FIRST EMPTINESS OF TWO DAMS IN PARALLEL[1]

By J. Gani

Australian National University

Summary. This paper considers the probabilities of first emptiness of two dams in parallel, both subject to a steady release at constant unit rate, and fed by a discrete additive input process such that unit inputs are always directed to the dam with lesser content. The problem is equivalent to that of the single dam fed by two ordered inputs, and a recurrence relation for the probabilities of first emptiness in this process is obtained. Equations for the generating functions of the probabilities are derived, and a formal solution to these is given.

A more convenient method of evaluating probabilities of first emptiness is found by reducing the process to an associated occupancy problem; it is shown how the probabilities of first emptiness for Poisson inputs are then obtained by a rapid computational procedure.

The paper concludes with a general formulation of the problem when the times of arrival for two ordered non-negative inputs of random size form a Poisson process.

1. Introduction. Probability distributions of times of first emptiness in a single dam (or times when the server is free in an equivalent queue) have been considered in a variety of cases by Takács [1], Kendall [2], Gani [3], and Gani and Prabhu [4] among others. Recently, Haight [5] has studied the stationary probability distribution p_{xy} of the number of customers x, y, waiting in two queues in parallel, such that new arrivals join the shorter queue, or a particular queue if the queues are equally long.

The problem of first emptiness in the present paper is based on a model related to Haight's when the dam inputs (or equivalent queue service times) are of constant size. Our concern, however, is with dam contents (or equivalent queue waiting times) rather than with numbers of customers; some time-dependent results are obtained which do not arise directly from Haight's considerations.

Let D_1, D_2 be two dams at initial levels z_1, z_2 respectively ($z_2 > z_1 > 0$), whose contents $Z_i(t)$ ($i = 1, 2$) at times $0 \leqq t < \infty$ are each subject to a steady release at constant unit rate until emptiness occurs, when the release ceases. There is a discrete non-negative input $X(t) = 0, 1, 2, \cdots$, during the interval of time $(0, t)$, unit inputs arriving one at a time and being fed into D_1 or D_2 according to a rule specified below. The process $X(t)$ is additive, with a probability distribution

$$f(j, \tau) = \Pr\{X(\tau) = j\} \qquad (j = 0, 1, 2, \cdots) \tag{1.1}$$

Received November 12, 1959; revised October 6, 1960.

[1] Research supported by the Office of Naval Research at Columbia University.

such that its probability generating function (p.g.f.) is of the form

$$\psi(\theta, \tau) = \sum_{j=0}^{\infty} \theta^j f(j, \tau) = \{\psi(\theta, 1)\}^\tau, \tag{1.2}$$

where $0 \leqq \theta \leqq 1$ for simplicity. When the time parameter t ranges continuously over $[0, \infty)$, the only distribution $f(j, \tau)$ which corresponds to a non-negative integer valued stochastic process $X(t)$ is the Poisson; however, for t restricted to the integers $0, 1, 2, \cdots$ other distributions $f(j, \tau)$ exist. All formulae in Sections 1, 2 apply to both the cases of discrete and continuous times, and it is for this reason that the general notation $f(j, \tau)$ is used in what follows.

The input rule is the following: $X(t)$ is first fed into D_1, which has the lesser initial content z_1, until the time ($t = t_1$ say) when $Z_1(t) \geqq Z_2(t)$; the next input is then diverted into D_2, and thereafter unit inputs are fed alternately into D_1 and D_2.

We are concerned with the time T of first emptiness of D_1 and D_2, $(z_1 \leqq T < \infty)$, at which $\text{Min}\{Z_1(T), Z_2(T)\} = 0$ for the first time. If for simplicity, $z_2 - z_1$ is assumed non-integral[2], the content of D_i ($i = 1, 2$) until time T may be written as

$$Z_i(t) = z_i + X_i(t) - t \qquad (0 \leqq t \leqq T; i = 1, 2), \tag{1.3}$$

where $X_1(t) + X_2(t) = X(t)$, and these inputs are given in $0 \leqq t \leqq T$ for $i = 1, 2$, by

$$X_i(t) = \begin{cases} \delta_{1i} X(t) & \text{for } X(t) \leqq [z_2 - z_1] + 1, \\ \delta_{1i}([z_2 - z_1] + 1) + [\tfrac{1}{2}\{X(t) - [z_2 - z_1] - \delta_{1i}\}] & \text{otherwise,} \end{cases} \tag{1.4}$$

with $\delta_{1i} = 1$ if $i = 1$, or 0 if $i = 2$, and $[y]$ indicates the integral part of y.

