RECURRENT GAMES AND THE PETERSBURG PARADOX[1]

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1. Introduction. A *recurrent game* $\mathcal{G}$ is defined by a sequence of trials of a certain, recurrent event $\mathcal{E}$ [1, pp. 238–242]. Let $X_1, X_2, \cdots$ be the sequence of recurrence times of $\mathcal{E}$, $S_n = X_1 + \cdots + X_n$ being the total number of trials up to and including the nth occurrence of $\mathcal{E}$. The X_n are independent random variables with positive integer values and a common distribution:

$$(1) \qquad p_i = P[X_n = i] \qquad (i, n = 1, 2, \cdots), \qquad p_i \geqq 0, \qquad \sum_1^\infty p_i = 1.$$

We assume that at each occurrence of $\mathcal{E}$ the player receives a *reward* which is a function of the number of trials since the previous occurrence of $\mathcal{E}$; thus at the kth occurrence of $\mathcal{E}$ the player receives the reward c_{X_k}, where $\{c_i\}$ is a given sequence of constants. The player also pays a *fee* f_k on the kth occurrence of $\mathcal{E}$, where $\{f_i\}$ is another given sequence of constants. On any trial on which $\mathcal{E}$ does not occur no money changes hands. With these rules the game $\mathcal{G}$ is determined by the three sequences of constants

$$(2) \qquad \mathcal{G} = \{p_i, c_i, f_i\}.$$

Let

$$(3) \qquad \begin{aligned} V_n &= \text{amount received by player at the } n\text{th trial} \\ &= \begin{cases} c_{X_k} \text{ if for some } k,\ S_k = n \\ 0 \text{ otherwise,} \end{cases} \\ W_n &= \text{amount paid by player at the } n\text{th trial} \\ &= \begin{cases} f_k \text{ if for some } k,\ S_k = n \\ 0 \text{ otherwise,} \end{cases} \end{aligned}$$

and let

$$(4) \qquad \begin{aligned} T_n &= \text{total amount received by player during the first } n \text{ trials} \\ &= V_1 + \cdots + V_n, \\ U_n &= \text{total amount paid by player during the first } n \text{ trials} \\ &= W_1 + \cdots + W_n. \end{aligned}$$

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If for some fixed n we have

$$(5) \qquad ET_n = EU_n$$

we shall say that $\mathcal{G}$ is *fair for that value of n*, and if (5) holds simultaneously for all $n = 1, 2, \cdots$ we shall say that $\mathcal{G}$ is *fair*. We shall derive methods for determining whether a game $\mathcal{G}$ is fair.

As an example we mention the classical *Petersburg game*: a coin with probability of heads $p = 1 - q$ is tossed repeatedly (p is usually taken to be $\frac{1}{2}$). At the appearance of the kth head the player receives 2^{X_k} dollars from the bank, and simultaneously pays a fee f_k to the bank. Thus

$$(6) \qquad p_i = pq^{i-1}, \qquad c_i = 2^i.$$

Problem: how should the schedule of fees $\{f_i\}$ be fixed to make the game fair?

The usual discussion of this game does not consider a fixed number of tosses of the coin, but rather assumes that the coin is tossed until heads first appears, the question being that of the proper fee for the player to pay for the privilege of making one such (random) number of tosses. The so-called Petersburg paradox arises from the fact that the expected reward to the player at the end of the first run (i.e., at the first appearance of heads) is infinite when $0 < p \leqq \frac{1}{2}$,

$$(7) \qquad \sum_1^{\infty} p_i c_i = 2p \sum_0^{\infty} (2q)^i = \infty \qquad \text{for } 0 < p \leqq \tfrac{1}{2},$$

so that the law of large numbers implies that in repeated runs the game would be favorable to the player for any fixed fee, no matter how large. W. Feller ([1], p. 235–237) has greatly illuminated the situation by showing that if when $p = \frac{1}{2}$ the player pays the cumulative fee $f_1 + \cdots + f_m = m \log_2 m$ for the privilege of making a fixed number m of such runs then the game is asymptotically fair in the sense that

$$(8) \qquad \lim_{m\to\infty} \frac{\textit{total reward after m runs}}{f_1 + \cdots + f_m} = 1$$

in probability. It should be noted that our definition of "fair" involves a fixed number of *tosses*, not of *runs*, and requires equality of expected reward received and fee paid rather than convergence to 1 of the ratio of reward to fee. We shall in fact show that to make the Petersburg game fair in our sense when $p = \frac{1}{2}$ the player should pay the cumulative fee $f_1 + \cdots + f_m = m(m + 1)$ if there are m (random) runs in n(fixed) tosses, since with this agreement the expected net gain of the player will be 0 for every n.

