ASYMPTOTIC POWER OF CERTAIN TEST CRITERIA (BASED ON FIRST AND SECOND DIFFERENCES) FOR SERIAL CORRELATION BETWEEN SUCCESSIVE OBSERVATIONS

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1. Introduction and summary. Given x_i, $i = 1, 2, \cdots, n$, a sequence of n observations, the following statistics for measuring dispersion are defined as usual:

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2, \qquad \left(\bar{x} = \sum_{i=1}^{n} x_i/n\right)$$

$$(1) \quad \delta^2 = \frac{1}{n-1}\sum_{i=1}^{n-1}(x_i - x_{i+1})^2, \qquad d = \frac{1}{n-1}\sum_{i=1}^{n-1}|x_i - x_{i+1}|,$$

$$\delta_2^2 = \frac{1}{n-2}\sum_{i=1}^{n-2}(x_i - 2x_{i+1} + x_{i+2}), \qquad d_2 = \frac{1}{n-2}\sum_{i=1}^{n-2}|x_i - 2x_{i+1} + x_{i+2}|.$$

Further, we define the following ratio criteria:

$$(2) \quad \begin{aligned} w^2 &= \delta^2/s^2, & W &= d/s, \\ w_2^2 &= \delta_2^2/s^2, & W_2 &= d_2/s, \\ u^2 &= \delta_2^2/\delta^2, & U &= d_2/d. \end{aligned}$$

Von Neumann [12] proposed the ratio[1] $w^2 = \delta^2/s^2$ for testing the randomness of the sequence against the alternatives of serial correlation or trend. $W = d/s$, which has the advantage of being simpler for computation, was suggested by Kamat [8] for the same purpose. Similarly, we can construct two more ratio criteria, $w_2^2 = \delta_2^2/s^2$ and $W_2 = d_2/s$, by using the second order differences. And, finally, using both first and second successive differences, it is possible to construct a third type of ratio criteria, $u^2 = \delta_2^2/\delta^2$ and $U = d_2/d$, which may also be used for detecting serial correlation or trend in successive observations of the sequence. (See Tintner [11].)

Under the hypothesis of randomness, when the x_i are independent normal (μ, σ), the distributions of the ratio statistics w^2, w_2^2, W and W_2 are known, either in exact or approximate form (Von Neumann [12], Kamat [8] and [9].) Under the same hypothesis, Kamat [9] has shown that they are all asymptotically normal. (See also Anderson [1], Hsu [6], and Dixon [3].) The distributions, however, do not appear to have been discussed under any alternative hypotheses of non-randomness.

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[1] It should be noted that both Von Neumann and Kamat used n instead of $n - 1$ in the denominator of $s^2 = \sum_i (x_i - \bar{x})^2/(n-1)$.

As to comparisons of their discriminating powers, Anderson [1] has shown that, against the alternatives defined by the density function

$$p(x_1, x_2, \cdots, x_n) = K \exp\left[-\frac{1}{2\sigma^2}\left\{(1 + \rho^2) \sum_{i=1}^{n} (x_i - \mu)^2 - 2\rho \sum_{i=1}^{n-1} (x_i - \mu)(x_{i+1} - \mu)\right\}\right],$$

or by

$$p(x_1, x_2, \cdots, x_n) = K \exp\left[-\frac{1}{2\sigma^2}\left\{(1 + \rho^2 - \rho)((x_1 - \mu)^2 + (x_n - \mu)^2) + (1 + \rho^2) \sum_{i=2}^{n-1} (x_i - \mu)^2 - 2\rho \sum_{i=1}^{n-1} (x_i - \mu)(x_{i+1} - \mu)\right\}\right],$$

a statistic linearly dependent on w^2 is the *best* criterion (in the Neymann-Pearson framework) to test the hypothesis of randomness, that is to test $\rho = 0$. As mentioned by Anderson, these density functions are modifications of the density function of the circular autoregressive model, $x_i - \mu = \rho(x_{i-1} - \mu) + e_i$, where the e_i are independent normal $(0, \sigma)$, and are obtained from it by modifying the end terms in squares and products. For the autoregressive model, however, he has shown that no such best criterion exists. We are not aware of any other work on the power of these criteria.