The processes (1.3) are illustrated in Figure 1; for all values of the time

$$t_0 < t \leqq T$$

(where t_0 is the first instant at which $X(t) = [z_2 - z_1]$), the dam contents differ by less than one unit, the differences being alternately

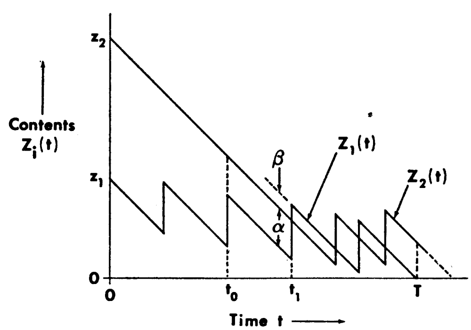

FIG. 1

$$\alpha = \{Z_2(t_0 + 0) - Z_1(t_0 + 0)\} < 1$$

[2] If $z_2 - z_1$ is integral, the only difference is that when both dams reach the same level a rule must be specified directing the next input into one particular dam.

and $\beta = 1 - \alpha < 1$. The time of first emptiness

$$(1.5)\quad \begin{aligned} T &= z_1, z_1 + 1, \cdots, z_1 + [z_2 - z_1]; \\ &\qquad z_1 + [z_2 - z_1] + [\tfrac{1}{2}(n + 1)]\alpha + [\tfrac{1}{2}n]\beta \quad (n = 1, 2, \cdots), \end{aligned}$$

is a point at which the minimal path Min $\{Z_1(t), Z_2(t)\}$ may touch the axis $z = 0$.

Probabilities of first emptiness at the times $T = z_1, z_1 + 1, \cdots, z_1 + [z_2 - z_1]$, are precisely those for a single dam (cf., Kendall [2], Gani [3]); they are given generally by

$$(1.6)\qquad g(z_1, T) = (z_1/T)f(T - z_1, T),$$

or, when the input distribution is Poisson, by

$$(1.7)\qquad g(z_1, T) = \frac{z_1}{T} e^{-\lambda T} \frac{(\lambda T)^{T-z_1}}{(T - z_1)!}.$$

There is thus no need to reconsider the problem for $t \leqq t_0$; we may, without loss of generality, start with initial contents $z_2 > z_1 > 0$ such that

$$z_2 - z_1 = \alpha < 1,$$

the times of first emptiness after n inputs then being

$$(1.8)\qquad T = z_1 + [\tfrac{1}{2}(n + 1)]\alpha + [\tfrac{1}{2}n]\beta \qquad (n = 0, 1, 2, \cdots).$$

We see readily from the minimal path in Figure 1 that the problem is equivalent to that of the single dam with ordered inputs $0 < \alpha, \beta < 1$ $(\alpha + \beta = 1)$; we shall discuss it more simply in this form.

2. First emptiness of the dam with ordered inputs. Consider a dam with initial content z, fed by ordered inputs $\alpha, \beta > 0$ $(\alpha + \beta = 1)$ such as that shown

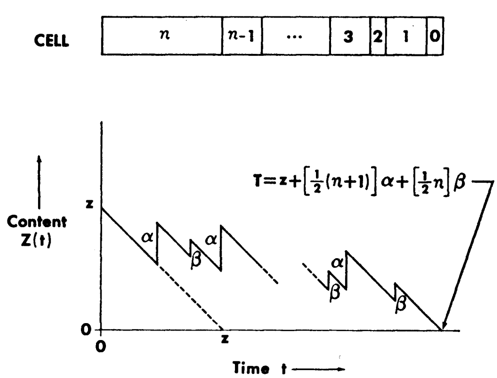

FIG. 2

in Figure 2, the input probability distribution being

$$\Pr\{X(\tau) = [\tfrac{1}{2}(j + 1)]\alpha + [\tfrac{1}{2}j]\beta\} = f(j, \tau) \qquad (j = 0, 1, 2, \cdots)$$

as in (1.1). When $\alpha = \beta$, this reduces to the simpler case of the dam with inputs of constant size $\alpha = \frac{1}{2}$, and the probabilities of first emptiness at times $T = z + n\alpha$ $(n = 0, 1, 2, \cdots)$ are of the form (1.6). These may be obtained from the discrete analogue of Kendall's [2] integral equation, namely

$$g(z, T) = \begin{cases} f(0, z) & (T = z), \\ \displaystyle\sum_{j=1}^{(T-z)\alpha^{-1}} f(j, z) g(j\alpha, T - z) & (T > z). \end{cases} \tag{2.1}$$

This states that starting with a content z, if there is an input $j\alpha > 0$ in time z, the probability of first emptiness at T is the convolution (over j) of the independent probabilities of an input $j\alpha$ in time z and of first emptiness in the remaining time $T - z$, starting with the new content $j\alpha$.