2. Expected reward and fee at the nth trial and conditions for their equality. Returning to the general recurrent game (2), with V_n defined by (3) for $n \geqq 1$, let

$$(9) \quad v_n = EV_n = \text{expected amount received by player at the } n\text{th trial.}$$

The conditional value of V_n given that the first occurrence time X_1 of $\mathcal{E}$ is ν is

$$(10)\qquad \begin{aligned} &0 \text{ if } \nu > n \\ &c_n \text{ if } \nu = n \\ &\tilde{V}_{n-\nu} \text{ if } \nu = 1, \cdots, n-1, \end{aligned}$$

where, since $\mathcal{E}$ is a recurrent event, $\tilde{V}_{n-\nu}$ is a random variable with the same distribution as $V_{n-\nu}$. Hence

$$(11)\qquad v_n = \sum_{\nu=1}^{\infty} p_\nu E[V_n \mid X_1 = \nu] = p_n c_n + \sum_{\nu=1}^{n-1} p_\nu v_{n-\nu}.$$

Setting

$$(12)\qquad p_0 = v_0 = 0$$

for convenience, we can write (11) in the form

$$(13)\qquad v_n = p_n c_n + \sum_{\nu=0}^{n} p_\nu v_{n-\nu},$$

valid for all $n \geqq 0$. The v_n are uniquely determined by (12) and (13), as is $ET_n = v_1 + \cdots + v_n$.

For the explicit solution of (13) it is convenient to introduce the formal power series

$$(14)\quad P(x) = \sum_0^\infty p_n x^n, \qquad G(x) = \sum_0^\infty p_n c_n x^n, \qquad V(x) = \sum_0^\infty v_n x^n,$$

in terms of which (13) becomes

$$(15)\qquad V(x) = G(x) + P(x)V(x),$$

and therefore we have

$$(16)\qquad V(x) = G(x)/[1 - P(x)].$$

For example, in the Petersburg game we have from (6)

$$(17)\qquad P(x) = px \sum_0^\infty (qx)^n = \frac{px}{1 - qx}, \qquad 1 - P(x) = \frac{1-x}{1-qx},$$
$$G(x) = P(2x),$$

so that

$$(18)\qquad V(x) = \frac{2px}{1 - 2qx} \cdot \frac{1 - qx}{1 - x}.$$

By expanding (18) in a power series around $x = 0$ we find that

$$(19)\qquad V_n = \begin{cases} (n+1)/2 & \text{for } p = \tfrac{1}{2}, \\ [2p/(2p-1)][p - q(2q)^{n-1}] & \text{for } p \neq \tfrac{1}{2}, \end{cases}$$

and by summation from 1 to n that

$$
(20) \qquad \begin{aligned} ET_n &= v_1 + \cdots + v_n \\ &= \begin{cases} [n(n+3)]/4 & \text{for } p = \tfrac{1}{2}, \\ \dfrac{2p}{(2p-1)^2}\{np(2p-1) + q[(2q)^n - 1]\} & \text{for } p \neq \tfrac{1}{2}. \end{cases} \end{aligned}
$$

We turn now to the computation of

(21) $w_n = EW_n =$ expected amount paid by player at the nth trial.

Let

(22) $P_j(n) = P[S_j = n] =$ coefficient of x^n in the series expansion of $[P(x)]^j$,

so that

$$
(23) \qquad [P(x)]^j = \sum_{n=j}^{\infty} P_j(n)x^n.
$$

Setting $w_n = f_n = 0$ by convention we can write

$$
(24) \qquad w_n = \sum_{j=0}^{n} f_j P_j(n).
$$

Introducing the formal power series

$$
(25) \qquad W(x) = \sum_0^\infty w_n x^n, \qquad F(x) = \sum_0^\infty f_n x^n
$$

we have from (23) and (24) the relation

$$
(26) \qquad \begin{aligned} W(x) &= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} f_j P_j(n)\right)x^n = \sum_{j=0}^{\infty} f_j\left(\sum_{n=j}^{\infty} P_j(n)x^n\right) \\ &= \sum_{j=0}^{\infty} f_j[P(x)]^j = F(P(x)). \end{aligned}
$$

Comparing (26) and (16) we see that the necessary and sufficient condition that $v_n = w_n$ for all n is that

$$
(27) \qquad F(P(x)) = G(x)/[1 - P(x)],
$$

and since $ET_n = v_1 + \cdots + v_n$, $EU_n = w_1 + \cdots + w_n$, this is also the condition that $\mathcal{G}$ be fair. Thus we have the

