ON A THEOREM OF RÉNYI CONCERNING MIXING SEQUENCES OF SETS

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I. Introduction. Let Ω be a set and $\mathcal{A}$ a σ-algebra of subsets of Ω. Let P be a probability measure defined on $\mathcal{A}$, i.e., P is a non-negative completely additive set function defined on $\mathcal{A}$ with $P(\Omega) = 1$. Let α be a number with $0 \leqq \alpha \leqq 1$ and let $\{A_n, n \geqq 1\}$ be a sequence of sets. (We shall assume from now on that every set under discussion is an element of $\mathcal{A}$.) We shall say that the sequence $\{A_n\}$ is strongly mixing with density α if for every set B we have

$$\lim_n P(A_n \cap B) = \alpha P(B).$$

Concerning such sequences, Rényi [1] has proved a result which we state here as

Theorem 1 (Rényi). *Let $\{A_n, n \geqq 1\}$ be a strongly mixing sequence of density α and let Q be a probability measure defined on $\mathcal{A}$ such that Q is absolutely continuous with respect to P. Then* $\lim_n Q(A_n) = \alpha$.

In Section 2 we prove some preliminary results and then show that the condition of absolute continuity of Q with respect to P may be replaced by a weaker condition. In Section 3 we apply this result to obtain limit distributions for normed sums of certain sequences of dependent random variables.

II. Generalization of Rényi's Theorem. Let P and Q be probability measures on the measurable space $(\Omega, \mathcal{A})$. In the following $\mathcal{B}_i$, $i = \infty, 1, 2, 3, \cdots$, is a σ-subalgebra of $\mathcal{A}$, and P_i and Q_i are the restrictions of P and Q to $\mathcal{B}_i$. It is well known from the Lebesgue decomposition theorem that there is a singular set $B_i \varepsilon \mathcal{B}_i$ of Q_i relative to P_i with $P_i(B_i) = 0$ and such that for any $A \varepsilon \mathcal{B}_i$, $P_i(A - B_i) = 0$ implies that $Q_i(A - B_i) = 0$; i.e., relative to P_i, Q_i is singular on B_i and absolutely continuous on the complement B_i^c of B_i.

Lemma 1. *If* $\mathcal{B}_1 \supset \mathcal{B}_2$, *then* $(P + Q)(B_2 - B_1) = 0$.

Proof. Since $P(B_2) = 0$, then $P(B_2 - B_1) = 0$. Now $B_2 \varepsilon \mathcal{B}_1$, hence

$$Q(B_2 - B_1) = Q_1(B_2 - B_1) = 0.$$

Lemma 2. *Let* $\mathcal{B}_1 \supset \mathcal{B}_2 \supset \cdots \supset \mathcal{B}_\infty = \bigcap_n \mathcal{B}_n$. *$Q_\infty$ is absolutely continuous with respect to P_∞ if and only if* $\lim_n Q(B_n) = 0$.

Proof. It follows from Lemma 1 that $Q(B_\infty) \leqq Q(B_n)$ for every n. Thus if $\lim_n Q(B_n) = 0$, then $Q(B_\infty) = 0$ and Q_∞ is absolutely continuous with respect to P_∞. Conversely we have $Q(\lim_n \sup B_n) = 0$ since $P(B_n) = 0$ for every n and $\lim_n \sup B_n \varepsilon \mathcal{B}_\infty$. Consequently $\lim_n Q(B_n) = 0$.

We can now generalize Rényi's theorem to obtain

Theorem 2. *Let $\{A_n, n \geqq 1\}$ be a strongly mixing sequence of density α with*

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respect to P. For each positive integer n let $\mathcal{B}_n$ be the minimal σ-algebra containing the sets $A_n, A_{n+1}, \cdots$, and let $\mathcal{B}_\infty$ be $\bigcap_n \mathcal{B}_n$. If Q is a probability measure on $\mathcal{A}$ such that Q_∞ is absolutely continuous with respect to P_∞, then $\lim_n Q(A_n) = \alpha$.

PROOF. For each positive integer n let B_n be the singular set in $\mathcal{B}_n$ of Q_n relative to P_n, and choose n so large that $Q(B_n) < \epsilon$, where ϵ is an arbitrary positive number. For every positive integer m we have

$$Q(A_m) = Q(A_m \cap B_n^c) + Q(A_m \cap B_n) = Q(B_n^c)Q'(A_m) + Q(A_m \cap B_n),$$

where Q' is the probability measure defined by $Q'(A) = Q(A \cap B_n^c)/Q(B_n^c)$. Clearly Q' is absolutely continuous with respect to P when both are confined to $\mathcal{B}_n$, and it follows from Rényi's theorem that $\lim_m Q'(A_m) = \alpha$. The theorem follows.

By strengthening the hypothesis, we may obtain a considerably stronger conclusion for arbitrary decreasing sequences of σ-algebras.

