ADMISSIBLE AND MINIMAX ESTIMATES OF PARAMETERS IN TRUNCATED SPACES[1]

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In this paper we investigate some properties of point estimates when an upper or lower bound for the parameter, or unknown state of nature, is given in advance. The point estimation problem is characterized as follows. On the basis of an observation of a random variable x, with distribution function of the form $P_\omega(x) = \int_{-\infty}^{x} p_\omega(t)\, d\mu(t)$, it is desired to estimate some function $h(\omega)$. Here p_ω is a density with respect to a fixed σ-finite measure μ. An estimate $\delta = \delta(x)$ of $h(\omega)$ is desirable according to a criterion which minimizes, in some sense, the risk. We take as loss function W, the square error, i.e., $W(\omega, \delta) = [\delta - h(\omega)]^2$, and consider two criteria of desirability of an estimate: minimaxity and admissibility.

It is not unreasonable to assume that, often, some information about the parameters in the form of a bound, is known before. These bounds may be fixed, or may be of the form of orderings of parameters. In this paper we deal with fixed bounds.

Let μ be a σ-finite measure on the real line with spectrum $\mathfrak{X}$. Assume $\mathfrak{X}$ is nondegenerate to avoid trivialities. We consider the exponential family of densities with respect to μ, that is, the family of densities p_ω, where

$$p_\omega(x) = \beta(\omega) \exp (x\omega),$$

all x, and $\omega \varepsilon T$, where $T = \{\omega \mid \beta(\omega)^{-1} = \int_{\mathfrak{X}} \exp (x\omega)\, d\mu(x) < \infty\}$. Assume x is a random variable distributed according to p_ω. We wish to estimate $\varphi(\omega) = E_\omega\{x\}$ from a single observation. There is no loss of generality in the restriction to a single observation, for a sufficient statistic for n observations from an exponential family is the sum of the observations, whose distribution is again a member of the exponential family.

Our main assumption is that

$$\Omega = \{\omega \mid \omega \geqq a\} \subset T, \tag{1}$$

where a is an interior point of T. For purposes of simplicity we take $a = 0$ and proceed to develop admissible estimates for parameters in such truncated parameter spaces. The proof for $a \neq 0$ (or $\Omega = \{\omega \mid \omega \leqq a\}$) follows the development below.

T is a connected set, and $\beta(\omega)^{-1} = \int \exp (x\omega)\, d\omega$ is positive and analytic at each interior point of T. For the exponential family we have

$$\varphi(\omega) = -\beta'(\omega)/\beta(\omega). \tag{2}$$

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$\varphi'(\omega)$ is the variance of x. Therefore $\varphi'(\omega) > 0$ and $\varphi(\omega)$ is a (strictly) increasing function of ω. For each real x, let $G(x) = \int_0^\infty \beta(\omega) \exp(x\omega)\, d\omega$, (possibly taking on the value $+\infty$). $G(x)$ is convex, and $\{x \mid G(x) < \infty\}$ is an interval with left hand endpoint $-\infty$ and right endpoint b. Let $\bar{x} = \sup \mathfrak{X}$. If $x < \bar{x}$, then it is simple to show that $G(x)$ is finite. $G(\bar{x})$, however, may not be finite. In particular, $G(x - 1/\sigma) < \infty$, for $\sigma > 0$.

If $x < \bar{x}$, then

$$\beta(\omega) \exp(x\omega) \to 0 \quad \text{as} \quad \omega \to \infty \tag{3}$$

as we show. Differentiation shows that $\beta(\omega) \exp(x\omega)$ is monotone in ω, for large ω. Thus, if $\beta(\omega) \exp(x\omega) \nrightarrow 0$, we have that $\lim_{\omega\to\infty} \inf \beta(\omega) \exp(x\omega) > 0$. Let ϵ be such that $x + \epsilon < \bar{x}$. There are positive numbers M, and A, such that $\beta(\omega) \exp(x\omega) > M$, for $\omega > A$. Since $\bar{x} \leqq b$, we have that

$$\infty > G(x + \epsilon) = \int_0^\infty \beta(\omega) \exp \omega(x + \epsilon)\, d\omega > M \int_0^\infty \exp(\omega\epsilon)\, d\omega = \infty.$$

Condition (3) follows from this contradiction.

