ESTIMATING THE PARAMETERS OF NEGATIVE EXPONENTIAL POPULATIONS FROM ONE OR TWO ORDER STATISTICS

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0. Summary. This paper discusses the use of order statistics in estimating the parameters of (negative) exponential populations. For the one-parameter exponential population, the best linear unbiased estimators, $\tilde{\sigma}_k = c_k x_k$ and $\tilde{\sigma}_{lm} = c_l x_l + c_m x_m$, of the parameter σ are given, based on one order statistic x_k and on two order statistics x_l and x_m. For samples of any size up through $n = 100$, a table is given of k, l, and m and of the coefficients c_k, c_l, and c_m, together with the coefficients of σ^2 in the variances V_k and V_{lm} of the estimators, and the corresponding efficiencies E_k and E_{lm} (relative to the best linear unbiased estimator based on all order statistics). For the two-parameter exponential population, the best linear unbiased estimators, $\tilde{\alpha} = c_{\alpha l} x_l + c_{\alpha m} x_m$, $\tilde{\sigma} = c_{\sigma l} x_l + c_{\sigma m} x_m$, and $\tilde{\mu} = c_{\mu l} x_l + c_{\mu m} x_m$, of the parameters α and σ and the mean $\mu = \alpha + \sigma$ are given, based on two order statistics x_l and x_m. For samples of any size up through $n = 100$, a table is given of m (l is always 1 for the best estimator) and of factors c_α and c_σ for computing the coefficients $c_{\alpha l} = 1 + c_\alpha$, $c_{\alpha m} = -c_\alpha$, $c_{\sigma l} = -c_\sigma$, $c_{\sigma m} = c_\sigma$, $c_{\mu l} = 1 + c_\alpha - c_\sigma$, and $c_{\mu m} = c_\sigma - c_\alpha$, together with the coefficients of σ^2 in the variances $V_{\tilde{\alpha}}$, $V_{\tilde{\sigma}}$, and $V_{\tilde{\mu}}$ of the estimators, and the corresponding efficiencies $E_{\tilde{\alpha}}$, $E_{\tilde{\sigma}}$, and $E_{\tilde{\mu}}$ (relative to the best linear unbiased estimators based on all order statistics).

1. Introduction. Since the publication, in 1946 and 1948 respectively, of papers by Mosteller [4] and by Wilks [10], a great deal of attention has been given to the use of order statistics in various statistical procedures, including the estimation of the parameters of various populations. Among the first to use this method for exponential populations was Halperin [2] in 1952. Since that time, Epstein and Sobel [1], Sarhan [5], [6], Sarhan and Greenberg [7], [8], and Sarhan, Greenberg, and Ogawa [9] have discussed various aspects of the use of order statistics in the estimation of the parameters of exponential populations. Most of these authors have considered best linear unbiased estimators based on all order statistics, or, in the case of truncated or censored samples, on all available order statistics. The last of the above papers includes simplified estimators based on two order statistics, with tables for samples of any size up through $n = 20$. The present paper contains more accurate tables for samples of any size up through $n = 100$, not only for estimators based on two order statistics, but also, in the case of the one-parameter exponential population, for estimators based on one order statistic.

In a previous paper, the author [3] studied estimators of the standard deviation

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of normal, rectangular, and one-parameter exponential populations based on sample ranges and quasi-ranges. The results were quite satisfactory for the symmetric populations, but not for the one-parameter exponential, for which it was found that estimators based on a single order statistic are more efficient. That discovery led to the further research reported in this paper.

2. Estimators of σ for the One-Parameter Exponential Population.

2.1. *Estimators based on one order statistic.* For the one-parameter exponential population with parameter σ, which is both the mean and the standard deviation of the population, the probability density function $f_1(x)$ is equal to $(1/\sigma)\exp(-x/\sigma)$ for $0 \leqq x < \infty$, and zero elsewhere. The sample mean $\bar{x}$, which has variance σ^2/n, is the minimum variance unbiased estimator, and also the maximum likelihood estimator, of the parameter σ. The expected value and the variance of the kth order statistic, x_k, of a sample of size n from this population are given (see Epstein and Sobel [1]) by

$$E(x_k) = \sigma \sum_1^k a_i \tag{1}$$

and

$$\operatorname{var} x_k = \sigma^2 \sum_1^k a_i^2, \tag{2}$$

where $a_i = 1/(n - i + 1)$. An unbiased estimator of the parameter σ, based on the order statistic x_k, is given by $\tilde{\sigma}_k = c_k x_k$, where

$$c_k = 1 \Big/ \sum_1^k a_i . \tag{3}$$

The variance of this estimator is given by

$$V_k = \sigma^2 \sum_1^k a_i^2 \Big/ \left(\sum_1^k a_i\right)^2, \tag{4}$$

and its efficiency (relative to the minimum variance unbiased estimator $\bar{x}$) is $E_k = \operatorname{var} \bar{x}/V_k$, where, as mentioned above, $\operatorname{var} \bar{x} = \sigma^2/n$. Thus the relative efficiency E_k is given by

$$E_k = \left(\sum_1^k a_i\right)^2 \Big/ \left(n \sum_1^k a_i^2\right). \tag{5}$$