In the case where $\alpha \neq \beta$, a similar equation holds, though it is now necessary to define two types of probabilities of first emptiness in time t, $g_\alpha(z, t)$ and $g_\beta(z, t)$, both starting from a content $z > 0$, but depending respectively on whether the first input is α (with α, β alternating) or β (with β, α alternating). Using precisely the same argument as above, it is clear that the summation formula for $g_\alpha(z, T)$ $(T = z + [\frac{1}{2}(n + 1)]\alpha + [\frac{1}{2}n]\beta,\ n = 0, 1, 2, \cdots)$ may be written

$$\begin{aligned} g_\alpha(z, T) &= g(z; [\tfrac{1}{2}(n + 1)[\alpha, [\tfrac{1}{2}n]\beta) \\ &= \begin{cases} f(0, z) \qquad (T = z), \\ \displaystyle\sum_{j=1}^{[\frac{1}{2}(n+1)]} f(2j - 1, z) g_\beta(j\alpha + j\beta - \beta, T - z) \\ \qquad + \displaystyle\sum_{j=1}^{[\frac{1}{2}n]} f(2j, z) g_\alpha(j\alpha + j\beta, T - z) \qquad \text{otherwise.} \end{cases} \end{aligned} \tag{2.2}$$

It is obvious from considerations of symmetry that for any initial content $z > 0$, one obtains $g_\beta(z, t)$ at times $t = z + [\frac{1}{2}(r + 1)]\beta + [\frac{1}{2}r]\alpha$ $(r = 0, 1, 2, \cdots)$, directly by an interchange of α and β in (2.2) so that

$$g_\beta(z, z + [\tfrac{1}{2}r]\alpha + [\tfrac{1}{2}(r + 1)]\beta) = g(z; [\tfrac{1}{2}(r + 1)]\beta, [\tfrac{1}{2}r]\alpha). \tag{2.3}$$

These equations can be used to evaluate the probabilities $g_\alpha(z, T)$ successively to any required value of T; the method will be illustrated later for Poisson inputs.

Let us now define the p.g.f.'s of $g_\alpha(z, t)$, $g_\beta(z, t)$ as $\phi_\alpha(\theta \mid z)$, $\phi_\beta(\theta \mid z)$ respectively, where

$$\begin{aligned} \phi_\alpha(\theta \mid z) &= \phi(\theta; \alpha, \beta \mid z) \\ &= \sum_{n=0}^{\infty} \theta^{z + [\frac{1}{2}(n+1)]\alpha + [\frac{1}{2}n]\beta}\, g_\alpha(z, z + [\tfrac{1}{2}(n + 1)]\alpha + [\tfrac{1}{2}n]\,\beta) \\ & \qquad\qquad (0 \leqq \theta \leqq 1) \end{aligned} \tag{2.4}$$

and $\phi_\beta(\theta \mid z) = \phi(\theta; \beta, \alpha \mid z)$. Then, it follows from (2.2) that

$$\begin{aligned} \phi_\alpha(\theta \mid z) &= \theta^z\{f(0, z) + \sum_{j=1}^{\infty} f(2j - 1, z)\phi_\beta(\theta \mid j\alpha + j\beta - \beta) \\ &\qquad + \sum_{j=1}^{\infty} f(2j, z)\phi_\alpha(\theta \mid j\alpha + j\beta)\}, \\ \phi_\beta(\theta \mid z) &= \theta^z\{f(0, z) + \sum_{j=1}^{\infty} f(2j - 1, z)\phi_\alpha(\theta \mid j\beta + j\alpha - \alpha) \\ &\qquad + \sum_{j=1}^{\infty} f(2j, z)\phi_\beta(\theta \mid j\alpha + j\beta)\}, \end{aligned} \tag{2.5}$$

where $\phi_\alpha(0 \mid z) = \phi_\beta(0 \mid z) = 0$.

We verify that where $\alpha = \beta$, the equation (2.5) reduces to the well-known form given by Takács [1]. For then $\phi_\alpha(\theta \mid z) = \phi_\beta(\theta \mid z) = \phi(\theta \mid z)$ and (2.5) becomes

$$\phi(\theta \mid z) = \theta^z\{f(0, z) + \sum_{j=1}^{\infty} f(j, z)\phi(\theta \mid j\alpha). \tag{2.6}$$

Now in this case the random variable T increases by independent increments when z increases, or

$$\phi(\theta \mid z) = \{\phi(\theta \mid 1)\}^z = \{\phi(\theta)\}^z. \tag{2.7}$$

It follows from (2.6) that $\phi(\theta)$ is the solution of

$$\{\phi(\theta)\}^z = \theta^z\{\sum_{j=0}^{\infty} f(j, z)\{\phi(\theta)\}^{j\alpha}\} = \theta^z\{\psi(\phi^\alpha(\theta))\}^z$$

or of the equation

$$\phi(\theta) = \theta\psi(\phi^\alpha(\theta)) \tag{2.8}$$

for which $\phi(0) = 0$.

Such a simplification is not possible when $\alpha \neq \beta$, since the increments of T corresponding to an increase of z are now no longer independent. We may, however, give a formal solution to the equations (2.5) for $\phi_\alpha(\theta \mid z)$ and $\phi_\beta(\theta \mid z)$. Consider the infinite row vector

$$\boldsymbol{\phi}' = \{\phi_\alpha(\theta \mid z_1), \phi_\beta(\theta \mid z_2), \phi_\alpha(\theta \mid z_3), \phi_\beta(\theta \mid z_4) \cdots\} \tag{2.9}$$

of p.g.f.'s appearing as coefficients in the expressions (2.5), where the

$$z_{2j-1}, z_{2j} \qquad (j = 1, 2, \cdots)$$

are respectively

$$\begin{aligned} z_{2j-1} &= [\tfrac{1}{2}(j + 1)]\beta + [\tfrac{1}{4}(2j + 1)]\alpha, \\ z_{2j} &= [\tfrac{1}{2}(j + 1)]\alpha + [\tfrac{1}{4}(2j + 1)]\beta. \end{aligned} \tag{2.10}$$