Now, take as *a priori* distribution,

$$\lambda_\sigma(\omega) = (1/\sigma) \exp(-\omega/\sigma), \qquad \omega \in \Omega, \qquad \lambda_\sigma(w) = 0, \quad \text{elsewhere.}$$

Then, the Bayes estimate of $\varphi(\omega)$ is given by

$$\begin{aligned} \delta_\sigma(x) &= [\textstyle\int_0^\infty \varphi(\omega)\beta(\omega) \exp \omega(x - 1/\sigma)\, d\omega]/[\int_0^\infty \beta(\omega) \exp \omega(x - 1/\sigma)\, d\omega \\ &= x - 1/\sigma + \beta(0)/G(x - 1/\sigma). \end{aligned} \tag{4}$$

The second step follows application of (2), and integration in the numerator. The risk is

$$\begin{aligned} \rho(\delta_\sigma, \omega) = \varphi'(\omega) &+ 2\beta(0)E_\omega\{x/G(x - 1/\sigma)\} - 2\varphi(\omega)\beta(0)E_\omega\{1/G(x - 1/\sigma)\} \\ &+ \beta^2(0)E_\omega\{1/G^2(x - 1/\sigma)\} - 2(\beta(0)/\sigma)E_\omega\{1/G(x - 1/\sigma)\} + 1/\sigma^2. \end{aligned}$$

It is easily verified that all these terms are finite. The Bayes risk of $\delta_\sigma(x)$ is given by $r(\delta_\sigma) = (1/\sigma) \int_0^\infty \exp(-\omega/\sigma)\rho(\delta_\sigma, \omega)\, d\omega$. Integrating, and using (2) we obtain

$$r(\delta_\sigma) = \frac{1}{\sigma}\int_0^\infty \varphi'(\omega) \exp(-\omega/\sigma)\, d\omega - \frac{\beta^2(0)}{\sigma}\int \frac{1}{G(x - 1/\sigma)}\, d\mu(x) + \frac{1}{\sigma^2}. \tag{5}$$

As $\sigma \to \infty$, $\delta_\sigma(x) \to \delta(x) = x + \beta(0)/G(x)$. $r(\delta)$, the average risk of δ with respect to λ_σ is readily calculated to be

$$\begin{aligned} &\frac{1}{\sigma}\int_0^\infty \varphi'(\omega) \exp(-\omega/\sigma)\, d\omega + \frac{\beta^2(0)}{\sigma}\int \frac{G(x - 1/\sigma)}{G^2(x)}\, d\mu(x) \\ &\qquad - 2\frac{\beta^2(0)}{\sigma}\int \frac{1}{G(x)}\, d\mu(x) + 2\frac{\beta(0)}{\sigma^2}\int \frac{G(x - 1/\sigma)}{G(x)}\, d\mu(x). \end{aligned} \tag{6}$$

We now show that $\delta(x)$ is admissible. The method is essentially that of Blyth [1]. Suppose, by way of contradiction, that $\delta(x)$ is not admissible. Then, there is an estimate $\delta^*(x)$, such that $\rho(\delta^*, \omega) \leqq \rho(\delta, \omega)$, for all $\omega \varepsilon \Omega$, and strict inequality for some ω. Now, $\rho(\delta^*, \omega)$ is continuous. Hence, for some $\epsilon > 0$, $\rho(\delta^*, \omega) < \rho(\delta, \omega) - \epsilon$, for ω belonging to some interval $(\underline{\omega}, \bar{\omega})$. Consider the quantity

$$[r(\delta) - r(\delta^*)]/[r(\delta) - r(\delta_\sigma)], \tag{7}$$

where $r(\delta)$, $r(\delta^*)$ are the average risks with respect to λ_σ, of the estimates $\delta(x)$ and $\delta^*(x)$, respectively.