THEOREM. *$\mathcal{G}$ is fair if and only if* (27) *holds.*

A trivial example is afforded by any game $\mathcal{G}$ of the form $\{p_i, 1, 1\}$; here $F(x) = x/(1-x)$, $G(x) = P(x)$, and (27) holds. More interesting is the case in which $p_1 \neq 0$ so that the series $P(x)$ has an inverse series $P^{-1}(x)$ near $x = 0$; then the solution of (27) for given P and G is

$$F(x) = G(P^{-1}(x))/(1 - x). \tag{28}$$

COROLLARY 1. *If* $p_1 \neq 0$ *then for any schedule of rewards* $\{c_i\}$ *there is a unique schedule of fees* $\{f_i\}$, *given by* (28), *whch makes* $\mathcal{G} = \{p_i, c_i, f_i\}$ *fair.*

For example, in the Petersburg game we have from (17)

$$P(x) = \frac{px}{1 - qx}, \qquad G(x) = P(2x), \qquad P^{-1}(x) = \frac{x}{p + qx}, \tag{29}$$

so that the fair F is

$$F(x) = \frac{G(P^{-1}(x))}{1 - x} = \frac{P(2x/p + qx)}{1 - x} = \frac{2x}{1 - x} \cdot \frac{1}{1 - \lambda x}, \tag{30}$$

where we have set

$$\lambda = q/p. \tag{31}$$

Expanding F about $x = 0$ we find that

$$f_n = \begin{cases} 2n & \text{for } p = \frac{1}{2}, \\ 2.\dfrac{1 - \lambda^n}{1 - \lambda} & \text{for } p \neq \frac{1}{2}. \end{cases} \tag{32}$$

Thus for $p = \frac{1}{2}$ the fair cumulative fee for the first m runs is

$$f_1 + \cdots + f_m = m(m + 1), \tag{33}$$

as stated in Section 1.

It is of some interest to express the condition (27) for fairness directly in terms of the f_i. To do this we observe that from (26) we have

$$[1 - P(x)]F(P(x)) = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} f_j[P_j(n) - P_{j+1}(n)] \right) x^n. \tag{34}$$

Thus (27) is equivalent to the condition that

$$\sum_{j=0}^{n} f_j[P_j(n) - P_{j+1}(n)] = p_n c_n \qquad (n \geqq 1). \tag{35}$$

The coefficient of f_n in (35) is

$$P_n(n) - P_{n+1}(n) = P_n(n) = p_1^n, \tag{36}$$

which is $\neq 0$ when $p_1 \neq 0$. Hence when $p_1 \neq 0$ the equations (35) have a unique solution $\{f_i\}$ for any $\{c_i\}$, as we have already seen.

If we set

$$b_n = \sum_{1}^{n} p_k c_k \tag{37}$$

and sum (35) from 1 to n we obtain the equivalent system of equations

$$
\begin{aligned}
b_n &= \sum_{k=1}^{n}\sum_{j=1}^{k} f_j[P_j(k) - P_{j+1}(k)] \\
&= \sum_{j=1}^{n} f_j \sum_{k=j}^{n} [P_j(k) - P_{j+1}(k)] \\
&= \sum_{j=1}^{n} f_j\{P[S_j \leqq n] - P[S_{j+1} \leqq n]\} \\
&= \sum_{j=1}^{n} f_j P[\mathcal{E} \text{ occurs exactly } j \text{ times in the first } n \text{ trials}].
\end{aligned} \tag{38}
$$

Thus we have

COROLLARY 2. $\mathcal{G}$ *is fair if and only if, setting*

$$H_n(j) = P[\mathcal{E} \textit{ occurs exactly } j \textit{ times in the first } n \textit{ trials}], \tag{39}$$

we have

$$\sum_{j=1}^{n} f_j H_n(j) = \sum_{j=1}^{n} p_j c_j \qquad (n \geqq 1). \tag{40}$$

3. Acknowledgments and remarks.

(a) The author's interest in recurrent games arose during a conversation with Professor L. Takács concerning the Petersburg game; Takács had obtained the equation (20) for $p = \frac{1}{2}$. The condition (28) for fairness is due to the referee; in the original version of the present paper (40) was used.

(b) For the Petersburg game with $p = \frac{1}{2}$ and $f_n = 2n$ it can be shown that the variances of T_n and U_n are given by the formulas

$$
\begin{aligned}
\text{Var } T_n &= 3(2^n - 1) - (n/24)(n^3 + 4n^2 + 8n + 35), \\
\text{Var } U_n &= (n/8)(2n^2 + 5n + 1).
\end{aligned} \tag{41}
$$

Since by (20) and (32)

$$ET_n = EU_n = n(n+3)/4, \tag{42}$$

it follows that $U_n/EU_n \to 1$ in probability as $n \to \infty$ and therefore that the ratio

$$T_n/U_n = \frac{T_n/ET_n}{U_n/EU_n}$$

has approximately the same distribution for large n as does the ratio T_n/ET_n, which has mean 1 but variance which becomes infinite with n. Thus even though this game is fair in our sense, it does not follow that the ratio T_n/U_n tends to 1 in probability as $n \to \infty$.