The best estimator of σ, based on one order statistic x_k, is the one for that value of k which minimizes V_k (maximizes E_k). The author is not aware of any analytical method for determining the value of k which yields the best estimator of σ; hence, for each value of n, V_k was computed for $k = 1(1)n$. When the best value of k for a given n had been found, the corresponding c_k and E_k were also computed. The computations were performed on the IBM 1620 computer. Table 1 gives, for $n = 1(1)100$, the value of k for the best estimator of σ, the coefficient c_k (to 6 significant figures), the coefficient, V_k/σ^2, of σ^2 in the variance V_k of the estimator (to 7 significant figures or 6 decimal places, whichever is less accurate), and the relative efficiency E_k (to 5 significant figures). The tabular values of c_k, V_k/σ^2, and E_k are accurate to within a unit in the last place given.

2.2. *Estimators based on two order statistics.* Unbiased linear estimators of the parameter σ, based on two order statistics x_l and x_m, are given by $\tilde{\sigma}_{lm} = c_l x_l + c_m x_m$, where $c_l E(x_l) + c_m E(x_m) = \sigma$, with $E(x_l)$ and $E(x_m)$ given by equation (1), if k takes the values l and m. The variance of such an estimator is given by

$$V_{lm} = c_l^2 \operatorname{var} x_l + c_m^2 \operatorname{var} x_m + 2c_l c_m \operatorname{cov}(x_l, x_m), \tag{6}$$

where var x_l and var x_m are given by equation (2), if k takes the values l and m, and $\operatorname{cov}(x_l, x_m)$ is given (see Sarhan [5]) by

$$\operatorname{cov}(x_l, x_m) = \sigma^2 \sum_1^l a_i^2 = \operatorname{var} x_l, \qquad (l < m). \tag{7}$$

It can be shown that, for given values of l and m, the values of c_l and c_m which yield the unbiased estimator $\tilde{\sigma}_{lm}$ with minimum variance are

$$c_l = 1 \Big/ \left(\sum_1^l a_i + \lambda \sum_1^m a_i \right) \tag{8}$$

and

$$c_m = \lambda c_l, \tag{9}$$

where

$$\lambda = \sum_{l+1}^m a_i \sum_1^l a_i^2 \Big/ \left(\sum_1^l a_i \sum_1^m a_i^2 - \sum_1^m a_i \sum_1^l a_i^2 \right). \tag{10}$$

By substituting from equations (2) and (7)–(9) into equation (6), one finds that the minimum variance of $\tilde{\sigma}_{lm}$, for given l and m, is

$$V_{lm} = \sigma^2 \left[(1 + 2\lambda) \sum_1^l a_i^2 + \lambda^2 \sum_1^m a_i^2 \right] \Big/ \left(\sum_1^l a_i + \lambda \sum_1^m a_i \right)^2. \tag{11}$$

The efficiency of this estimator (relative to the minimum variance unbiased estimator $\bar{x}$) is $E_{lm} = \operatorname{var} \bar{x}/V_{lm}$, where, as before, $\operatorname{var} \bar{x} = \sigma^2/n$. Thus the relative efficiency E_{lm} is given by

$$E_{lm} = \left(\sum_1^l a_i + \lambda \sum_1^m a_i \right)^2 \Big/ n \left[(1 + 2\lambda) \sum_1^l a_i^2 + \lambda^2 \sum_1^m a_i^2 \right]. \tag{12}$$

The best estimator of σ, based on two order statistics x_l and x_m, is the one for those values of l and m which minimize V_{lm} (maximize E_{lm}). The author is not aware of any analytical method for determining these values of l and m; hence, for each value of n, V_{lm} was computed for $l = 1(1)(n-1)$ and $m = (l+1)(1)n$. When the best values of l and m for a given n had been found, the corresponding c_l, c_m, and E_{lm} were also computed. The computations were performed on the IBM 1620 computer. Table 1 gives, for $n = 2(1)100$, the values of l and

m for the best estimator of σ, the coefficients c_l and c_m (to 6 significant figures), the coefficient, V_{lm}/σ^2, of σ^2 in the variance V_{lm} of the estimator (to 7 decimal places), and the relative efficiency E_{lm} (to 5 significant figures). The tabular values of c_l, c_m, V_{lm}/σ^2, and E_{lm} are accurate to within a unit in the last place given.

3. Estimators or Parameters α, σ and Mean μ for the Two-Parameter Exponential Population from Two Order Statistics. For the two-parameter exponential population with parameters α and σ, the probability density function $f_2(x)$ is $(1/\sigma)\exp[-(x - \alpha)/\sigma]$ for $\alpha \leqq x < \infty$, and zero elsewhere. The mean of this population, which will be denoted by μ, is equal to $\alpha + \sigma$. For a sample of size n from this population, the expected value of the kth order statistic, x_k, exceeds by α the value given in equation (1) for the one-parameter exponential population, and thus is given by

$$E(x_k) = \alpha + \sigma \sum_1^k a_i . \tag{13}$$

The variance of x_k and the covariance of x_l and x_m are the same as for the one-parameter exponential population, and hence are given by equations (2) and (7), respectively.