We obtain from (2.5) for these various values of z that

$$\phi = \mathbf{A} + \mathbf{B}\phi, \tag{2.11}$$

where $\mathbf{A}$ is the infinite column vector $\{\theta^{z_r} f(0, z_r)\}$ $(r = 1, 2, \cdots)$, and $\mathbf{B}$ the infinite matrix $\{b_{rk}\}$ defined by

$$\mathbf{B} = \begin{bmatrix} 0 & \theta^{\beta} f(1, \beta) & \theta^{\beta} f(2, \beta) & 0 & \cdots \\ \theta^{\alpha} f(1, \alpha) & 0 & 0 & \theta^{\alpha} f(2, \alpha) & \cdots \\ 0 & \theta^{\alpha+\beta} f(1, \alpha+\beta) & \theta^{\alpha+\beta} f(2, \alpha+\beta) & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \tag{2.12}$$

where in row $2j - 1$ $(j = 1, 2, \cdots)$ the elements are

$$b_{2j-1,4m-3} = b_{2j-1,4m} = 0,$$
$$b_{2j-1,4m-2} = \theta^{z_{2j-1}} f(2m - 1, z_{2j-1})$$
$$b_{2j-1,4m-1} = \theta^{z_{2j-1}} f(2m, z_{2j-1}) \qquad (m = 1, 2, \cdots)$$

while in row $2j$ $(j = 1, 2, \cdots)$

$$b_{2j,4m-3} = \theta^{z_{2j}} f(2m - 1, z_{2j}),$$
$$b_{2j,4m-2} = b_{2j,4m-1} = 0,$$
$$b_{2j,4m} = \theta^{z_{2j}} f(2m, z_{2j}) \qquad (m = 1, 2, \cdots).$$

It follows that

$$(\mathbf{I} - \mathbf{B})\phi = \mathbf{A}. \tag{2.13}$$

Now since $\sum_{k=1}^{\infty} b_{rk} = \theta^{z_r}\{1 - f(0, z_r)\} < \theta^{z_r}$ for all r, and $\theta^{z_r} \leqq 1$ for all

$$0 \leqq \theta \leqq 1,$$

the matrix $\mathbf{I} + \sum_{n=1}^{\infty} \mathbf{B}^n$ exists in this range and is the unique two-sided reciprocal of $(\mathbf{I} - \mathbf{B})$ so that formally

$$\phi = (\mathbf{I} + \sum_{n=1}^{\infty} \mathbf{B}^n)\mathbf{A}. \tag{2.14}$$

Since the coefficients in the expressions of $\phi_\alpha(\theta \mid z)$, $\phi_\beta(\theta \mid z)$ are now known, we see from (2.5) that these p.g.f.'s are fully defined.

3. Probabilities of first emptiness for ordered Poisson inputs. Suppose that the input process $X(t)$ is Poisson with constant parameter $\lambda > 0$, such that

$$f(j, \tau) = e^{-\lambda\tau}(\lambda\tau)^j/j! \qquad (j = 0, 1, 2, \cdots) \tag{3.1}$$

with the p.g.f.

$$\psi(\theta, \tau) = \{e^{-\lambda(1-\theta)}\}^{\tau} \qquad (0 \leqq \theta \leqq 1). \tag{3.2}$$

We illustrate the evaluation of $g_\alpha(z, T)$ from equation (2.2) for values of

$$T = z + [\tfrac{1}{2}(n + 1)]\alpha + [\tfrac{1}{2}n]\beta, \qquad n = 0, 1, 2, 3, 4, 5, 6.$$

We first have from (2.2) and (2.3) that

$$g_\alpha(z, z) = e^{-\lambda z} = g_\beta(z, z) \tag{3.3}$$

so that from (2.2)

$$g_\alpha(z, z + \alpha) = e^{-\lambda(z+\alpha)}\lambda z. \tag{3.4}$$

It follows that $g_\beta(z, z + \beta) = e^{-\lambda(z+\beta)}\lambda z$, and thus for $z = \alpha$ that

$$g_\beta(\alpha, \alpha + \beta) = e^{-\lambda(\alpha+\beta)}\lambda\alpha,$$

so that

$$\begin{aligned} g_\alpha(z, z + \alpha + \beta) &= e^{-\lambda z}\{\lambda z g_\beta(\alpha, \alpha + \beta) + ((\lambda z)^2/2!)g_\alpha(\alpha + \beta, \alpha + \beta)\} \\ &= e^{-\lambda(z+\alpha+\beta)}(\lambda^2 / 2!)z(z + 2\alpha). \end{aligned} \tag{3.5}$$