(c) The following simple example is instructive. Let $p_1 = \epsilon$, $p_2 = p_3 = (1 - \epsilon)/2$, $p_i = 0$ for $i > 3$, where ϵ is a parameter, $0 \leqq \epsilon \leqq 1$, and let

c_1, c_2, c_3 be fixed, with $c_2 \neq c_3$. It is easily seen that

$$(43)\qquad \begin{aligned} H_1(1) &= \epsilon, \quad H_2(1) = \tfrac{1}{2} + (\epsilon/2) - \epsilon^2, \quad H_2(2) = \epsilon^2 \\ H_3(1) &= 1 - \epsilon, \quad H_3(2) = \epsilon - \epsilon^3, \quad H_3(3) = \epsilon^3. \end{aligned}$$

By Corollary 1, if $\epsilon > 0$ this game can be made fair. To do this we must by Corollary 2 choose f_1, f_2, f_3 in accordance with the equations

$$(44)\qquad \begin{aligned} &f_1\epsilon = \epsilon c_1 \\ &f_1[\tfrac{1}{2} + (\epsilon/2) - \epsilon^2] + f_2\epsilon^2 = \epsilon c_1 + [(1 - \epsilon)/2]c_2 \\ &f_1(1 - \epsilon) + f_2(\epsilon - \epsilon^3) + f_3\epsilon^3 = \epsilon c_1 + [(1 - \epsilon)/2](c_2 + c_3). \end{aligned}$$

For $\epsilon > 0$ these equations give for f_2 the value

$$(45)\qquad f_2 = \epsilon^{-2}[(-\tfrac{1}{2} + (\epsilon/2) + \epsilon^2)c_1 + [(1 - \epsilon)/2]c_2].$$

It follows that

(i) If $c_1 < c_2$ then $f_2 \to \infty$ as $\epsilon \to 0$

(ii) If $c_1 > c_2$ then $f_2 \to -\infty$ as $\epsilon \to 0$

(iii) If $\epsilon = 0$ then equations (44) yield $f_1 = c_2 = (c_2 + c_3)/2$, a contradiction.

Thus this game can be made fair for any $\epsilon > 0$, but as $\epsilon \to 0$ the fair fee f_2 approaches ∞ or $-\infty$, according as $c_1 < c_2$ or $c_1 > c_2$.

(d) Consider the Petersburg game with an unbiased coin, $p = \frac{1}{2}$. We have shown that if the player pays the fee $f_i = 2i$ at the conclusion of the ith run (i.e., at the appearance of the ith head), then the game becomes fair in the sense that $ET_n = EU_n$ for every n. Under the definition of a recurrent game which we have used it will be observed that for a fixed total number of tosses n it will usually happen that the last head will appear before the final toss, leaving a string of tails as the final segment of the sequence of tosses which is without effect in computing the rewards and fees. A player who ends his sequence of n tosses with a long string of tails might therefore feel unhappy, since if his last toss had been a head he would have received a large additional reward, although he would also have had to pay an additional fee. Accordingly, we can modify the rules to provide that *the nth toss shall always be regarded as a head*. Let T'_n, U'_n be respectively the total reward and fee for this modified game, using the same schedule of fees, $f_i = 2i$. It is curious that *the modified game remains fair*, $ET'_n = EU'_n$ for all n.

To see this we consider the increment of total reward after n tosses for the modified game over the original one, $\Delta T_n = T'_n - T_n$. We have

$$\begin{aligned} E(\Delta T_n) &= P\,[\text{no heads in } n \text{ tosses}]\cdot 2^n \\ &\quad + \sum_{k=1}^{n-1} P\,[\text{last head at } k\text{th toss}]\cdot 2^{n-k} \\ &= 1 + \sum_{k=1}^{n-1} \frac{1}{2^{n-k+1}} \cdot 2^{n-k} = 1 + \frac{n-1}{2}. \end{aligned}$$

Similarly,

$$E(\Delta U_n) = \sum_{i=0}^{n-1} P\,[i \text{ heads in first } n - 1 \text{ tosses, last toss tail}]\cdot 2(i + 1)$$

$$= \sum_{i=0}^{n-1} \binom{n-1}{i} \cdot \frac{1}{2^{n-1}} \cdot (i + 1) = \frac{n-1}{2} + 1,$$

so that $E(\Delta T_n) = E(\Delta U_n)$ and hence $ET'_n = EU'_n$.

REFERENCE

[1] William Feller, *An Introduction to Probability Theory and its Applications*, Vol. 1, 2nd ed., John Wiley and Sons, New York, 1957.