The denominator of (7) is clearly non-negative. We will show that for some, sufficiently large values of σ, the ratio (7) > 1. This implies $r(\delta^*) < r(\delta_\sigma)$, a contradiction. Noting that $G(x - 1/\sigma)$ is an increasing function of σ, we have

$$r(\delta) - r(\delta_\sigma) \leqq \frac{\beta^2(0)}{\sigma} \int \left[\frac{1}{G(x - 1/\sigma)} - \frac{1}{G(x)}\right] d\mu(x) + \frac{1}{\sigma^2}. \tag{8}$$

To establish that the first term of (8) tends to zero as $\sigma \to \infty$, we notice that the integrand is positive, tends to zero as $\sigma \to \infty$, and, for $\sigma > 1$, is bounded by $1/G(x - 1)$. Since $1/G(x - 1)$ is integrable, the desired result follows by the Lebesgue dominated convergence theorem. This implies that $r(\delta) - r(\delta_\sigma)$ is $o(1/\sigma)$ as $\sigma \to \infty$. The numerator of (7) $> (\epsilon/\sigma) \int_{\underline{\omega}}^{\bar{\omega}} \exp(-\omega/\sigma)\, d\omega > K/\sigma$, where K is a positive constant. Thus, (7) $\geqq K/[\sigma o(1/\sigma)] \to \infty$ as $\sigma \to \infty$. For some σ, sufficiently large, (7) > 1. This is the required contradiction.

In conclusion we state the result more generally.

THEOREM 1: *If condition* (1) *is satisfied, then an admissible estimate for* $\varphi(\omega) = E_\omega\{x\}$ *is*

$$\delta(x) = x + \beta(a) \cdot [\exp(ax)]/[\textstyle\int_a^\infty \beta(\omega) \exp(\omega x)\, d\omega].$$

In order to investigate minimaxity, we rely upon the following theorem (c.f., Lehmann [6]). *Let* λ_σ, $\sigma > 0$, *be a set of distributions over* Ω. *Let* δ_σ *be the Bayes solution and* $r(\delta_\sigma)$ *the Bayes risk corresponding to* λ_σ. *If* $r(\delta_\sigma) \to r$, *as* $\sigma \to \infty$, *and* δ *is any estimate with* $\rho(\delta, \omega) \leqq r$, *then* δ *is minimax.*

Binomial: $\Omega = \{p \mid p \geqq a\}$ Straightforward calculation gives

$$\delta(x) = x + \frac{a^x(1 - a)^{n-x}}{\int_a^1 p^{x-1}(1 - p)^{n-x-1}\, dp}, \qquad x = 0, 1, \cdots n.$$

A simple example, $n = 1$, $a = \frac{1}{2}$, shows that minimaxity cannot be concluded by the cited theorem.

Poisson: $\Omega = \{\lambda \mid \lambda \geqq a\}$. This estimate is

$$\delta(x) = x + \frac{(na)^x e^{-na}}{n \int_a^\infty (n\lambda)^{x-1} e^{-na}\, d\lambda}, \qquad x = 0, 1, 2, \cdots.$$

Examining the condition for minimaxity, it is readily shown that every estimate is minimax. In the untruncated case, this is a well-known property of estimates, if the loss is square error.

Normal: $\Omega = \{\omega \mid \omega \geqq 0\}$. The estimate is

$$\delta(x) = x + \frac{\exp(-x^2/2)}{\int_{-\infty}^{x} \exp(-t^2/2)\, dt} = x + \nu(x).$$

The properties of ν are summarized in the following lemma.

LEMMA 1.: (i) $x + \nu(x) > 0$,
(ii) $\nu'(x) = -\nu(x)[x + \nu(x)]$,
(iii) $1 - \nu(x)[x + \nu(x)] = \delta'(x) \geqq 0$,
(iv) *ν is convex.*

PROOF: (i) This is obvious, from the way $\delta(x)$ was derived.
(ii) Differentiation readily gives this relationship.
(iii) and (iv) Sampford [7], has proved that the function $g(x) = F'(x)/[1 - F(x)]$, where $F(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{x} \exp(-t^2/2)\, dt$ is convex and $0 < g'(x) < 1$. We have that $\nu(x) = F'(x)/F(x) = g(-x)$. Therefore, $0 > \nu'(x) = -g'(x) > -1$, yielding (iii). Similarly $\nu''(x) = g''(-x) > 0$, all x, and thus ν is convex.

Instead of proving that the estimate is minimax, by the method of the previous examples, we proceed as follows.