Unbiased linear estimators of the parameters α and σ and the mean μ may be obtained from any two order statistics x_l and x_m. These estimators are of the form

$$\tilde{\alpha} = c_{\alpha l} x_l + c_{\alpha m} x_m , \tag{14}$$

$$\tilde{\sigma} = c_{\sigma l} x_l + c_{\sigma m} x_m , \tag{15}$$

and

$$\tilde{\mu} = c_{\mu l} x_l + c_{\mu m} x_m . \tag{16}$$

It has been shown (see Sarhan, Greenberg, and Ogawa [9]) that, for given l and m, the coefficients in the best linear estimators based on two order statistics x_l and x_m are given by

$$c_{\alpha l} = 1 + c_\alpha , \; c_{\alpha m} = -c_\alpha , \tag{17}$$

$$c_{\sigma l} = -c_\sigma , \; c_{\sigma m} = c_\sigma , \tag{18}$$

and

$$c_{\mu l} = 1 + c_\alpha - c_\sigma , \; c_{\mu m} = c_\sigma - c_\alpha , \tag{19}$$

where

$$c_\alpha = \sum_1^l a_i \Big/ \sum_{l+1}^m a_i \tag{20}$$

and

$$c_\sigma = 1 \Big/ \sum_{l+1}^m a_i . \tag{21}$$

The variance of the estimator $\tilde{\sigma}$ is given by

$$V_{\tilde{\sigma}} = \sigma^2 \sum_{l+1}^{m} a_i^2 \Big/ \left(\sum_{l+1}^{m} a_i\right)^2, \tag{22}$$

and the variances of the estimators $\tilde{\alpha}$ and $\tilde{\mu}$ are given by

$$\begin{aligned} V_{\tilde{\alpha}} &= \sigma^2 \sum_{1}^{l} a_i^2 + \left(\sum_{1}^{l} a_i\right)^2 V_{\tilde{\sigma}} \\ &= \sigma^2 \left[\sum_{1}^{l} a_i^2 + \left(\sum_{1}^{l} a_i\right)^2 \sum_{l+1}^{m} a_i^2 \Big/ \left(\sum_{l+1}^{m} a_i\right)^2\right] \end{aligned} \tag{23}$$

and

$$\begin{aligned} V_{\tilde{\mu}} &= \sigma^2 \sum_{1}^{l} a_i^2 + \left(\sum_{1}^{l} a_1 - 1\right)^2 V_{\tilde{\sigma}} \\ &= \sigma^2 \left[\sum_{1}^{l} a_i^2 + \left(\sum_{1}^{l} a_i - 1\right)^2 \sum_{l+1}^{m} a_i^2 \Big/ \left(\sum_{l+1}^{m} a_i\right)^2\right]. \end{aligned} \tag{24}$$

The best linear unbiased estimators of α, σ, and μ based on all order statistics (see Sarhan and Greenberg [7]) are

$$\hat{\alpha} = \left[(n^2 - 1)x_1 - \sum_{2}^{n} x_i\right] \Big/ [n(n - 1)], \tag{25}$$

$$\hat{\sigma} = \left[\sum_{2}^{n} x_i - (n - 1)x_1\right] \Big/ (n - 1), \tag{26}$$

and

$$\hat{\mu} = \sum_{1}^{n} x_i/n = \bar{x}. \tag{27}$$

These are also the maximum likelihood estimators. Their variances are

$$V_{\hat{\alpha}} = \sigma^2/[n(n - 1)], \tag{28}$$

$$V_{\hat{\sigma}} = \sigma^2/(n - 1), \tag{29}$$

and

$$V_{\hat{\mu}} = \sigma^2/n. \tag{30}$$

The efficiencies of the estimators $\tilde{\alpha}$, $\tilde{\sigma}$, and $\tilde{\mu}$ (relative to the best linear unbiased estimators $\hat{\alpha}$, $\hat{\sigma}$, and $\hat{\mu}$ based on all order statistics) are given by

$$\begin{aligned} E_{\tilde{\alpha}} = V_{\hat{\alpha}}/V_{\tilde{\alpha}} = \left(\sum_{l+1}^{m} a_i\right)^2 \Big/ n(n - 1) \\ \cdot \left[\sum_{1}^{l} a_i^2 \left(\sum_{l+1}^{m} a_i\right)^2 + \left(\sum_{1}^{l} a_i\right)^2 \sum_{l+1}^{m} a_i^2\right], \end{aligned} \tag{31}$$

$$E_{\tilde{\sigma}} = V_{\hat{\sigma}}/V_{\tilde{\sigma}} = \left(\sum_{l+1}^{m} a_i\right)^2 \Big/ (n - 1) \sum_{l+1}^{m} a_i^2, \tag{32}$$

and

$$(33) \qquad E_{\tilde{\mu}} = V_{\hat{\mu}}/V_{\tilde{\mu}} = \left(\sum_{l+1}^{m} a_i\right)^2 \Big/ n\left[\sum_{1}^{l} a_i^2 \left(\sum_{l+1}^{m} a_i\right)^2 + \left(\sum_{1}^{l} a_i - 1\right)^2 \sum_{l+1}^{m} a_i^2\right].$$

