A COMBINATORIAL LEMMA FOR COMPLEX NUMBERS[1]

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Although combinatorial lemmas have been used quite successfully in analyzing sums of random variables [1, 2], to the best of our knowledge these considerations have been restricted to the case of real numbers and real variables. It is our purpose in this note to show by a simple example that combinatorial lemmas for complex numbers can also be given and applied to analyzing random walks in the plane.

1. Random walks in the plane. Let $\{Z_k\}$ be a sequence of independent, identically distributed complex-valued random variables. Let $S_0 = 0$, and let $S_n = Z_1 + \cdots + Z_n$, $n \geqq 1$. We call $S_0, S_1, \cdots, S_n, \cdots$ a *random walk in the plane*. The combinatorial lemmas given below are concerned with the *convex hull* of the random walk. Specifically, every walk $S_0, \cdots, S_n$ ($n + 1$ points in the plane) determines a smallest closed, convex set containing these points. The boundary of this set is called the (convex) hull[2] of $S_0, \cdots, S_n$. Later, we will be concerned with three properties of the hull of a walk. We list these properties in the form of variables for later reference.

(1)

K_n: the number of variables $Z_1, \cdots, Z_n$ which are line segments in the hull of $S_0, \cdots, S_n$,

H_n: the number of line segments (sides) in the hull of $S_0, \cdots, S_n$,

L_n: the length of the hull of $S_0, \cdots, S_n$.

2. Combinatorics. Let $z_1, z_2, \cdots, z_n$ be a set of n complex numbers and let $s_k = z_1 + \cdots + z_k$. If σ: $i_1, i_2, \cdots, i_n$ is any permutation of $1, 2, \cdots, n$, we let $s_k(\sigma) = z_{i_1} + \cdots + z_{i_k}$. The notation $\vec{z}_A$ will represent the sum of the vectors in a subset A of $z_1, \cdots, z_n$ while z_A will denote the (non-directed) line segment corresponding to $\vec{z}_A$. We need an important definition which seems to be the natural analogue of "rational independence" for real numbers.

Definition. Let $z_1, \cdots, z_n$ be complex numbers with partial sums $s_0, \cdots, s_n$. We say the vectors $z_1, \cdots, z_n$ are *skew* if z_A is parallel to z_B only when $A = B$.

Every vector z in the plane, when extended along its length, determines two half-planes which we call the right and left half-planes of z, respectively. We include the line itself in both of the half-planes.

Lemma 1. *Let $z_1, \cdots, z_n$ be skew vectors with sum z. Then, there exists exactly*

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[2] Some authors call this the boundary of the hull.

one cyclic permutation σ of $z_1, \cdots, z_n$ such that the points $s_0(\sigma)$, $s_1(\sigma)$, $\cdots$ $s_n(\sigma)$ all lie in the left (*right*) *half-plane of z.* (See Fig. 1).

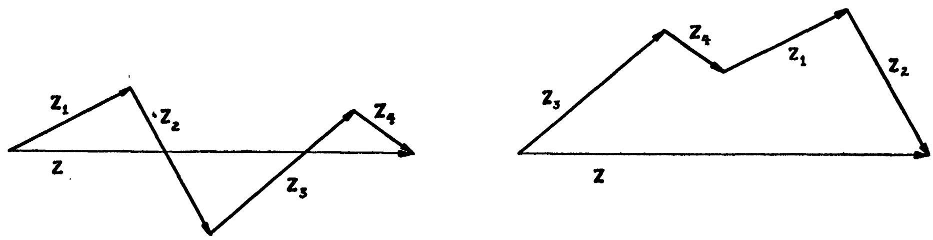

FIG. 1

PROOF. Since $z_1, \cdots, z_n$ are skew vectors, there is at most one point among $s_0, \cdots, s_{n-1}$ in the right half-plane of z which is a maximum distance (possibly zero) from the line determined by z. If s_k is this point ($k = 0, 1, \cdots, n - 1$), we take $\sigma: k + 1, \cdots, n, 1, \cdots, k$. The uniqueness of σ follows from the uniqueness of the index k. Note that among all $n!$ permutations of $z_1, \cdots, z_n$ exactly $(n - 1)!$ are such that $s_0(\sigma), \cdots, s_n(\sigma)$ lie in the left-half plane of z.

