

A NOTE ON UNBIASED TESTS<sup>1</sup>

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The now standard derivation [1] of many most powerful unbiased tests requires obtaining a test which has a certain form (4) and satisfies certain side conditions (2) involving conditional distributions. The purpose of this note is to call attention to the fact that tests already known to be unbiased, such as the ordinary  $t$ -tests, automatically satisfy the side conditions, because the derivation already implies that all unbiased tests do. This by-passes the inconvenient argument otherwise required to satisfy the side conditions. It may also make it worthwhile to restate the required form of the test in terms of unconditional distributions (4'), so as to avoid entirely the computation of conditional distributions.

The general situation will now be stated explicitly and then illustrated by the one-sample  $t$ -test.

Suppose  $X$ , the set of observations, has a (possibly discrete) density function  $f(x; \theta, \delta)$ , where  $\theta$  may represent the parameter (or conceivably parameters) of interest and  $\delta$  the nuisance parameters. Let  $\phi$  denote a (randomized) test, rejecting the null hypothesis with probability  $\phi(x)$  when  $x$  is observed. Suppose, as holds in many problems, the null and alternative hypotheses are such that if a test  $\phi$  is unbiased at level  $\alpha$ , it must satisfy one or more conditions

$$(1) \quad E_{\theta_j, \delta} \{ \phi(X) h_j(X) \} = c_j \quad \text{for all } \delta, \quad j = 1, \dots, J,$$

where each  $\theta_j$  is a particular value of  $\theta$ . Suppose, for each  $j$ ,  $T$  is a complete sufficient statistic for the family of distributions given by  $\theta = \theta_j$ ,  $\delta$  arbitrary. Then (1) is equivalent to

$$(2) \quad E_{\theta_j, \delta} \{ \phi(X) h_j(X) \mid T \} = c_j,$$

where the left-hand side now doesn't depend on  $\delta$  (which could therefore be chosen arbitrarily) because of the sufficiency of  $T$  when  $\theta = \theta_j$ .

Consider a particular alternative  $\theta', \delta'$ . The conditional power against this alternative is

$$(3) \quad E_{\theta', \delta'} \{ \phi(X) \mid T \}.$$

Let  $f(x; \theta, \delta, t)$  be the density of the conditional distribution of  $X$  given  $T = t$ . Then, according to the Neyman-Pearson Lemma (for one or more side conditions), the conditional power (3) will be maximized, subject to (2), by a test

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of the following form: for some constants  $k_j(\theta', \delta', t)$ ,  $\phi(x) = 1$  or  $0$  respectively, according as

$$(4) \quad f(x; \theta', \delta', t) > \quad \text{or} \quad < \sum_{j=1}^J k_j(\theta', \delta', t) h_j(x) f(x; \theta_j, \delta_j, t),$$

where the  $\delta_j$  are arbitrary and indeed superfluous because of the sufficiency of  $T$  when  $\theta = \theta_j$ .

The unbiased test most powerful against  $\theta', \delta'$  might now be obtained by seeking directly constants  $k_j$  such that (2) is satisfied when  $\phi$  is defined by (4). (At  $x$ 's where the two sides of (4) are equal,  $\phi(x)$  can be chosen arbitrarily in some problems, but must be chosen carefully in others because of discreteness.)

Sometimes, however, it is more convenient to use the fact that any level  $\alpha$ , unbiased test of the form (4) is most powerful against  $\theta', \delta'$  among level  $\alpha$ , unbiased tests. This follows because any level  $\alpha$ , unbiased test satisfies (2), that being how (2) was derived, and any test of the form (4) satisfying (2) maximizes (3) subject to (2) by the Neyman-Pearson Lemma.

When this argument is used, it may also be worth while to avoid entirely the computation of conditional densities by rewriting (4) as

$$(4') \quad f(x; \theta', \delta') > \quad \text{or} \quad < \sum_{j=1}^J k_j(\theta', \delta', t) h_j(x) f(x; \theta_j, \delta_j),$$

where the  $\delta_j$  are no longer superfluous but are still arbitrary and may be chosen in any convenient way. The constants  $k_j(\theta', \delta', t)$  in (4') are in general different from those in (4). To see that (4) and (4') are equivalent, let  $g(t; \theta, \delta)$  be the density of  $T$  and note  $f(x; \theta, \delta, t) = f(x; \theta, \delta) / g(t; \theta, \delta)$  (possibly multiplied by a function of  $x$ , depending on what measures the three densities are computed with respect to). Thus multiplying (4) by  $g(t; \theta', \delta')$  and incorporating the factor  $g(t; \theta', \delta') / g(t; \theta_j, \delta_j)$  into  $k_j(\theta', \delta', t)$  gives (4'), and vice versa.

To summarize:

**THEOREM.** *If every unbiased, level  $\alpha$  test satisfies (1) and if  $T$  is, for each  $j$ , a complete sufficient statistic for the family of distributions given by  $\theta = \theta_j$ ,  $\delta$  arbitrary, then any unbiased, level  $\alpha$  test of the form (4) or (4') is most powerful against  $\theta', \delta'$  among unbiased, level  $\alpha$  tests.*

Of course, it follows immediately that if an unbiased, level  $\alpha$  test has the form (4) or (4') for every alternative  $\theta', \delta'$ , then it is uniformly most powerful unbiased at level  $\alpha$ .

The one-sample  $t$ -test is conveniently treated in this way, for example. Let  $X = (X_1, \dots, X_n)$  be a sample of  $n$  from a normal distribution with mean  $\theta$  and variance  $\delta$ , both unknown, and consider testing the null hypothesis  $\theta = 0$  against the alternative  $\theta \neq 0$ . The usual continuity and differentiation arguments show that any level  $\alpha$ , unbiased test  $\phi$  satisfies

$$(5) \quad E_{0,\delta}\{\phi(X)\} = \alpha \quad \text{for all } \delta,$$

$$(6) \quad E_{0,\delta}\{\phi(X) \sum X_i\} = 0 \quad \text{for all } \delta.$$

Thus (1) holds with  $J = 2$ ,  $\theta_1 = \theta_2 = 0$ ,  $h_1(X) = 1$ ,  $h_2(X) = \sum X_i$ ,  $c_1 = \alpha$ ,  $c_2 = 0$ .  $T = \sum X_i^2$  is a complete sufficient statistic for the family of distributions given by  $\theta = \theta_j = 0$ ,  $\delta$  arbitrary, as required. With  $\delta_1 = \delta_2 = \delta'$ , (4) becomes

$$(7) \quad (2\pi\delta')^{-1n} \exp \left\{ -\sum (x_i - \theta')^2 / 2\delta' \right\} > \quad \text{or} \quad < \\ [k_1(\theta', \delta', t) + k_2(\theta', \delta', t) \sum x_i] (2\pi\delta')^{-1n} \exp \left\{ -\sum x_i^2 / 2\delta' \right\},$$

where  $t$  denotes a value of  $T$ , not the  $t$ -statistic. Relation (7) is equivalent to

$$(8) \quad \exp \left\{ \frac{\theta'}{\delta'} \sum x_i - \frac{n\theta'^2}{2\delta'} \right\} > \quad \text{or} \quad < k_1(\theta', \delta', t) + k_2(\theta', \delta', t) \sum x_i.$$

It is easily verified that the ordinary two-tailed  $t$ -test with equal tails has this form (for every  $\theta' \neq 0$  and every  $\delta' > 0$ ), and is unbiased. It follows it is (uniformly) most powerful unbiased.

#### REFERENCE

- [1] LEHMANN, E. L. (1959). *Testing Statistical Hypotheses*. Wiley, New York.