The best estimators of α, σ, and μ, based on two order statistics x_l and x_m, are those for the values of l and m which minimize $V_{\tilde{\alpha}}$, $V_{\tilde{\sigma}}$, and $V_{\tilde{\mu}}$ (maximize $E_{\tilde{\alpha}}$, $E_{\tilde{\sigma}}$, and $E_{\tilde{\mu}}$). It can be shown that, for a fixed value of m, the variances of the estimators are smallest when $l = 1$. It can be seen from equations (23) and (24) that, for a fixed value of l, the value of m which minimizes $V_{\tilde{\sigma}}$ also minimizes $V_{\tilde{\alpha}}$ and $V_{\tilde{\mu}}$. The author is not aware of any purely analytical method of determining the best value of m; hence, for each value of n, $V_{\tilde{\sigma}}$ was computed for $l = 1$ and $m = 2(1)n$. When the best value of m for a given n had been found, the corresponding c_α, c_σ, $V_{\tilde{\alpha}}$, $V_{\tilde{\mu}}$, $E_{\tilde{\alpha}}$, $E_{\tilde{\sigma}}$, and $E_{\tilde{\mu}}$ were also computed. The computations were performed on the IBM 1620 computer. Table 2 gives, for $n = 2(1)100$, the value of m for the best estimators of α, σ, and μ, the factors c_α and c_σ (to 6 significant figures or 6 decimal places, whichever is less accurate), the coefficient, $V_{\tilde{\alpha}}/\sigma^2$, of σ^2 in the variance $V_{\tilde{\alpha}}$ (to 7 significant figures or 9 decimal places, whichever is less accurate), the coefficients, $V_{\tilde{\sigma}}/\sigma^2$ and $V_{\tilde{\mu}}/\sigma^2$, of σ^2 in the variances $V_{\tilde{\sigma}}$ and $V_{\tilde{\mu}}$ (to 7 significant figures or 7 decimal places, whichever is less accurate), and the relative efficiencies $E_{\tilde{\alpha}}$, $E_{\tilde{\sigma}}$, and $E_{\tilde{\mu}}$ (to 5 significant figures). The tabular values of c_α, c_σ, $V_{\tilde{\alpha}}/\sigma^2$, $V_{\tilde{\sigma}}/\sigma^2$, $V_{\tilde{\mu}}/\sigma^2$, $E_{\tilde{\alpha}}$, $E_{\tilde{\sigma}}$, and $E_{\tilde{\mu}}$ are accurate to within a unit in the last place given.

4. Remarks.

(i) The variance of the best estimator of σ for the two-parameter exponential population based on two order statistics x_1 and x_m from a sample of size n is the same as the variance of the best estimator of σ for the one-parameter exponential population based on one order statistic x_k from a sample of size $n - 1$, with $k = m - 1$. This can be seen by a comparison of equations (4) and (22), though the author did not observe this fact until confronted with equal numerical values. The relative efficiencies of these estimators are also equal, since in each case the variance of the best linear unbiased estimator based on all order statistics is $\sigma^2/(n - 1)$.

(ii) For the one-parameter exponential population, the values $k = 80$, $l = 64$, and $m = 93$, with coefficients $c_k = 0.629074$, $c_l = 0.522657$, and $c_m = 0.181399$ and relative efficiencies $E_k = 65.093\%$ and $E_{lm} = 82.460\%$, for $n = 100$ may be compared with the results obtained by Sarhan, Greenberg, and Ogawa [9], whose corresponding asymptotic values are $0.7968n$, $0.6386n$, $0.9266n$, 0.6275, 0.5232, 0.1790, 64.76%, and 82.03%.

(iii) For the two-parameter exponential population, it can be seen from equations (20) and (21) that, since $l = 1$ for the best estimators, $c_\alpha = a_1 c_\sigma = c_\sigma/n$. For convenience, however, separate columns for c_α and c_σ are given in Table 2.