A complete class of estimates for the risk function

$$\rho(\delta, \omega) = (2\pi)^{-\frac{1}{2}} \int_0^\infty [\delta(x) - \omega]^2 \exp[-\tfrac{1}{2}(x - \omega)^2]\, dx$$

is the collection of all monotone increasing functions. Let $\delta(x)$ be monotone increasing. We recall that $p_\omega(x) = 2\pi^{-\frac{1}{2}} \exp[-\frac{1}{2}(x - \omega)^2]$ has a monotone likelihood ratio, i.e., for $x_1 > x_2$ and $\omega^* > \omega$ we have $p_{\omega^*}(x_1)p_\omega(x_2) \geqq p\omega(x_1)p_{\omega^*}(x_2)$. Let $A = \rho(\delta, \omega^*) - \rho(\delta, \omega)$, where $\omega^* > \omega$. Applying (9), with $x_1 = x$, $x_2 = 0$, it follows that $A \geqq 0$. Thus $\rho(\delta, \omega)$ is monotone increasing in ω. The differential inequality procedure of Hodges and Lehmann [3] shows that a minimax procedure has risk $\leqq 1$. Since $\rho(x + \nu(x), \omega) = 1 - \omega E_\omega\{\nu(x)\}$ is monotone increasing for $\omega > 0$, and tends to 1 as $\omega \to \infty$, it follows that the estimate $x + \nu(x)$ is minimax. In the same way, it is easy to show that the estimate $\delta^* = \max[0, x]$, with

$$\rho(\delta^*, \omega) = 1 - (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{0} (x^2 - 2x\omega) \exp[-\tfrac{1}{2}(x - \omega)^2]\, dx,$$

is minimax.

We now take a second approach to Theorem 1. Theorem 2, below, gives a sufficient condition for admissibility a.e., in the nontruncated case. The proof is omitted. It is essentially embodied in a theorem by Karlin [4]. Theorem 3 generalizes Theorem 2 to the case of parameters in truncated spaces.

THEOREM 2: *$p_\omega(x)$ is a density with respect to μ, jointly measurable in x and ω. Ω is an interval with ends points $\underline{\omega}$, $\bar{\omega}$. $Q(\omega)$ is a positive measurable function on Ω.*

δ is an estimate with bounded risk. c is an interior point of Ω. If

$$\infty > \int_c^b \frac{1}{Q(\omega)}\, d\omega \to \infty \quad \text{as} \quad b \to \bar{\omega},$$

$$\infty > \int_a^c \frac{1}{Q(\omega)}\, d\omega \to \infty \quad \text{as} \quad a \to \underline{\omega},$$

and if there is a $K > 0$, such that

$$\left| \int_a^b [\ (x) - \varphi(\omega)]Q(\omega)p_\omega(x)\, d\omega \right| \leqq K[Q(b)p_b(x) + Q(a)p_a(x)],$$

for all $\underline{\omega} < a < b < \bar{\omega}$, and all x, then if δ^* is an estimate satisfying $\rho(\delta^*, \omega) \leqq \rho(\delta, \omega)$, all $\omega \in \Omega$, we have $\rho(\delta^*, \omega) = \rho(\delta, \omega)$, a.e. *(Lebesgue).*

THEOREM 3: *$p_\omega(x)$ is a density with respect to μ, jointly measurable in x and ω. Ω is an interval $[a, \bar{\omega})$. δ is an estimate with bounded risk. $Q(\omega)$ is a positive, measurable function on Ω, and*

$$\infty > \int_a^b \frac{1}{Q(\omega)}\, d\omega \to \infty \quad \text{as} \quad b \to \bar{\omega}.$$

If there is a $K > 0$ such that

$$\left| \int_a^b [\delta(x) - \varphi(\omega)]Q(\omega)p_\omega(x)\, d\omega \right| \leqq KQ(b)p_b(x)$$

for all $b \in (a, \bar{\omega})$, and all x, then if δ^* is an estimate satisfying

$$\rho(\delta^*, \omega) \leqq \rho(\delta, \omega), \qquad \text{all } \omega \in \Omega, \tag{10}$$

we have $$\rho(\delta^*, \omega) = \rho(\delta, \omega) \qquad \text{a.e.}$$

PROOF: (10) implies

$$\int_{-\infty}^{\infty} [\delta^*(x) - \delta(x)]^2 p_\omega(x)\, d\mu(x) \leqq 2 \int_{-\infty}^{\infty} [\delta(x) - \delta^*(x)][\delta(x) - \varphi(\omega)]p_\omega(x)\, d\mu(x).$$

Let $T(\omega) = \int_{-\infty}^{\infty} [\delta^*(x) - \delta(x)]^2 p_\omega(x)\, d\mu(x)$. Note $T(\omega)$ is measurable and finite.

