THE NONPARAMETRIC ORDERING: (1001) → (0110)

BY JOHN S. WHITE

General Motors Research Laboratories

Let $Z = (Z_1, Z_2, \cdots, Z_N)$ be a random vector with $Z_i = 1(0)$ if the ith smallest in absolute value in a sample of N from the density $f(x)$ is positive (negative). Then

$$P(Z = z) = N! \int_{0 \leqq y_1 \leqq \cdots \leqq y_N} \prod_{i=1}^{N} [f^{1-z_i}(-y_i) f^{z_i}(y_i)\, dy_i].$$

In the case of normal slippage to the right (i.e., $f(x) = f(x, \mu)$ is $N(\mu, 1)$, $\mu > 0$), Savage [1] obtains a simple ordering of the 2^N possible values of Z for $N = 3$, namely:

$$111 \to 011 \to 101 \to 001 \to 110 \to 010 \to 100 \to 000,$$

where Prob $(Z = z) >$ Prob $(Z = z')$ if and only if $z \to z'$.

For $N = 4$, Savage gives a partial ordering of the 2^4 possible values of Z. The following theorem orders two more values of Z.

THEOREM 1: *If* X_1, X_2, X_3, X_4 *are* NID $(\mu, 1)$, with $\mu > 0$, *then*

$$D = \text{Prob}\,[Z = (1001)] - \text{Prob}\,[Z = (0110)] > 0.$$

PROOF:

$$D = \text{Prob}\,[Z = (1001)] - \text{Prob}\,[Z = (0110)]$$

$$= 4!/(2\pi)^2 \int_{0 \leqq y_1 \leqq \cdots \leqq y_4} \exp\left(-\tfrac{1}{2}\Sigma y^2 - 2\mu^2\right)\{\exp\left(\mu(y_4 - y_3 - y_2 + y_1)\right)$$

$$- \exp\left(-\mu(y_4 - y_3 - y_2 + y_1)\right)\} \prod dy_i .$$

$$D = 4!e^{-2\mu^2}/(2\pi)^2 \int_{0 \leqq y_1 \leqq \cdots \leqq y_4} 2 \sinh \mu(y_4 - y_3 - y_2 + y_1) e^{-\Sigma^2_{y_i}/2} \prod dy_i .$$

Now make the transformation $y_i = \sum_{j=1}^{i} w_j$. The Jacobian is 1 and the region of integration becomes $0 \leqq w_i \leqq \infty$, $i = 1, \cdots, 4$. Hence,

$$D = c \iiiint_0^\infty \sinh \mu(w_4 - w_2) \exp\left(-\Sigma(\Sigma w^2)/2\right) \prod dw_i ,$$

where

$$c = 2 \cdot 4! e^{-2\mu^2}/(2\pi)^2.$$

Noting that the integral is positive for $w_4 > w_2$ and negative for $w_2 < w_4$, break up the region of integration into two parts in two different ways as follows

Received July 1, 1960.

$$D = \int_{w_1=0}^{\infty}\int_{w_3=0}^{\infty}\int_{w_2=0}^{\infty}\int_{w_4=w_2}^{\infty} + \int_{w_1=0}^{\infty}\int_{w_3=0}^{\infty}\int_{w_2=0}^{\infty}\int_{w_4=0}^{w_2} = (say)\ D_1^+ + D_1^-,$$

$$D = \int_{w_1=0}^{\infty}\int_{w_3=0}^{\infty}\int_{w_4=0}^{\infty}\int_{w_2=0}^{w_4} + \int_{w_1=0}^{\infty}\int_{w_3=0}^{\infty}\int_{w_4=0}^{\infty}\int_{w_2=w_4}^{\infty} = (say)\ D_2^+ + D_2^-.$$

Combining these two results, D may be written as $D = D_1^+ + D_2^-$. Setting $w_2 = s$ and $w_4 = t$ in D_1^+ and $w_2 = t$, $w_4 = s$ in D_2^- gives

$$D = c\int_0^\infty\int_0^\infty\int_0^\infty\int_s^\infty \sinh \mu(t-s)\exp(-\tfrac{1}{2}\{(w_1^2 + (w_1+s)^2 + (w_1+s+w_3)^2 + (w_1+s+w_3+t)^2)\})\,dt\,ds\,dw_3\,dw_1$$
$$- c\int_0^\infty\int_0^\infty\int_0^\infty\int_s^\infty \sinh \mu(t-s)\exp(-\tfrac{1}{2}\{(w_1^2 + (w_1+t)^2 + (w_1+t+w_3)^2 + (w_1+t+w_3+s)^2)\})\,dt\,ds\,dw_3\,dw_1,$$

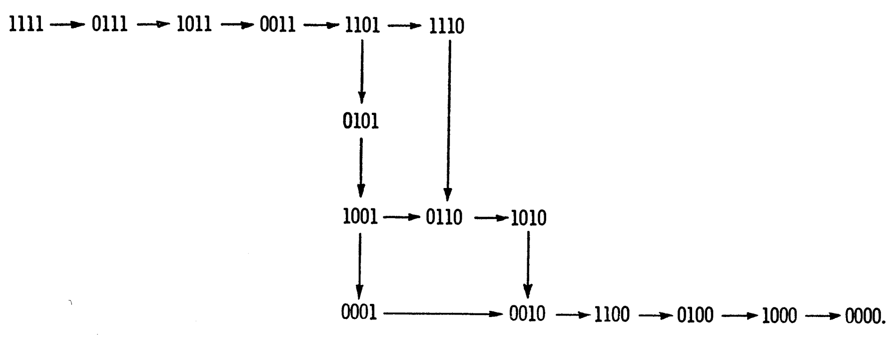
$$D = c\int_0^\infty\int_0^\infty\int_0^\infty\int_s^\infty \sinh \mu(t-s)\exp(-\tfrac{1}{2}\{(w_1^2 + (w_1+s+w_3+t)^2)\})$$
$$[\exp(-\tfrac{1}{2}\{(w_1+s)^2 + (w_1+s+w_3)^2\}) - \exp(-\tfrac{1}{2}\{(w_1+t)^2 + (w_1+t+w_3)^2\})]\,dt\,ds\,dw_3\,dw_1.$$

Thus D is positive if the difference of the exponentials in the square brackets is positive, but this is obviously true since $t > s$ over the range of integration.

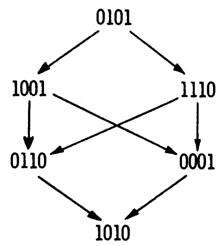
THEOREM 2: *If* $X_1, \cdots, X_N$ $(N \geqq 4)$ *are* NID $(\mu, 1)$ *with* $\mu > 0$, *then* $D = \text{Prob}(Z = z) - \text{Prob}(Z = z') > 0$ *where* z *and* z' *are identical except that* $z_1 = z_2' = z_3' = z_4 = 1$ *and* $z_1' = z_2 = z_3 = z_4' = 0$.

PROOF: The proof of this Theorem follows from Theorem 1 in exactly the same way that Savage's Theorem 6.1 follows from Sobel's Theorem ([1], footnote p. 1024).

Using the results of Theorem 1 and Savage's Theorem 6.1, the partial ordering for $N = 4$ becomes, for normal slippage to the right,



Some preliminary Monte Carlo analysis suggests that the following orderings may be valid, although no analytic proofs are available:



REFERENCE

[1] I. Richard Savage, "Contributions to the theory of rank order statistics—the one-sample case," *Ann. Math. Stat.*, Vol. 30 (1959), pp. 1018–1023.