THEOREM 3. *Let $\mathcal{B}_1 \supset \mathcal{B}_2 \supset \cdots \mathcal{B}_\infty = \bigcap_n \mathcal{B}_n$ be an arbitrary decreasing sequence of σ-subalgebras of $\mathcal{A}$, and Q be a probability measure on $\mathcal{A}$. Then $Q_\infty = P_\infty$ if and only if* $\lim_n (Q_n - P_n) = 0$ *uniformly over* $\mathcal{B}_n$.

PROOF. If $\lim_n [Q_n - P_n] = 0$ uniformly over $\mathcal{B}_n$ then clearly $Q_\infty = P_\infty$. Conversely assume that this is the case. Let $\mu = Q - P$ and for each positive integer n let C_n be the Hahn set for μ in $\mathcal{B}_n$, i.e., $\mu(C_n) = \sup_{C \epsilon \beta_n} \mu(C)$. Now if $B \varepsilon \mathcal{B}_n$ it can easily be seen that $\mu(B) \leqq \mu(C_n \cup B)$. Suppose now there exists $\epsilon > 0$ and an infinite sequence $\{k_n\}$ of integers such that $\mu(C_{k_n}) \geqq \epsilon$. From the remark above it follows that $\mu(C_{k_1} \cup C_{k_2}) \geqq \mu(C_{k_2}) \geqq \epsilon$. Similarly

$$\mu(C_{k_1} \cup C_{k_2} \cup C_{k_3}) \geqq \epsilon,$$

etc. Thus $\mu(\bigcup_{j=1}^\infty C_{k_j}) \geqq \epsilon$ and by the same argument $\mu(\bigcup_{j=n}^\infty C_{k_j}) \geqq \epsilon$ for every n. Hence $\mu(\lim_n \sup C_{k_n}) \geqq \epsilon$. But $\lim_n \sup C_{k_n} \varepsilon \mathcal{B}_\infty$ and by hypothesis μ vanishes on $\mathcal{B}_\infty$, which is a contradiction. The same argument applies to the set function $P - Q$, and the theorem is proved. For the application we have in mind we shall need a result which is an immediate consequence of Theorem 3.

COROLLARY. *Let $\{\mathcal{B}_n, n \geqq 1\}$ be a sequence of σ-algebras with $\mathcal{A} \supset \mathcal{B}_1 \supset \cdots$, and let $\mathcal{B}_\infty = \bigcap_n \mathcal{B}_n$. Let Q be a probability measure on $\mathcal{A}$ and let $\{A_n, n \geqq 1\}$ be a sequence of sets. Suppose for each positive integer k there exists a sequence of sets $\{A_{n,k}, n \geqq 1\}$ with $A_{n,k} \varepsilon \mathcal{B}_k$ for n sufficiently large such that*

$$\lim_n [P(A_{n,k}) - P(A_n)] = \lim_n [Q(A_{n,k}) - Q(A_n)] = 0.$$

Then if $Q_\infty = P_\infty$ we have $\lim_n [P(A_n) - Q(A_n)] = 0$.

III. Application. Let $\{X_n, n \geqq 1\}$ be a sequence of real random variables and let P be the probability measure defined on the Borel sets of infinite-dimensional Euclidean space induced by the finite-dimensional distributions of the process $\{X_n\}$. For each positive integer n let $\mathcal{B}_n$ be the smallest σ-algebra of Borel sets with respect to which the random variables $X_n, X_{n+1}, \cdots$, are measurable. The sequence $\{\mathcal{B}_n\}$ is then decreasing and we define $\mathcal{B}_\infty = \bigcap_n \mathcal{B}_n$. Let $\{a_n, n \geqq 1\}$

be a sequence of real numbers and let $\{b_n, n \geqq 1\}$ be a sequence of positive numbers with $\lim_n b_n = \infty$. For each integer n define the set $A_n(x)$ by

$$A_n(x) = \{(S_n/b_n) - a_n \leqq x\}$$

where x is an arbitrary real number and $S_n = \sum_{i=1}^n X_i$.

Now suppose Q is the probability measure induced by the finite-dimensional distributions defined by

$$Q(X_{i_1} \leqq a_1, \cdots, X_{i_k} \leqq a_k) = \prod_{j=1}^{k} P(X_{i_j} \leqq a_j).$$

Assume now that there exists a probability distribution $F(x)$ such that

$$\lim_n Q[A_n(x)] = F(x)$$

for every x which is a continuity point for $F(x)$. We shall be interested in conditions on P such that $\lim_n P[A_n(x)] = F(x)$ at continuity points of $F(x)$. As we shall show consequently this will in fact follow from the condition $P_\infty = Q_\infty$. Thus we first prove

THEOREM 4. *Suppose for every $\epsilon > 0$ there exists a positive integer n_o depending only on ϵ, and suppose that for every choice of nonnegative integers $i_1, \cdots, i_k$ with $n_o \leqq i_1 < \cdots < i_k$ there exists a k-dimensional probability measure R which may depend on ϵ, n_o, and k, such that for every k-dimensional rectangle*

$$\{a_1 < x_1 \leqq b_1, \cdots, a_k < x_k \leqq b_k\}$$

we have

$$|P(a_1 < X_{i_1} \leqq b_1, \cdots, a_k < X_{i_k} \leqq b_k) - \prod_{j=1}^{k} P(a_j < X_{i_j} \leqq b_j)|$$
$$< \epsilon R(a_1 < x_1 \leqq b_1, \cdots, a_k < x_k \leqq b_k).$$

Then $P_\infty = Q_\infty$.