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TABLE 1

Best Estimators of Parameter μ of 1-Parameter Exponential Population

n	From 1 Order Statistic, x_k				From 2 Order Statistics, x_l and x_m					
	k	c_k	V_k/σ^2	$E_k(\%)$	l	m	c_l	c_m	V_{lm}/σ^2	$E_{lm}(\%)$
1	1	1.00000	1.000000	100.00						
2	2	.666667	.5555556	90.000	1	2	.500000	.500000	.5000000	100.00
3	3	.545455	.4049587	82.313	2	3	.447368	.342105	.3421053	97.436
4	4	.480000	.3280000	76.220	3	4	.413043	.265217	.2652174	94.262
5	5	.437956	.2807289	71.243	3	5	.527997	.256818	.2140154	93.451
6	5	.689655	.2337165	71.311	4	6	.493892	.216654	.1805452	92.313
7	6	.627803	.2017178	70.820	5	7	.467528	.188618	.1571815	90.887
8	7	.582121	.1787245	69.940	5	8	.553626	.187760	.1393976	89.672
9	8	.546756	.1613595	68.859	6	9	.527062	.167990	.1247200	89.088
10	8	.699806	.1468046	68.118	7	10	.505032	.152501	.1132201	88.324
11	9	.657948	.1333457	68.175	8	11	.486391	.140031	.1039622	87.444
12	10	.623748	.1225454	68.002	8	12	.552632	.140623	.0960924	86.722
13	11	.595191	.1136773	67.668	9	13	.533452	.130469	.0891540	86.281
14	12	.570919	.1062578	67.222	10	14	.516702	.121903	.0833001	85.748
15	12	.673448	.0994728	67.020	11	15	.501916	.114575	.0782930	85.150
16	13	.646247	.0932310	67.038	10	15	.519329	.217012	.0735428	84.984
17	14	.622580	.0878686	66.945	11	16	.506584	.204426	.0692778	84.910
18	15	.601766	.0832093	66.766	12	17	.495052	.193425	.0655497	84.753
19	16	.583292	.0791212	66.520	12	18	.531320	.193368	.0621307	84.711
20	16	.660325	.0752377	66.456	13	19	.519816	.183870	.0590788	84.633
21	17	.640194	.0716497	66.461	14	20	.509272	.175398	.0563567	84.496
22	18	.622092	.0684545	66.401	14	21	.542190	.175587	.0539094	84.317
23	19	.605709	.0655900	66.288	15	22	.531650	.168092	.0516082	84.247
24	20	.590798	.0630065	66.131	16	23	.521895	.161307	.0495251	84.132
25	20	.652475	.0605006	66.115	17	24	.512834	.155136	.0476304	83.980
26	21	.636502	.0581740	66.115	17	25	.542341	.155498	.0458907	83.811
27	22	.621843	.0560556	66.072	18	26	.533236	.149905	.0442401	83.718
28	23	.608333	.0541184	65.993	19	27	.524718	.144765	.0427233	83.594
29	24	.595834	.0523395	65.883	18	27	.529588	.203622	.0412679	83.558
30	24	.647255	.0505915	65.887	19	28	.521922	.196822	.0398898	83.564
31	25	.634017	.0489616	65.884	20	29	.514693	.190530	.0386145	83.539
32	26	.621699	.0474550	65.852	21	30	.507861	.184689	.0374308	83.487
33	27	.610203	.0460582	65.793	21	31	.529266	.184782	.0363033	83.471
34	28	.599445	.0447593	65.711	22	32	.522441	.179406	.0352472	83.444
35	28	.643532	.0434715	65.724	23	33	.515963	.174384	.0342605	83.395
36	29	.632230	.0422665	65.721	23	34	.536163	.174543	.0333365	83.325
37	30	.621609	.0411405	65.694	24	35	.529688	.169881	.0324462	83.298
38	31	.611604	.0400859	65.649	25	36	.523519	.165502	.0316097	83.252
39	31	.651175	.0390941	65.588	26	37	.517635	.161380	.0308224	83.189
40	32	.640744	.0381083	65.603	26	38	.536491	.161608	.0300780	83.117

TABLE 1 (*Continued*)