Proceeding step by step in this manner, we derive further that

$$g_\alpha(z, z + 2\alpha + \beta) = e^{-\lambda(z+2\alpha+\beta)}(\lambda^3/3!)z\{z^2 + 3z(\alpha + \beta) + 3(\alpha^2 + 2\alpha\beta)\} \tag{3.6}$$

$$\begin{aligned} g_\alpha(z, z + 2\alpha + 2\beta) &= e^{-\lambda(z+2\alpha+2\beta)}(\lambda^4/4!)z \\ &\times \{z^3 + 4z^2(2\alpha + \beta) + 6z(3\alpha^2 + 4\alpha\beta + \beta^2) + 4(4\alpha^3 + 9\alpha^2\beta + 3\alpha\beta^2)\} \end{aligned} \tag{3.7}$$

$$\begin{aligned} g_\alpha(z, z + 3\alpha + 2\beta) &= e^{-\lambda(z+3\alpha+2\beta)}(\lambda^5/5!)z \times \{z^4 + 5z^3(2\alpha + 2\beta) \\ &+ 10z^2(4\alpha^2 + 8\alpha\beta + 3\beta^2) + 10z(7\alpha^3 + 21\alpha^2\beta + 18\alpha\beta^2 + 4\beta^3) \\ &+ 5(11\alpha^4 + 44\alpha^3\beta + 54\alpha^2\beta^2 + 16\alpha\beta^3)\} \end{aligned} \tag{3.8}$$

$$\begin{aligned} g_\alpha(z, z + 3\alpha + 3\beta) &= e^{-\lambda(z+3\alpha+3\beta)}(\lambda^6/6!)z \times \{z^5 + 6z^4(3\alpha + 2\beta) \\ &+ 15z^3(8\alpha^2 + 12\alpha\beta + 4\beta^2) \\ &+ 20z^2(20\alpha^3 + 48\alpha^2\beta + 33\alpha\beta^2 + 7\beta^3) \\ &+ 15z(43\alpha^4 + 144\alpha^3\beta + 162\alpha^2\beta^2 + 72\alpha\beta^3 + 11\beta^4) \\ &+ 6(81\alpha^5 + 350\alpha^4\beta + 520\alpha^3\beta^2 + 290\alpha^2\beta^3 + 55\alpha\beta^4)\}. \end{aligned} \tag{3.9}$$

The evaluation of $g_\alpha(z, T)$ can be continued to any required value of T.

The p.f.g.'s $\phi_\alpha(\theta \mid z)$, $\phi_\beta(\theta \mid z)$ for this process will be defined by the equation (2.14) for the coefficients of the vector $\boldsymbol{\phi}$, where $f(j, \tau)$ is now given by (3.1).

It will be noted that for $\alpha = \beta$, the equations (3.3)–(3.9) reduce to the result (1.7) of the form

$$g(z, z + n\alpha) = e^{-\lambda(z+n\alpha)}(\lambda^n/n!)z(z + n\alpha)^{n-1} \qquad (n = 0, 1, \cdots). \tag{3.10}$$

with the p.g.f. $\phi(\theta)$ which is the solution of

$$\phi(\theta) = \theta\{e^{-\lambda(1-\phi^\alpha(\theta))}\} \tag{3.11}$$

such that $\phi(0) = 0$. It has not proved possible to obtain an expression as simple as (3.10) for $g_\alpha(z, z + [\frac{1}{2}(n + 1)]\alpha + [\frac{1}{2}n]\beta)$. However, we present an alternative approach, in which the interpretation of the process as an occupancy problem leads to a simpler method of evaluating these probabilities.

4. First emptiness as an occupancy problem: Poisson inputs. When the inputs are of uniform size ($\alpha = \beta$), it has been shown (cf., Gani, [3]) that first emptiness may be characterized as an occupancy problem subject to certain restrictions. With some minor modifications, the same formulation can be used in the case of ordered inputs ($\alpha \neq \beta$).

Consider Figure 2, where the time of first emptiness is

$$T = z + [\tfrac{1}{2}(n + 1)]\alpha + [\tfrac{1}{2}n]\beta;$$

let the region between times $t = 0$ and $t = z$ be thought of as cell n, and that between $t = z + [\frac{1}{2}(n - j)]\alpha + [\frac{1}{2}(n - j - 1)]\beta$ and

$$t = z + [\tfrac{1}{2}(n - j + 1)]\alpha + [\tfrac{1}{2}(n - j)]\beta$$

as cell j ($j = 0, 1, \cdots, n - 1$). Then the non-negative number of inputs x_j in each cell ($j = 0, 1, \cdots n$) must satisfy the conditions

$$\sum_{j=0}^{i} x_j \leqq i \qquad (x_j \geqq 0; i = 0, \cdots, n - 1) \tag{4.1}$$

together with

$$\sum_{j=0}^{n} x_j = n. \tag{4.2}$$

The probability of such input arrangements is the sum of all those coefficients of terms $\theta_0^{x_0} \cdots \theta_n^{x_n}$ for which the x_j satisfy conditions (4.1)–(4.2) in the p.g.f.

$$P_{n+1}(\theta_0, \cdots, \theta_n) = \psi(\theta_n, z) \prod_{i=1}^{n} \psi(\theta_{n-i}, \tau_{n-i}) \quad (0 \leqq \theta_0, \cdots, \theta_n \leqq 1), \tag{4.3}$$

where $\tau_{n-(2j-1)} = \alpha$, $\tau_{n-2j} = \beta$ ($j = 1, 2, \cdots, [\frac{1}{2}(n + 1)]$), and $\tau_0 = \alpha$ or β depending on whether n is odd or even.

