

the differentiation with respect to w being justified by the uniform convergence of the latter integral in $-\infty < w < \infty$, a fact also allowing integration to give

$$\phi(t/c_2)/\phi(t/c_1) = \int \exp(itu)\, dG^*(u),$$

a characteristic function.

THEOREM 4. *If conditions* (i) *and* (ii) *of Theorem 3 hold and if additionally all cumulants of* $f[\cdot]$ *exist, a necessary condition that* $\mathcal{E}(c_1) \succ \mathcal{E}(c_2)(R^1)$ *whenever* $c_1 > c_2$ *is that either* (i) $f[\cdot]$ *is a normal probability density function or* (ii) *the even-order cumulants of* $f[\cdot]$ *are positive.*

PROOF. Take $c_1 = c > 1$ and $c_2 = 1$. If k_r are the cumulants of $f[\cdot]$,

$$\phi(t)/\phi(t/c) = \exp\,[k_1(1 - c^{-1})it + k_2(1 - c^{-2})(it)^2/2! + \cdots].$$

Write $k_r(c) = (1 - c^{-r})k_r$. Then, by Theorem 3, $k_r(c)$ are the cumulants of some distribution. Write $\mu_r'(c)$ for the corresponding moments. Then it is necessary that the doubly-infinite matrix

$$(3.3) \qquad \begin{pmatrix} 1 & \mu_1'(c) & \mu_2'(c) & \cdot & \cdot \\ \mu_1'(c) & \mu_2'(c) & \mu_3'(c) & \cdot & \cdot \\ \mu_2'(c) & \mu_3'(c) & \mu_4'(c) & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \end{pmatrix}$$

be positive-semi-definite (see [4]). Now $\mu_r'(c) - k_r(c)$ is a polynomial in $k_1(c), \cdots, k_{r-1}(c)$ with terms of degree greater than one and when $c \cong 1$, $k_r(c) \cong r(c-1)k_r$. Therefore $\mu_r'(c) \cong r(c-1)k_r$ when $c \cong 1$. Substituting in all but the first row and column of (3.3), it is therefore necessary that the doubly-infinite matrix

$$(3.4) \qquad \begin{pmatrix} 2k_2 & 3k_3 & 4k_4 & \cdot & \cdot \\ 3k_3 & 4k_4 & 5k_5 & \cdot & \cdot \\ 4k_4 & 5k_5 & 6k_6 & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \end{pmatrix}$$

be positive-semi-definite. This firstly implies $k_{2r} \geqq 0$, $r = 1, 2, \cdots$. The case $k_2 = 0$ corresponds to the degenerate limiting case when, for some α, $X(c) = \theta + \alpha$ with zero variance for all c. With $k_2 > 0$, either (i) $k_4 = 0$ or (ii) $k_4 > 0$. (i) If $k_4 = 0$, it is readily verified that for (3.4) to be positive-semi-definite, $k_r = 0$ for $r > 2$; that is, $f[\cdot]$ is a normal probability density function.

(ii) If $k_4 > 0$ and $k_5 \neq 0$ then $4k_4 \cdot 6k_6 - (5k_5)^2 \geqq 0$ implies $k_6 > 0$. If $k_4 > 0$ and $k_5 = 0$ then

$$\begin{vmatrix} 2k_2 & 3k_3 & 4k_4 \\ 3k_3 & 4k_4 & 5k_5 \\ 4k_4 & 5k_5 & 6k_6 \end{vmatrix} \geqq 0$$

implies $k_6 > 0$. Thus $k_4 > 0$ implies $k_6 > 0$. Similarly $k_6 > 0$ implies $k_8 > 0$ and so on. Therefore $k_{2r} > 0$ for $r \geqq 1$ and the theorem is established.

The following comments on Theorem 4 seem appropriate.

(a) Condition (i) is sufficient as well as necessary. For $\phi(t) = \exp(\mu it - \frac{1}{2}\sigma^2 t^2)$ implies

$$\phi(t/c_2)/\phi(t/c_1) = \exp\left[\mu(c_2^{-1} - c_1^{-1})it - \tfrac{1}{2}\sigma^2(c_2^{-2} - c_1^{-2})t^2\right]$$

which is the characteristic function of another normal distribution. Hence $\mathcal{E}(c_1) \succ \mathcal{E}(c_2)$ for $c_1 > c_2$.

(b) It is possible that condition (ii) is inconsistent with $\mathcal{E}(c_1) \succ \mathcal{E}(c_2)(R^1)$ whenever $c_1 > c_2$; in which event, yet another characterisation of the normal distribution would be provided.

(c) The theorem is not necessarily true unless all cumulants of $f[\cdot]$ exist. For the Cauchy distribution given by $f[u] = 1/[\pi(1 + u^2)]$ has no cumulants but $\phi(t/c_2)/\phi(t/c_1) = \exp[-(c_2^{-1} - c_1^{-1})\,|t|\,]$ which is the characteristic function of another Cauchy distribution.