$$\begin{aligned} \int_a^b T(\omega)Q(\omega)\, d\omega &\leqq 2 \int_a^b Q(\omega) \int_{-\infty}^{\infty} [\delta(x) - \delta^*(x)][\delta(x) - \varphi(\omega)]p_\omega(x)\, d\mu\, d\omega \\ &= 2 \int_{-\infty}^{\infty} [\delta(x) - \delta^*(x)] \int_a^b Q(\omega)[\delta(x) - \varphi(\omega)]p_\omega(x)\, d\omega\, d\mu(x) \\ &\leqq 2KQ(b) \int_{-\infty}^{\infty} |\delta(x) - \delta^*(x)| p_b(x)\, d\mu(x) \leqq 2KQ(b)[T(b)]^{\frac{1}{2}}. \end{aligned} \tag{11}$$

The last step follows by Schwarz' inequality.

We now show that $H(b) = \int_a^b T(\omega)Q(\omega)\, d\omega = 0$ for all $b \in (a, \bar{\omega})$. This implies that $T(\omega) = 0$ for almost all ω, further implying, $\rho(\delta^*, \omega) = \rho(\delta, \omega)$ for almost all ω, the desired result.

Suppose on the contrary, that there is a number $c \in (a, \bar{\omega})$, such that $H(c) > 0$.

Inequality (11) implies that

$$\frac{1}{4K^2}\cdot\frac{1}{Q(b)} \leqq \frac{Q(b)T(b)}{H^2(b)}, \qquad c \leqq b < \bar{\omega},$$

or that

$$\frac{1}{4K^2}\int_c^B \frac{1}{Q(b)}\,db \leqq \int_c^B \frac{Q(b)T(b)}{H^2(b)}\,db, \qquad c \leqq B < \bar{\omega}. \tag{12}$$

The left hand side of (12) approaches $+\infty$ as $B \to \bar{\omega}$. We now show that the right hand side is bounded as $B \to \bar{\omega}$, which gives a contradiction. Let

$$G(B) = \int_c^B \frac{Q(b)T(b)}{H^2(b)}\,db + \frac{1}{H(B)}, \quad \text{for} \quad c \leqq B < \bar{\omega}.$$

Then $G'(B)$ exists and is equal to 0, for almost all $B \varepsilon [c, \bar{\omega})$. Also, G is absolutely continuous on each interval $[c, d]$ where $c < d < \bar{\omega}$. Hence, G is constant on $[c, \bar{\omega})$, implying that the right hand side of (12) is $1/H(c) - 1/H(B)$, which remains bounded as $B \to \bar{\omega}$.

We deduce, quite simply, Theorem 1 from Theorem 3. Take $Q(\omega) \equiv 1$. Let $f(s) = [\beta(s) \exp (xs)]/[\int_s^\infty \beta(\omega) \exp (x\omega)\, d\omega]$. Then

$$\int_a^b [x + f(a) - \varphi(\omega)]\beta(\omega) \exp (x\omega)\, d\omega = \int_b^\infty \beta(\omega) \exp^{x\omega} d\omega[f(b) - f(a)]. \tag{13}$$

We have $f'(s) = f(s)[x - \varphi(s) + f(s)]$. We show that $f'(s) > 0$, that is, f is a (strictly) increasing function. This implies that (13) $< \beta(b) \exp (xb)$, the condition for admissibility a.e. Since φ is an increasing function

$$\varphi(s) < \frac{\int_s^\infty \varphi(\omega)\beta(\omega) \exp (\omega x)\, d\omega}{\int_s^\infty \beta(\omega) \exp (x\omega)\, d\omega} = x + f(s).$$

Thus $x + f(s) - \varphi(s) > 0$, proving $f'(s) > 0$.

As a further application of Theorem 3, consider the normal distribution with mean ω, $\omega > a$, and variance 1. Let λ be an arbitrary positive number, and $m = 1/(\lambda + 1)$. Let $F(x) = \int_{-\infty}^x \exp (-t^2/2)\, dt$. With $Q(\omega) = \beta(\omega)^\lambda = \exp (-\omega^2\lambda/2)$, it follows, in the same way as above, that the estimate

$$a + m(x - a) + m^{\frac{1}{2}}[F'((x - a)m^{\frac{1}{2}})]/[F((x - a)m^{\frac{1}{2}})]$$

is admissible.

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