In order to illustrate the method clearly, we consider ordered inputs having the Poisson distribution (3.1), with p.g.f. (3.2). Then

$$\begin{aligned} P_{n+1}(\theta_0, \cdots, \theta_n) &= \exp\{-\lambda(z + [\tfrac{1}{2}(n + 1)]\,\alpha + [\tfrac{1}{2}n]\,\beta)\} \\ &\times \begin{cases} \exp\{\lambda(\theta_n z + \theta_{n-1}\alpha + \theta_{n-2}\beta + \cdots + \theta_0\beta)\} & \text{if } n = 2r \\ \exp\{\lambda(\theta_n z + \theta_{n-1}\alpha + \theta_{n-2}\beta + \cdots + \theta_0\alpha)\} & \text{if } n = 2r + 1 \end{cases} \end{aligned} \tag{4.4}$$

where $r = 0, 1, \cdots$.

Let us consider the case where n is even ($n = 2r$) for which $\tau_0 = \beta$, and define the following set of polynomials

$$\begin{aligned} &H_{\beta i}(\theta) \\ &= \exp\{-\lambda\{[\tfrac{1}{2}(i + 1)]\alpha + [\tfrac{1}{2}(i + 2)]\beta\}\} \sum_{j=0}^{i} C_{\beta ij}\theta^j/j! \qquad (i = 0, 1, \cdots) \end{aligned} \tag{4.5}$$

such that

$$\begin{aligned} H_{\beta 0}(\theta) &= e^{-\lambda\beta} \\ H_{\beta i}(\theta) &= \langle \exp\{-\lambda(1 - \theta) \\ &\quad \{([\tfrac{1}{2}(i + 1)] - [\tfrac{1}{2}i])\alpha + ([\tfrac{1}{2}(i + 2)] - [\tfrac{1}{2}(i + 1)])\beta\}\} \\ &\qquad H_{\beta, i-1}(\theta)\rangle \qquad (i = 1, 2, \cdots) \end{aligned} \tag{4.6}$$

where the brackets $\langle \ \rangle$ indicate the truncation of all terms in θ of degree higher than the ith. Let us define further, for $n = 2r$, the polynomial $H_{\beta,2r}(\theta, z)$ as

$$(4.7) \qquad H_{\beta,2r}(\theta, z) = \langle e^{-\lambda z(1-\theta)} H_{\beta,2r-1}(\theta) \rangle;$$

it is clear then that $H_{\beta,2r}(\theta, z)$, itself not a p.g.f., is that part of

$$\Phi_{2n+1}(\theta) = P_{2r+1}(\theta, \theta, \cdots, \theta)$$

satisfying the conditions (4.1), while the probability of the input arrangements also subject to (4.2) is

$$(4.8) \qquad g_\alpha(z, z + r(\alpha + \beta)) = e^{-\lambda\{z+r(\alpha+\beta)\}} C_{\beta,2r,2r}(z)/(2r)!$$

$C_{\beta,2r,2r}(z)$ being the coefficient of θ^{2r} in $H_{\beta,2r}(\theta, z)$.

If the number of cells n is odd ($n = 2r + 1$), so that $\tau_0 = \alpha$, the corresponding set of polynomials

$$(4.9) \qquad H_{\alpha i}(\theta) = \exp\{-\lambda\{[\tfrac{1}{2}(i+2)]\,\alpha + [\tfrac{1}{2}(i+1)]\,\beta\}\} \sum_{j=0}^{i} C_{\alpha ij} \frac{\theta^j}{j!} \qquad (i = 0, 1, \cdots)$$

may be obtained directly from (4.6) by an interchange of α and β. For

$$n = 2r + 1,$$

the polynomial $H_{\alpha,2r+1}(\theta, z)$ is then

$$(4.10) \qquad H_{\alpha,2r+1}(\theta, z) = \langle e^{-\lambda z(1-\theta)} H_{\alpha,2r}(\theta) \rangle$$

and the probability

$$(4.11) \qquad g_\alpha(z, z + r(\alpha + \beta) + \alpha) = e^{-\lambda\{z+r(\alpha+\beta)+\alpha\}} C_{\alpha,2r+1,2r+1}(z)/(2r+1)!$$

where $C_{\alpha,2r+1,2r+1}(z)$ is the coefficient of θ^{2r+1} in (4.10).