(d) As an example of the use of the theorem, if $f[u] = 1$ for $0 < u < 1$ and $f[u] = 0$ elsewhere then all cumulants exist. However such a distribution has $k_4 < 0$. Therefore it is not possible that $\mathcal{E}(c_1) \succ \mathcal{E}(c_2)(R^1)$ whenever $c_1 > c_2$.

(e) A possible alternative approach, not requiring the condition on the cumulants, is to relate the problem to that of the determination of the indefinitely divisible laws (Lévy, [2]). On p. 159 of [2], the basic equation of such laws is given as

$$F^*(x, t_1) = \int F^*(x - y, t_0)\, d_y F(y, t_0, t_1), \qquad t_0 < t_1, \tag{3.5}$$

where $F^*(x, t)$ is the distribution function of a stochastic random variable $X(t)$ at the time t and $F(y, t_0, t_1)$ is the distribution function of the increment $X(t_1) - X(t_0)$. In Theorem 3, we have established that $\mathcal{E}(c_1) \succ \mathcal{E}(c_2)(R^1)$ whenever $c_1 > c_2$ implies the existence of a distribution function on R^1, $G^*(u)$, more accurately written $G^*(u, c_1, c_2)$, such that

$$F[c_2 w] = \int F[c_1(w - u)]\, d_u G^*(u, c_1, c_2). \tag{3.6}$$

(Only condition (ii) of Theorem 3 is needed for this.) That (3.6) is a special case of (3.5) can be seen by writing $c_1 = t_0^{-1}$, $c_2 = t_1^{-1}$, $w = x$, $u = y$, and observing that the $F^*(x, t)$ of (3.5) has been specialised to $F[x/t]$. The "expansion" factor c^{-1} therefore takes the place of time, t. Lévy shows that if $X(0) = 0$, the distribution functions $F^*(x, t)$ are continuous in t and

$$\psi(z, t) = \log\left[\int \exp(izu)\, d_u F^*(u, t)\right]$$

then the most general solution of (3.5) is given by

$$\psi(z, t) = f(t)iz - \frac{1}{2}g(t)z^2 + \int \left[\exp(izu) - 1 - \frac{izu}{1 + u^2}\right] d_u n(t, u) \tag{3.7}$$

with certain conditions on $f(t)$, $g(t)$ and $n(t, u)$. In our specialization of this, $F^*(x, t) = F[x/t]$ which, being absolutely continuous, is therefore continuous with respect to t. Also as $t \to 0$, $F[x/t] \to H(x)$ where $H(x) = 1$, $x > 0$, and $H(x) = 0$, $x < 0$, so that, formally, $X(0) = 0$. Also $\psi(z, t) = \log \phi(zt)$ so that the restrictions on $f(t)$, $g(t)$ and $n(t, u)$ must be increased to make the right-hand-side of (3.7) a function of zt. The solution is, however, left very general. For example, putting $f(t) = t$, $g(t) = t^2$ and $n(t, u) = h(u/t)$ where $h(v)$ is a bounded non-decreasing function of v which is antisymmetrical about $v = 0$ and obeys the condition $h'(v) + v\,h''(v) < 0$, the necessary conditions are satisfied.

4. Theorem 5. *If $f[\cdot]$ is bounded and differentiable and Θ is any finite interval of R^1, a sufficient condition that $\mathcal{E}(c_1)S \geqq \mathcal{E}(c_2)(\Theta)$ whenever $c_1 > c_2$ is that $f[\cdot]$ be unimodal.*

Proof. (The extension of the theorem to the case $\Theta = R^1$ is direct but tedious and will not be given here. The conditions of uniform convergence necessary to justify local differentiation of certain integrals will be assumed.) For a prior distribution function for θ, $F(\theta)$,

$$\mathcal{I}[\mathcal{E}(c), F(\theta)]$$
$$= \int\int cf[c(x-\theta)] \log \{cf[c(x-\theta)]/ \int cf[c(x-\phi)]dF(\phi)\}\, dx\, dF(\theta)$$
$$= \int f[u] \log f[u]\, du - \int g(v, c) \log g(v, c)\, dv,$$

where $g(v, c) = \int f[v - c\theta]\, dF(\theta)$. Therefore

$$\frac{d}{dc} \mathcal{I}[\mathcal{E}(c), F(\theta)] = -\int \frac{\partial}{\partial c} [g(v, c) \log g(v, c)]\, dv$$
$$= -\int \frac{\partial}{\partial c} g(v, c) \cdot \log g(v, c)\, dv - \int \frac{\partial}{\partial c} g(v, c)\, dv.$$