Let $z_1, \cdots, z_n$ be a fixed set of skew vectors. Every permutation σ determines a "path" $s_0(\sigma)$, $s_1(\sigma)$, $\cdots$, $s_n(\sigma)$. Since each line segment of the hull of this path connects two points of the path, each line segment of the hull is a sum of a subset of the vectors $z_1, \cdots, z_n$. Moreover, this subset uniquely determines the line segment. The next lemma tells us how often a particular segment is likely to appear in the hull of a path. To avoid having to adopt a convention for degenerate polygons when $n = 1$, we will assume that $n \geqq 2$ from now on.

LEMMA 2. *Let $z_1, \cdots, z_n$ be fixed skew vectors and let A be a fixed subset of m of these vectors. Then, the line segment z_A appears in the hull of exactly*

$$2(m - 1)!(n - m)!$$

of the $n!$ paths $s_0(\sigma), \cdots, s_n(\sigma)$ as σ ranges over all permutations.

PROOF. Let $z_{n+1} = -s_n$ and let A' denote the complement of A in $(z_1, \cdots, z_n, z_{n+1})$. We call $s_0(\sigma), \cdots, s_n(\sigma), s_{n+1}(\sigma) = s_0(\sigma)$ the completed path associated with $z_{i_1}, \cdots, z_{i_n}$. In order that z_A (or equivalently $z_{A'}$) appears in the hull of $s_0(\sigma), \cdots, s_n(\sigma)$, it is necessary that $\vec{z}_A = s_{k+m}(\sigma) - s_k(\sigma)$ for some k. We can thus think of any completed path $s_0(\sigma), \cdots, s_{n+1}(\sigma)$ whose hull contains z_A as subdivided naturally into two ordered sets of vectors, $(z_{i_{k+1}}, \cdots, z_{i_{k+m}})$ and $(z_{i_{k+m+1}}, \cdots, z_{i_n}, z_{n+1}, \cdots, z_{i_k})$. The paths corresponding to each of these ordered sets of vectors must lie in the same half-plane of $\vec{z}_A$. Moreover any ordering of the vectors in A and A' subject to the condition that their paths lie in the same half-plane of $\vec{z}_A$ gives rise to a completed path $s_0(\sigma), \cdots, s_{n+1}(\sigma)$, the origin (and hence the value of k) being determined from the position of z_{n+1} in the ordering of A'. Thus, we need only to count how many different pairs of orderings of A and A' there are such that both subpaths lie in the same half-plane of $\vec{z}_A$. From Lemma 1 we find that there are $(m - 1)!(n - m)!$ ways of ordering A and A' so that the subpaths *both* lie in the left half-plane of $\vec{z}_A$. Taking into account also the right half-plane of $\vec{z}_A$ the proof is completed.

3. Application to random walks in the plane. In the applications $Z_k = X_k + iY_k$, where X_k and Y_k are real-valued random variables with a joint density function. This implies that, with probability one, $Z_1, \cdots, Z_n$ are skew vectors. If σ: $i_1, \cdots, i_n$ is a permutation of $1, \cdots, n$, then $K_n(\sigma)$, $H_n(\sigma)$, and $L_n(\sigma)$ are defined as in (1) in terms of the sums $S_0(\sigma), \cdots, S_n(\sigma)$ of the permuted vectors $Z_{i_1}, \cdots, Z_{i_n}$.

EXAMPLE 1.