In this paper, we discuss the distributions of these ratio criteria under another alternative of serial dependence between successive observations, specified in (4) below, when the size of the sample sequence is large. It should be noted that this alternative of serial dependence is different from the alternatives considered by Anderson, and that it is not related to the autoregressive model mentioned above. First, in Section 2, we show that all these ratio criteria are asymptotically normally distributed under this alternative hypothesis. In Section 3, we obtain the means and the variances of the limiting normal distributions. In Section 4, we present numerically the power for three sample sizes, $n = 100$, 200 and 400, and for some positive values of the serial correlation coefficient. Power curves are also exhibited for $n = 200$ and 400. Finally, in Section 5, we compare the relative efficiencies of these ratio criteria by using the Pitman criterion of asymptotic relative efficiency, as extended by Noether [10].

2.1. Some theorems. We state below two theorems from Hoeffding and Robbins [5] which will be used in proving the asymptotic normality of the ratio statistics. (See also Fraser [4].)

THEOREM 1. *Let* $X_1, X_2, \cdots,$ *be a sequence of random variables which is m-dependent* (i.e., $(X_1, X_2, \cdots, X_r)$ *is always independent of* $(X_s, X_{s+1}, \cdots,)$ *for* $s - r > m$). *If*

(a) $E(X_i) = \mu_i$, *and the second and the third moments exist for all* X_i, *and*

(b) *the* $\lim_{p\to\infty} p^{-1} \sum_{h=1}^{p} A_{i+h} = A$ *exists uniformly for all* i, *where*

$$A_i = \text{Var}\,(X_{i+m}) + 2\sum_{j=0}^{m-1} \text{Cov}\,(X_{i+j}, X_{i+m}),$$

then $\sum_{i=1}^{n} X_i$ *is asymptotically normally distributed with mean* $\mu = \sum_{i=1}^{n} \mu_i$ *and variance* nA.

THEOREM 2. *Let* (P_n, Q_n) $(n = 1, 2, \cdots)$ *and* (P, Q) *be random variables over* R^2. *If*

(a) (P_n, Q_n) *converges in distribution to* (P, Q) *as* $n \to \infty$,

(b) d_n $(n = 1, 2, \cdots)$ *is a sequence of non-zero real constants such that* $\lim_{n\to\infty} d_n = 0$,

(c) $H(x, y)$ *is a function of real variables* (x, y) *which has a total differential at* $(0, 0)$, *with*

$$H_1 = \left.\frac{\partial H(x, y)}{\partial x}\right|_{(0,0)}, \qquad H_2 = \left.\frac{\partial H(x, y)}{\partial y}\right|_{(0,0)},$$

which are not both zero, then as $n \to \infty$ *the random variable*

$$W_n = d_n^{-1}\{H(d_nP_n, d_nQ_n) - H(0, 0)\}$$

converges in distribution to $H_1P + H_2Q$.

Further, if (P, Q) *is normally distributed with mean* $(0, 0)$ *and a non-singular covariance matrix* $\begin{bmatrix} A & B \\ B & C \end{bmatrix}$, *then* W_n *will have a limiting normal distribution with mean* 0 *and variance* $H_1^2A + 2H_1H_2B + H_2^2C$.

From Theorem 2 we get the following

COROLLARY 1. *If* T *and* V *are random variates with* $E(T) = T_o$ *and* $E(V) = V_o \neq 0$, *and if* $n^{\frac{1}{2}}(T - T_o)$ *and* $n^{\frac{1}{2}}(V - V_o)$ *have a distribution which is in the limit bivariate normal with a non-singular covariance matrix* $\begin{bmatrix} A & B \\ B & C \end{bmatrix}$, *then* $n^{\frac{1}{2}}(T/V - T_o/V_o)$ *will have a limiting normal distribution with mean zero and variance*

$$V_o^{-2}A - 2T_oV_o^{-3}B + T_o^2V_o^{-4}C. \tag{3}$$

The result follows by taking $H(x, y) = (x + T_o)/(y + V_o)$ and $d_n = n^{-\frac{1}{2}}$.

2.2. Definition of the alternative hypothesis: bounds for ρ. Suppose x_i, $i = 1, 2, \cdots, n$, is a sequence of normal variates with common mean, μ, common standard deviation, σ, and common correlation coefficient, ρ, between successive observations x_i and x_{i+1}, and zero correlation otherwise. Since the ratio statistics are independent of μ and σ, we may, without loss of generality, define the alternative hypothesis as follows:

$$E(x_i) = 0, \qquad E(x_i^2) = 1, \qquad E(x_ix_{i+1}) = \rho, \qquad E(x_ix_j) = 0 \tag{4}$$

otherwise, for all i and j.