PROOF. Let $\epsilon > 0$ and choose n_o accordingly. Let $\mu = P - Q$. Then if S is a finite-dimensional rectangle in $\mathcal{B}_{n_o}$ it follows from the hypothesis that there exists a probability measure R such that $|\mu(S)| < \epsilon R(S) \leqq \epsilon$. Now let

$$\{S_m, m \geqq 1\}$$

be a sequence of disjoint rectangles in $\mathcal{B}_{n_o}$ of uniformly bounded dimension and let $S = \bigcup_m S_m$. Then we may choose a probability measure R for which

$$|\mu(S_m)| < \epsilon R(S_m)$$

simultaneously for each m, and it follows from the complete additivity of μ and R that $|\mu(S)| \leqq \epsilon$. Now if A is a finite-dimensional cylinder set in $\mathcal{B}_{n_o}$ and if δ is a positive number there is a set S which is the union of a denumerable number of disjoint rectangles of uniformly bounded dimension such that

$$|\mu(A - S)| + |\mu(S - A)| < \delta.$$

Consequently $|\mu(A)| \leqq \epsilon + \delta$. If B is an arbitrary set in $\mathcal{B}_{n_0}$ we may approximate it arbitrarily closely by finite-dimensional cylinder sets and consequently $\lim_n \sup_{B \in \mathcal{B}_n} |\mu(B)| = 0$. The theorem follows from Theorem 3.

Now let $\{X_n, n \geqq 1\}$ be a stochastic process satisfying the conditions of Theorem 4. Define the sets $A_{n,k}(x)$ for $k = 2, 3, \cdots$, and $n \geqq k$ by

$$A_{n,k}(x) = \{[(S_n - S_{k-1})/b_n] - a_n \leqq x\}.$$

Then if x is a continuity point of $F(x)$ it is easily verified that

$$\lim_n [Q(A_n(x)) - Q(A_{n,k}(x))] = 0$$

for every k, and obviously $A_{n,k}(x) \in \mathcal{B}_k$. It follows from Theorem 3 and Theorem 4 that $\lim_k [P(A_{n,k}(x)) - Q(A_{n,k}(x))] = 0$ uniformly in $n \geqq k$. From this it is again easy to verify that $\lim_n [P(A_n(x)) - P(A_{n,k}(x))] = 0$ and we obtain $\lim_n P(A_n(x)) = F(x)$.

We summarize in

Theorem 5. *Let $\{X_n, n \geqq 1\}$ be a stochastic process satisfying the conditions of Theorem 4. Let $F(x)$ be a distribution function and suppose*

$$\lim_n Q((S_n/b_n) - a_n \leqq x) = F(x)$$

at every continuity point of $F(x)$. Then $\lim_n P((S_n/b_n) - a_n \leqq x) = F(x)$ at such continuity points.

Révész, [2], arrived at the conclusion of Theorem 5, using conditions somewhat stronger than those imposed by Theorem 4. However, his derivation is incorrect, since he concludes that under his conditions P is absolutely continuous with respect to Q. The following simple example shows that this is in fact not the case. Let Q be the probability measure corresponding to the stochastic process $\{X_n, n \geqq 1\}$ where the X_i are independent identically distributed random variables with mean zero, variance one, and continuous distributions. Let P be the probability measure corresponding to the process $\{Y_n, n \geqq 1\}$ where

$$Y_1 = Y_2 = X_1, \qquad Y_n = X_n \quad \text{for} \quad n > 2.$$

Then P is not absolutely continuous with respect to Q since

$$Q(x_1 = x_2) = 0 = 1 - P(x_1 = x_2).$$

However, it is easily verified, that the conditions of Révész's theorem apply to the process $\{Y_n\}$. Actually his conditions imply that $P_\infty = Q_\infty$ and consequently his theorem remains valid.

REFERENCES

[1] A. Rényi, "On mixing sequences of sets," *Acta Math. Acad. Sci. Hung.*, Vol. 9 (1958), pp. 215–228.

[2] P. Révész, "A limit distribution theorem for sums of dependent random variables," *Acta Math. Acad. Sci. Hung.*, Vol. 10 (1959), pp. 125-131.