Expectation of K_n . By the identical distribution property, $E\{K_n\} = E\{K_n(\sigma)\}$ for any permutation σ. Thus,

$$n!E\{K_n\} = E\{\sum_{(\sigma)} K_n(\sigma)\}. \tag{2}$$

For any skew vector values of $Z_1, \cdots, Z_n$, the summation on the right in (2) equals the total number of times than any of the n possible one point sets $A = \{Z_k\}$ determines a segment Z_A in the hull of $S_0(\sigma), \cdots, S_n(\sigma)$ as σ ranges over all permutations. This means

$$\sum_{\sigma} K_n(\sigma) = 2 \sum_{m=1}^{n} (n - 1)! = 2n!. \tag{3}$$

Thus, we expect to find exactly 2 of the vectors $Z_1, \cdots, Z_n$ as line segments in the convex hull of $S_0, \cdots, S_n$. We note in passing that (3) is a *universal* relation, valid for any values of the skew vectors.

EXAMPLE 2.

Expectation of H_n . Once again we have $E\{H_n\} = E\{H_n(\sigma)\}$ for every permutation σ. Thus,

$$n!E\{H_n\} = E\{\textstyle\sum_{(\sigma)} H_n(\sigma)\}. \tag{4}$$

For skew vector values of $Z_1, \cdots, Z_n$ the summation on the right in (4) is equal to the total number of lines in the $n!$ hulls of the paths $S_0(\sigma), \cdots, S_n(\sigma)$ as σ ranges over all permutations. Equivalently, from Lemma 2

$$\begin{aligned} \sum_{\sigma} H_n(\sigma) &= \sum_{A} 2(m - 1)!(n - m)! \\ &= 2 \sum_{m=1}^{n} (m - 1)!(n - m)! \binom{n}{m} \\ &= 2n! \sum_{m=1}^{n} 1/m. \end{aligned} \tag{5}$$

Finally,

$$E\{H_n\} = 2 \sum_{m=1}^{n} 1/m \cong 2 \log n. \tag{6}$$

Once again we note that (5) is a *universal* relation valid for any sequence of skew vectors.

EXAMPLE 3.

Expectation of L_n. (Spitzer and Widom [3])[3]. It is easy to see that

$$n!E\{L_n\} = E\{\textstyle\sum_{(\sigma)} L_n(\sigma)\}. \tag{7}$$

By an argument similar to that leading to (5), we find

$$\textstyle\sum_{(\sigma)} L_n(\sigma) = \sum_A 2(m-1)!(n-m)!|\vec{Z}_A|. \tag{8}$$

Thus,

$$\begin{aligned} E\{L_n\} &= \sum_A 2(m-1)!(n-m)!E\{|\vec{Z}_A|\}/n! \\ &= \sum_{m=1}^{n} 2(m-1)!(n-m)!\binom{n}{m} E\{|S_m|\}/n! \\ &= \sum_{m=1}^{n} E\{|S_m|\}/m. \end{aligned}$$

REFERENCES

[1] ERIK SPARRE ANDERSEN, "On the fluctuations of sums of random variables," *Math. Scand.*, Vol. 1 (1953), pp. 263–285.

[2] FRANK SPITZER, "A combinatorial lemma and its application to probability theory," *Trans. Amer. Math. Soc.*, Vol. 82 (1956), pp. 323–339.

[3] F. SPITZER AND H. WIDOM, "The circumference of a convex polygon," *Proc. Amer. Math. Soc.*, Vol. 12 (1961), pp. 506–509.

[3] By a limiting argument which we could also employ in this example Spitzer and Widom remove the condition that $Z_k = X_k + iY_k$ have a density.

A COMBINATORIAL DERIVATION OF THE DISTRIBUTION OF THE TRUNCATED POISSON SUFFICIENT STATISTIC[1]

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Let $X_1, \cdots, X_n$ be independently distributed with the Poisson distribution truncated away from zero, i.e.,

$$P(x) = \frac{e^{-\lambda}}{1-e^{-\lambda}}\frac{\lambda^x}{x!}, \qquad x = 1, 2, \cdots\cdots. \tag{1}$$

Tate and Goen showed [2] that $T = \sum_{i=1}^n X_i$ has the distribution

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