We find that

$$(4.12) \qquad \begin{aligned}
H_{\beta 0}(\theta) &= e^{-\lambda\beta} \\
H_{\beta 1}(\theta) &= e^{-\lambda(\alpha+\beta)}\{1 + (\lambda\alpha)\theta\} \\
H_{\beta 2}(\theta) &= e^{-\lambda(\alpha+2\beta)}\{1 + \lambda(\alpha + \beta)\theta + (\lambda^2/2!)(\beta^2 + 2\alpha\beta)\theta^2\} \\
H_{\beta 3}(\theta) &= e^{-\lambda(2\alpha+2\beta)}\{1 + \lambda\{2\alpha + \beta\} + (\lambda^2/2!)(3\alpha^2 + 4\alpha\beta + \beta^2)\theta^2 \\
&\qquad + (\lambda^3/3!)(4\alpha^3 + 9\alpha^2\beta + 3\alpha\beta^2)\theta^3\} \\
H_{\beta 4}(\theta) &= e^{-\lambda(2\alpha+3\beta)}\{1 + \lambda(2\alpha + 2\beta)\theta + (\lambda^2/2!)(4\beta^2 + 8\beta\alpha + 3\alpha^2)\theta^2 \\
&\qquad + (\lambda^3/3!)(4\alpha^3 + 18\alpha^2\beta + 21\alpha\beta^2 + 7\beta^3)\theta^3 \\
&\qquad + (\lambda^4/4!)(11\beta^4 + 44\beta^3\alpha + 54\beta^2\alpha^2 + 16\beta\alpha^3)\theta^4\} \\
H_{\beta 5}(\theta) &= e^{-\lambda(3\alpha+3\beta)}\{1 + \lambda(3\alpha + 2\beta)\theta + (\lambda^2/2!)(4\beta^2 + 12\alpha\beta + 8\alpha^2)\theta^2 \\
&\qquad + (\lambda^3/3!)(20\alpha^3 + 48\alpha^2\beta + 33\alpha\beta^2 + 7\beta^3)\theta^3 \\
&\qquad + (\lambda^4/4!)(11\beta^4 + 72\beta^3\alpha + 162\beta^2\alpha^2 + 144\beta\alpha^3 + 43\alpha^4)\theta^4 \\
&\qquad + (\lambda^5/5!)(81\alpha^5 + 350\alpha^4\beta + 520\alpha^3\beta^2 + 290\alpha^2\beta^3 + 55\alpha\beta^4)\theta^5\},
\end{aligned}$$

and these, together with the corresponding set of polynomials

$$H_{\alpha 0}(\theta), \cdots, H_{\alpha 5}(\theta),$$

on using (4.7) and (4.10) result in precisely those values of $g_\alpha(z, T)$ given in (3.3)–(3.9).

A recurrence relation for the $C_{\beta ij}$ (j fixed) in the polynomials $H_{\beta i}(\theta)$ (or $C_{\alpha ij}$ in $H_{\alpha i}(\theta)$) permits a rapid evaluation of these coefficients. For it is readily seen from (4.6) that for any $i > j$ ($j = 1, 2, \cdots$)

$$\begin{aligned} C_{\beta ij} &= \text{coefficient of } \theta^j/j! \text{ in} \\ &\quad \{\exp\{\lambda\theta\{[\tfrac{1}{2}(i+1)]\alpha + [\tfrac{1}{2}(i+2)]\beta - [\tfrac{1}{2}(j+1)]\,\alpha \\ &\qquad - [\tfrac{1}{2}(j+2)]\beta\}\}H_{\beta j}(\theta)\}, \\ &= \sum_{k=0}^{j} \binom{j}{k} C_{\beta jk}\, \lambda^{j-k}\{([\tfrac{1}{2}(i+1)] - [\tfrac{1}{2}(j+1)])\alpha \\ &\qquad + ([\tfrac{1}{2}(i+2)] - [\tfrac{1}{2}(j+2)])\beta\}^{j-k}, \\ &= \sum_{k=0}^{j-1} \binom{j}{k} C_{\beta ,j-1,k}\, \lambda^{j-k}\{([\tfrac{1}{2}(i+1)] - [\tfrac{1}{2}j])\alpha \\ &\qquad + ([\tfrac{1}{2}(i+2)] - [\tfrac{1}{2}(j+1)]\beta\}^{j-k}. \end{aligned} \tag{4.13}$$

Thus, given the coefficients $C_{\beta,j-1,k}$ ($k = 0, \cdots, j-1$) in $H_{\beta,j-1}(\theta)$, it is possible by a straightforward algebraic procedure to obtain all the coefficients $C_{\beta ij}$ in the polynomials $H_{\beta i}(\theta)$ ($i = j, j+1, \cdots$).