n	From 1 Order Statistic, x_k				From 2 Order Statistics, x_l and x_m					
	k	c_k	V_k/σ^2	$E_k(\%)$	l	m	c_l	c_m	V_{lm}/σ^2	$E_{lm}(\%)$
41	33	.630885	.0371813	65.598	27	39	.530595	.157745	.0293590	83.076
42	34	.621548	.0363080	65.577	27	39	.511693	.194092	.0286687	83.051
43	35	.612692	.0354837	65.539	27	40	.527739	.194098	.0280016	83.052
44	35	.647770	.0346995	65.497	28	41	.522497	.189685	.0273649	83.053
45	36	.638578	.0339230	65.508	29	42	.517464	.185500	.0267612	83.039
46	37	.629834	.0331879	65.503	30	43	.512624	.181525	.0261877	83.013
47	38	.621506	.0324908	65.485	30	44	.527790	.181619	.0256330	83.005
48	39	.613561	.0318289	65.454	31	45	.522953	.177867	.0251035	82.990
49	39	.645063	.0311934	65.425	32	46	.518293	.174292	.0245990	82.963
50	40	.636847	.0305660	65.432	32	47	.532844	.174417	.0241175	82.927
51	41	.628992	.0299688	65.427	33	48	.528185	.171029	.0236490	82.912
52	42	.621475	.0293996	65.412	34	49	.523688	.167791	.0232014	82.886
53	43	.614272	.0288564	65.386	35	50	.519344	.164695	.0227732	82.851
54	43	.642858	.0283310	65.365	35	51	.533183	.164857	.0223621	82.812
55	44	.635431	.0278136	65.370	35	51	.518358	.192720	.0219586	82.800
56	45	.628302	.0273189	65.366	36	52	.514374	.189295	.0215684	82.793
57	46	.621452	.0268453	65.352	36	53	.526631	.189331	.0211879	82.794
58	47	.614863	.0263915	65.329	37	54	.522652	.186069	.0208246	82.793
59	47	.641028	.0259499	65.315	38	55	.518794	.182935	.0204738	82.784
60	48	.634252	.0255159	65.319	39	56	.515050	.179922	.0201366	82.768
61	49	.627725	.0250994	65.314	39	57	.526789	.180005	.0198076	82.763
62	50	.621434	.0246992	65.302	40	58	.523047	.177123	.0194905	82.753
63	51	.615363	.0243145	65.282	41	59	.519411	.174347	.0191850	82.737
64	51	.639486	.0239381	65.273	41	60	.530779	.174447	.0188902	82.715
65	52	.633255	.0235689	65.275	42	61	.527144	.171785	.0186019	82.704
66	53	.627237	.0232133	65.271	43	62	.523609	.169216	.0183238	82.688
67	54	.621420	.0228707	65.260	44	63	.520168	.166736	.0180552	82.665
68	55	.615792	.0225404	65.242	43	63	.522477	.191905	.0177944	82.643
69	55	.638167	.0222158	65.236	44	64	.519185	.189124	.0175365	82.644
70	56	.632401	.0218979	65.238	45	65	.515976	.186434	.0172871	82.638
71	57	.626818	.0215909	65.234	45	66	.525891	.186478	.0170434	82.639
72	58	.621409	.0212943	65.224	46	67	.522686	.183891	.0168070	82.638
73	59	.616163	.0210076	65.208	47	68	.519559	.181386	.0165781	82.631
74	59	.637027	.0207248	65.204	48	69	.516508	.178961	.0163564	82.619
75	60	.631661	.0204481	65.206	48	70	.526081	.179033	.0161388	82.617
76	61	.626455	.0201804	65.201	49	71	.523031	.176693	.0159279	82.609
77	62	.621399	.0199211	65.192	50	72	.520052	.174423	.0157233	82.597
78	63	.616488	.0196699	65.178	50	73	.529377	.174507	.0155244	82.583
79	63	.636031	.0194214	65.177	51	74	.526398	.172314	.0153294	82.575
80	64	.631014	.0191784	65.177	52	75	.523487	.170185	.0151399	82.563

TABLE 1 (*concluded*)

n	From 1 Order Statistic, x_k				From 2 Order Statistics, x_l and x_m					
	k	c_k	V_k/σ^2	E_k (%)	l	m	c_l	c_m	V_{lm}/σ^2	E_{lm} (%)
81	65	.626136	.0189428	65.173	53	76	.520640	.168116	.0149559	82.547
82	66	.621392	.0187142	65.165	52	76	.522467	.189026	.0147749	82.540
83	67	.616774	.0184924	65.152	53	77	.519723	.186749	.0145969	82.539
84	67	.635154	.0182722	65.152	54	78	.517038	.184536	.0144240	82.535
85	68	.630443	.0180572	65.153	54	79	.525362	.184580	.0142541	82.536
86	69	.625856	.0178483	65.149	55	80	.522678	.182437	.0140886	82.534
87	70	.621385	.0176452	65.141	56	81	.520050	.180352	.0139276	82.529
88	71	.617028	.0174478	65.129	56	82	.528185	.180407	.0137707	82.520
89	71	.634376	.0172514	65.131	57	83	.525558	.178385	.0136163	82.518
90	72	.629936	.0170598	65.130	58	84	.522984	.176415	.0134660	82.512
91	73	.625605	.0168733	65.127	59	85	.520461	.174496	.0133195	82.503
92	74	.621380	.0166917	65.119	59	86	.528366	.174567	.0131763	82.493
93	75	.617256	.0165150	65.109	60	87	.525843	.172703	.0130356	82.487
94	75	.633681	.0163387	65.111	61	88	.523368	.170884	.0128984	82.478
95	76	.629482	.0161668	65.111	62	89	.520941	.169111	.0127645	82.466
96	77	.625382	.0159993	65.107	61	89	.522452	.186991	.0126315	82.466
97	78	.621376	.0158360	65.100	62	90	.520100	.185065	.0125015	82.465
98	78	.637131	.0156767	65.091	63	91	.517792	.183185	.0123745	82.461
99	79	.633057	.0155177	65.093	63	92	.524964	.183228	.0122493	82.462
100	80	.629074	.0153627	65.093	64	93	.522657	.181399	.0121271	82.460