But $(\partial/\partial c)g(v, c) = -\int \theta f'[v - c\theta]dF(\theta) = -(\partial/\partial v) \int \theta f[v - c\theta]\, dF(\theta) = -(\partial/\partial v)h(v, c)$, say, while $\int (\partial/\partial c)g(v, c)\, dv = (\partial/\partial c) \int g(v, c)\, dv = 0$ since $\int g(v, c)\, dv = 1$. Therefore

$$\frac{d}{dc} \mathcal{I}[\mathcal{E}(c), F(\theta)] = \int \frac{\partial}{\partial v} h(v, c) \cdot \log g(v, c)\, dv$$
$$= [h(v, c) \log g(v, c)]_{-\infty}^{\infty} - \int h(v, c) \frac{\partial}{\partial v} \log g(v, c)\, dv$$

by parts. By the conditions of the theorem, $f[\cdot]$ and $\int \theta^2 dF(\theta)$ are bounded by M and K respectively say. Therefore, using Schwarz's inequality,

$$h(v, c)\, |\log g(v, c)| = \int \theta f[v - c\theta]\, dF(\theta) \cdot |\log g(v, c)|$$
$$\leqq [\int \theta^2 dF(\theta)]^{\frac{1}{2}} [\int f[v - c\theta]^2\, dF(\theta)]^{\frac{1}{2}} |\log g(v, c)|$$
$$\leqq 2K^{\frac{1}{2}} M^{\frac{1}{2}}\, |g(v, c)^{\frac{1}{2}} \log g(v, c)^{\frac{1}{2}}|\,.$$

But $g(v, c) \to 0$ as $v \to \pm\infty$; therefore $h(v, c)\, |\log g(v, c)|$ does likewise. Hence

$$(4.1) \qquad \frac{d}{dc} \mathcal{I}[\mathcal{E}(c), F(\theta)] = -\int h(v, c) \frac{\partial}{\partial v} \log g(v, c)\, dv.$$

Consider any point v_1 at which $(\partial/\partial v) \log g(v, c) > 0$. Let v_2 be the least v with $v > v_1$ and $g(v, c) = g(v_1, c)$. Then at v_2, $g(v, c)$ will be non-increasing. Since $f[\cdot]$ is unimodal, there exists θ^* such that

$$f[v_1 - c\theta] - f[v_2 - c\theta] \geqq 0, \qquad \theta < \theta^*,$$
$$f[v_1 - c\theta] - f[v_2 - c\theta] \leqq 0, \qquad \theta > \theta^*,$$

or $(\theta - \theta^*)(f[v_2 - c\theta] - f[v_1 - c\theta]) \geqq 0$. Therefore

$$h(v_2, c) - h(v_1, c) = \int \theta(f[v_2 - c\theta] - f[v_1 - c\theta])\, dF(\theta)$$
$$= \int (\theta - \theta^*)(f[v_2 - c\theta] - f[v_1 - c\theta])\, dF(\theta),$$

using $\int \theta^*(f[v_2 - c\theta] - f[v_1 - c\theta])\, dF(\theta) = \theta^*[g(v_2, c) - g(v_1, c)] = 0$. Hence $h(v_2, c) \geqq h(v_1, c)$. Now R^1 for v can be divided by division points $d_1 < d_2 < \cdots < d_{2p+1}$ where $d_1 = -\infty$, $d_{2p+1} = \infty$ and possibly $p = \infty$, such that each interval (d_i, d_{i+1}) is a member of a pair of intervals in each of which $g(v, c)$ varies monotonically between the same two values, increasing in the lower interval and decreasing in the upper. Then

$$\int h(v, c) \frac{\partial}{\partial v} \log g(v, c)\, dv = \sum_{k=1}^{p} \int_k h(v, c) \frac{\partial}{\partial v} \log g(v, c)\, dv,$$

where $\int_k$ denotes the integral over the kth pair of intervals. Since $\log g(v, c)$ is non-increasing in the upper interval,

$$\begin{aligned} \int_k h(v, c) \frac{\partial}{\partial v} \log g(v, c)\, dv &= \int_k h(v,c)\, d_v \log g(v, c) \\ &= \int_{d_{i(k)}}^{d_{i(k)+1}} [h(v_1, c) - h(v_2, c)]\, d_v \log g(v_1, c), \end{aligned} \tag{4.2}$$

where v_2 is related to v_1 as explained and $(d_{i(k)}, d_{i(k)+1})$ is the lower interval of the kth pair. But in $(d_{i(k)}, d_{i(k)+1})$, $\log g(v, c)$ is non-decreasing and $h(v_1, c) \leqq h(v_2, c)$. Therefore (4.2) is non-positive and (4.1) gives

$$(d/dc)\mathcal{I}[\mathcal{E}(c), F(\theta)] \geqq 0$$

for all c. Therefore $\mathcal{I}[\mathcal{E}(c_1), F(\theta)] \geqq \mathcal{I}[\mathcal{E}(c_2), F(\theta)]$ whenever $c_1 > c_2$ and the theorem is proved.

5. Theorem 6. *When the Fisher informations are definable,* $\mathcal{E}(c_1)F \geqq \mathcal{E}(c_2)(R^1)$ *whenever* $c_1 > c_2$.

Proof. The Fisher information for θ and $\mathcal{E}(c)$ is

$$\begin{aligned} I(\theta, c) &= \int \text{cf}\,[c(x - \theta)] \left\{\frac{\partial}{\partial \theta} \log \text{cf}\,[c(x - \theta)]\right\}^2 dx \\ &= c^2 \int f[u] \left\{\frac{d}{du} \log f[u]\right\}^2 du \end{aligned}$$

which increases with c.

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