5. A general formulation of the dam with ordered inputs. Consider the dam with initial content z, fed by ordered inputs whose alternate magnitudes

$$x_\alpha, x_\beta > 0$$

are random, with distribution functions (d.f.) $H_\alpha(u)$ and $H_\beta(u)$ respectively, and such that their times of arrival form a Poisson process with constant parameter λ. Then, precisely as in (2.2), we may obtain the probability distribution of first emptiness times $dG_\alpha(z, T)$ ($z \leqq T < \infty$), starting with an α-type jump, as

$$dG_\alpha(z,T) = \begin{cases} e^{-\lambda z} \; (T = z) \\ e^{-\lambda z} \displaystyle\sum_{n=0}^{\infty} \left\{ \int_0^{T-z} dG_\beta(u, T-z)\, \frac{(\lambda z)^{2n+1}}{(2n+1)!}\, dH_{\alpha\alpha}^{(2n+1)}(u) \right. \\ \qquad \left. + \displaystyle\int_0^{T-z} dG_\alpha(u, T-z)\, \frac{(\lambda z)^{2n+2}}{(2n+2)!}\, dH_{\alpha\beta}^{(2n+2)}(u) \right\} \quad (T > z) \end{cases} \tag{5.1}$$

where $H_{\alpha\alpha}^{(2n+1)}(u)$, $H_{\alpha\beta}^{(2n+2)}(u)$ indicate d.f.'s for the $(2n+1)$th and $(2n+2)$th convolutions of the type

$$\begin{aligned} H_{\alpha\alpha}^{(2n+1)}(u) &= H_\alpha * H_\beta * \cdots * H_\alpha, \\ H_{\alpha\beta}^{(2n+2)}(u) &= H_\alpha * H_\beta * \cdots * H_\beta. \end{aligned} \tag{5.2}$$

An equation similar to (5.1) with α and β interchanged holds for $dG_\beta(z, T)$.

The generating function for $dG_\alpha(z, t)$ is

$$\phi_\alpha(\theta/z) = \theta^z e^{-\lambda z} + \int_{z+0}^{\infty} \theta^T \, dG_\alpha(z, T) \qquad (0 \leqq \theta \leqq 1)$$

$$(5.3) \qquad = e^{-\lambda z}\left\{\theta^z + \int_z^{\infty}\left[\int_0^{T-z} \theta^T \, dG_\beta(u, T - z) \sum_{n=0}^{\infty} \frac{(\lambda z)^{2n+1}}{(2n+1)!} \, dH_{\alpha\alpha}^{(2n+1)}(u) + \int_0^{T-z} \theta^T dG_\alpha(u, T - z) \sum_{n=0}^{\infty} \frac{(\lambda z)^{2n+2}}{(2n+2)!} \, dH_{\alpha\beta}^{(2n+2)}(u)\right]\right\}$$

and on changing the order of integration, this reduces to

$$(5.4) \qquad \phi_\alpha(\theta \mid z) = \theta^z e^{-\lambda z}\left\{1 + \int_0^{\infty} \phi_\beta(\theta \mid u) \sum_{n=0}^{\infty} \frac{(\lambda z)^{2n+1}}{(2n+1)!} \, dH_{\alpha\alpha}^{(2n+1)}(u) + \int_0^{\infty} \phi_\alpha(\theta \mid u) \sum_{n=0}^{\infty} \frac{(\lambda z)^{2n+2}}{(2n+2)!} \, dH_{\alpha\beta}^{(2n+2)}(u)\right\},$$

where $\phi_\alpha(0 \mid z) = 0$. A similar equation with α and β interchanged holds for $\phi_\beta(\theta \mid z)$.

It is seen directly that these give the well-known equation for the p.g.f. in the case where $H_\alpha(u) = H_\beta(u)$. For here, $\phi_\alpha(\theta \mid u) = \phi_\beta(\theta \mid u) = \{\phi(\theta)\}^u$, so that from (5.4)

$$\{\phi(\theta)\}^z = (\theta e^{-\lambda})^z\left\{1 + \int_0^{\infty} \{\phi(\theta)\}^u \sum_{n=1}^{\infty} \frac{(\lambda z)^n}{n!} \, dH^{(n)}(u)\right\}$$

$$= (\theta e^{-\lambda})^z\left\{\sum_{n=0}^{\infty} \frac{(\lambda z)^n}{n!} \psi^n(\phi(\theta))\right\}$$

$$= \{\theta \exp\{-\lambda\{1 - \psi(\phi(\theta))\}\}\}^z$$

or

$$(5.5) \qquad \phi(\theta) = \theta \exp\{-\lambda\{1 - \psi(\phi(\theta))\}\},$$

such that $\phi(0) = 0$.

6. Acknowledgment. I am grateful to Dr. J. L. Mott of Edinburgh University for helpful discussions on this problem during the summer of 1959.

REFERENCES

[1] L. Takács, "Investigation of waiting time problems by reduction to Markov processes," *Acta Math. Acad. Sci. Hung.* Vol. 6 (1955), pp. 101–129.

[2] D. G. Kendall, "Some problems in the theory of dams," *J. Roy. Stat. Soc.*, Ser. B., Vol. 19 (1957), pp. 207–212.

[3] J. Gani, "Elementary methods for an occupancy problem of storage," *Math. Annalen*, Vol. 136 (1958), pp. 454–465.

[4] J. Gani and N. U. Prabhu, "The time-dependent solution for a storage model with Poisson input," *J. Math. and Mechanics*, Vol. 8 (1959), pp. 653–663.

[5] F. A. Haight, "Two queues in parallel," *Biometrika*, Vol. 45 (1958), pp. 401–410.