TABLE 2

Best Estimators from 2 Order Statistics x_1, x_m of Parameters α, σ and Mean μ of 2-Parameter Exponential Population

n	m	c_α	c_σ	$V\tilde{\alpha}/\sigma^2$	$E\tilde{\alpha}(\%)$	$V\tilde{\sigma}/\sigma^2$	$E\tilde{\sigma}(\%)$	$V\tilde{\mu}/\sigma^2$	$E\tilde{\mu}(\%)$
2	2	.500000	1.00000	.5000000	100.00	1.000000	100.00	.5000000	100.00
3	3	.222222	.666667	.1728395	96.429	.5555556	90.000	.3580247	93.103
4	4	.136364	.545455	.08780992	94.902	.4049587	82.313	.2902893	86.121
5	5	.096000	.480000	.05312000	94.127	.3280000	76.220	.2499200	80.026
6	6	.072993	.437956	.03557580	93.697	.2807289	71.243	.2227284	74.830
7	6	.098522	.689655	.02517789	94.565	.2337165	71.311	.1921182	74.359
8	7	.078475	.627803	.01877684	95.102	.2017178	70.820	.1700652	73.501
9	8	.064680	.582121	.01455215	95.442	.1787246	69.940	.1535601	72.357
10	9	.054676	.546756	.01161360	95.673	.1613595	68.859	.1407012	71.073
11	9	.063619	.699806	.009477724	95.919	.1468046	68.118	.1295906	70.151
12	10	.054829	.657948	.007870456	96.256	.1333457	68.175	.1189919	70.033
13	11	.047981	.623748	.006642280	96.507	.1225454	68.002	.1103346	69.718
14	12	.042514	.595191	.005682027	96.700	.1136773	67.668	.1031197	69.268
15	13	.038061	.570919	.004916701	96.852	.1062578	67.222	.0970068	68.724
16	13	.042091	.673448	.004294816	97.016	.0994728	67.020	.0913336	68.430
17	14	.038015	.646247	.003782806	97.189	.0932310	67.038	.0860455	68.363
18	15	.034588	.622580	.003357619	97.330	.0878686	66.945	.0814630	68.197
19	16	.031672	.601766	.003000580	97.447	.0832093	66.766	.0774510	67.955
20	17	.029165	.583292	.002697803	97.545	.0791212	66.520	.0739069	67.653
21	17	.031444	.660325	.002438181	97.653	.0752377	66.456	.0705104	67.535
22	18	.029100	.640195	.002214152	97.758	.0716497	66.461	.0673502	67.490
23	19	.027047	.622092	.002019763	97.847	.0684545	66.401	.0645217	67.385
24	20	.025238	.605709	.001849983	97.925	.0655900	66.288	.0619741	67.232
25	21	.023632	.590798	.001700810	97.993	.0630065	66.131	.0596668	67.039
26	21	.025095	.652475	.001568788	98.067	.0605006	66.115	.0574155	66.988
27	22	.023574	.636502	.001451542	98.137	.0581740	66.115	.0553164	66.955
28	23	.022209	.621843	.001347010	98.199	.0560557	66.072	.0533987	66.882
29	24	.020977	.608333	.001253411	98.254	.0541184	65.993	.0516395	66.776
30	25	.019861	.595834	.001169266	98.303	.0523395	65.883	.0500195	66.641
31	25	.020879	.647255	.001093227	98.357	.0505915	65.887	.0484208	66.620
32	26	.019813	.634017	.001024377	98.408	.0489616	65.884	.0469259	66.594
33	27	.018839	.621699	.000961850	98.453	.0474551	65.852	.0455408	66.540
34	28	.017947	.610203	.000904895	98.494	.0460582	65.793	.0442538	66.462
35	29	.017127	.599445	.000852865	98.531	.0447593	65.711	.0430545	66.361
36	29	.017876	.643532	.000805148	98.572	.0434715	65.724	.0418616	66.356
37	30	.017087	.632230	.000761334	98.610	.0422665	65.721	.0407431	66.335
38	31	.016358	.621609	.000721011	98.644	.0411405	65.694	.0396962	66.293
39	32	.015682	.611604	.000683817	98.676	.0400859	65.649	.0387140	66.232
40	32	.016279	.651175	.000649434	98.705	.0390941	65.588	.0377889	66.157

TABLE 2 (*continued*)

n	m	c_α	c_σ	$V_{\tilde{\alpha}}/\sigma^2$	$E_{\tilde{\alpha}}(\%)$	$V_{\tilde{\sigma}}/\sigma^2$	$E_{\tilde{\sigma}}(\%)$	$V_{\tilde{\mu}}/\sigma^2$	$E_{\tilde{\mu}}(\%)$
41	33	.015628	.640744	.000617554	98.737	.0381083	65.602	.0368669	66.158
42	34	.015021	.630885	.000587971	98.767	.0371813	65.598	.0359988	66.140
43	35	.014455	.621548	.000560469	98.794	.0363080	65.577	.0351797	66.106
44	36	.013925	.612692	.000534857	98.819	.0354837	65.539	.0344057	66.057
45	36	.014395	.647770	.000510963	98.843	.0346995	65.497	.0336683	66.003
46	37	.013882	.638578	.000488621	98.868	.0339230	65.508	.0329368	66.003
47	38	.013401	.629835	.000467717	98.892	.0331879	65.503	.0322434	65.987
48	39	.012948	.621506	.000448130	98.914	.0324909	65.485	.0315852	65.959
49	40	.012522	.613562	.000429750	98.934	.0318289	65.454	.0309596	65.919
50	40	.012901	.645063	.000412477	98.954	.0311934	65.425	.0303581	65.880
51	41	.012487	.636847	.000396219	98.975	.0305661	65.432	.0297636	65.879
52	42	.012096	.628992	.000380906	98.994	.0299689	65.427	.0291971	65.865
53	43	.011726	.621475	.000366465	99.012	.0293996	65.412	.0286567	65.841
54	44	.011375	.614272	.000352831	99.029	.0288564	65.385	.0281405	65.807
55	44	.011688	.642858	.000339944	99.046	.0283310	65.365	.0276407	65.779
56	45	.011347	.635431	.000327747	99.063	.0278136	65.370	.0271480	65.777
57	46	.011023	.628302	.000316195	99.079	.0273189	65.366	.0266765	65.765
58	47	.010715	.621452	.000305245	99.094	.0268453	65.352	.0262249	65.744
59	48	.010421	.614864	.000294855	99.109	.0263915	65.329	.0257918	65.715
60	48	.010684	.641029	.000284986	99.123	.0259499	65.315	.0253699	65.695
61	49	.010398	.634253	.000275602	99.137	.0255159	65.319	.0249549	65.692
62	50	.010125	.627726	.000266675	99.151	.0250994	65.314	.0245564	65.682
63	51	.009864	.621434	.000258176	99.164	.0246992	65.302	.0241733	65.663
64	52	.009615	.615364	.000250077	99.176	.0243145	65.282	.0238047	65.638
65	52	.009838	.639486	.000242352	99.188	.0239381	65.272	.0234439	65.623
66	53	.009595	.633256	.000234979	99.200	.0235689	65.275	.0230896	65.620
67	54	.009362	.627237	.000227938	99.212	.0232133	65.271	.0227483	65.611
68	55	.009139	.621420	.000221209	99.223	.0228708	65.260	.0224193	65.595
69	56	.008925	.615792	.000214774	99.234	.0225404	65.242	.0221018	65.573
70	56	.009117	.638167	.000208615	99.244	.0222159	65.236	.0217897	65.562
71	57	.008907	.632402	.000202717	99.255	.0218979	65.238	.0214838	65.559
72	58	.008706	.626818	.000197066	99.265	.0215909	65.234	.0211882	65.550
73	59	.008512	.621409	.000191648	99.275	.0212943	65.224	.0209025	65.536
74	60	.008327	.616164	.000186451	99.284	.0210076	65.208	.0206263	65.516
75	60	.008494	.637027	.000181462	99.294	.0207248	65.204	.0203536	65.508
76	61	.008311	.631662	.000176670	99.303	.0204481	65.206	.0200867	65.505
77	62	.008136	.626455	.000172066	99.312	.0201804	65.201	.0198283	65.497
78	63	.007967	.621399	.000167640	99.320	.0199211	65.192	.0195779	65.484
79	64	.007804	.616488	.000163382	99.328	.0196699	65.178	.0193353	65.467
80	64	.007950	.636031	.000159285	99.337	.0194214	65.177	.0190951	65.462

TABLE 2 (*concluded*)

n	m	c_α	c_σ	$V\tilde{\alpha}/\sigma^2$	$E\tilde{\alpha}(\%)$	$V\tilde{\sigma}/\sigma^2$	$E\tilde{\sigma}(\%)$	$V\tilde{\mu}/\sigma^2$	$E\tilde{\mu}(\%)$
81	65	.007790	.631015	.000155339	99.345	.0191784	65.177	.0188602	65.459
82	66	.007636	.626137	.000151538	99.353	.0189428	65.173	.0186324	65.451
83	67	.007487	.621392	.000147875	99.360	.0187143	65.165	.0184112	65.440
84	68	.007343	.616774	.000144344	99.367	.0184924	65.152	.0181964	65.424
85	68	.007472	.635155	.000140937	99.375	.0182722	65.152	.0179832	65.420
86	69	.007331	.630444	.000137650	99.382	.0180572	65.152	.0177749	65.418
87	70	.007194	.625856	.000134476	99.389	.0178483	65.149	.0175725	65.411
88	71	.007061	.621386	.000131411	99.396	.0176453	65.141	.0173756	65.400
89	72	.006933	.617029	.000128449	99.402	.0174478	65.129	.0171842	65.385
90	72	.007049	.634377	.000125587	99.409	.0172515	65.130	.0169937	65.384
91	73	.006922	.629936	.000122818	99.415	.0170598	65.130	.0168077	65.381
92	74	.006800	.625606	.000120141	99.421	.0168733	65.127	.0166266	65.374
93	75	.006682	.621380	.000117550	99.427	.0166918	65.119	.0164503	65.364
94	76	.006567	.617256	.000115042	99.433	.0165150	65.109	.0162786	65.351
95	76	.006670	.633682	.000112614	99.439	.0163387	65.111	.0161074	65.351
96	77	.006557	.629483	.000110261	99.445	.0161668	65.111	.0159403	65.348
97	78	.006447	.625382	.000107982	99.451	.0159993	65.107	.0157774	65.342
98	79	.006341	.621376	.000105772	99.456	.0158360	65.100	.0156186	65.333
99	79	.006436	.637131	.000103630	99.461	.0156767	65.091	.0154636	65.321
100	80	.006331	.633057	.000101552	99.467	.0155177	65.093	.0153089	65.321