It should be stated here that the admissible upper bound for $|\rho|$ is not 1 but is $\frac{1}{2}$. The $n \times n$ correlation matrix of the alternative (4),

$$\begin{bmatrix} 1 & \rho & 0 & 0 & \cdots & 0 & 0 \\ \rho & 1 & \rho & 0 & \cdots & 0 & 0 \\ 0 & \rho & 1 & \rho & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & \rho & 1 \end{bmatrix},$$

must be positive definite and this imposes the restriction

$$|\rho| < (\tfrac{1}{2}) \sec (\pi/(n + 1)).$$

It follows that the upper bound for $|\rho|$, valid for all n, is $\frac{1}{2}$. The various correlation coefficients which are introduced in Section 3 below, and which contain the parameter ρ in their expressions, also impose different bounds on $|\rho|$. Examining these bounds it is found that they are all covered by the restriction $|\rho| \leqq \frac{1}{2}$.

2.3. Asymptotic normality of the ratio statistics. We now show that the ratio statistics defined in (2) are asymptotically normally distributed under the alternative of serial dependence between successive observations, as specified in (4) above. δ^2 and d are sums of 2-dependent sequences, and δ^2 and d_2 are sums of 3-dependent sequences, and they all satisfy the conditions of Theorem 1. It follows that they are all therefore asymptotically normally distributed under the alternative (4).

To show that s^2 and s are also asymptotically normally distributed under the alternative (4), we use Theorem 2 above. Let

$$P_n = n^{-\frac{1}{2}} \sum_i (x_i^2 - 1), \qquad Q_n = n^{\frac{1}{2}}\bar{x}, \qquad (\bar{x} = \sum_i x_i/n).$$

Taking $H(x, y) = x - y^2 + 1$, and $d_n = n^{-\frac{1}{2}}$, we have

$$H(0, 0) = 1, \qquad H_1 = 1, \qquad H_2 = 0.$$

By Theorem 2, the random variate $n^{-\frac{1}{2}}[(n - 1)(s^2/n) - 1]$ has the same limiting distribution as that of P_n. Under the alternative (4), however, P_n is a sum of 1-dependent sequence and it is therefore asymptotically normal by Theorem 1. It follows that s^2 is asymptotically normally distributed with mean unity under the alternative (4). Similarly, by taking $H(x, y) = (x - y^2 + 1)^{\frac{1}{2}}$ it can be shown that s also is asymptotically normally distributed with mean unity under this alternative.

Finally, by Corollary 1, the ratio statistics w^2, w_2^2, W and W_2 are asymptotically normally distributed. Since δ^2 and d have non-zero means (see Section 3), it follows, by the same Corollary, that u^2 and U are also asymptotically normally distributed.

2.4. Notation. If X_n is asymptotically normally distributed with mean a_n and variance b_n, a_n may not necessarily be near $E(X_n)$ and b_n may not be near

Var (X_n), for large n. The notation $E_{\to}(X_n)$ and $\text{Var}_{\to}(X_n)$ will therefore be used in this paper to denote a_n and b_n respectively.

3.1. Limit distributions. We now apply Corollary 1 to the ratio statistics defined in (2) to obtain the means and variances of their limiting normal distributions. Since n is large, terms of orders $O(1)$ and $O(n^{-1})$ alone are retained in the expressions for the means and the variances respectively.

3.2. Limit distribution of w^2. In order to find the mean and the variance of the limiting distribution of w^2 under the alternative defined in (4), the first and second order moments of s^2 and δ^2 must be determined. This consists of evaluating the expectations

$$E(s^2), E(s^4), E(\delta^2), E(\delta^4), E(\delta^2 s^2), \tag{5}$$

which in turn depends on the evaluation of the expectations

$E(x_i^2)$, $E(x_i^4)$, $E(\bar{x}^2)$, $E(\bar{x}^4)$, $E(x_i^2 x_j^2)$, $E(y_i^2)$, $E(y_i^4)$, $E(y_i^2 y_j^2)$, $E(\bar{x}^2 y_i^2)$, etc.

where, for the sake of simplicity, we have put $y_i = x_i - x_{i+1}$. Now x_i, y_i and $\bar{x}$ are normally distributed with means zero and variances 1, $2(1 - \rho)$, and $n^{-2}[n + 2(n - 1)\rho]$ respectively. Their mutual correlation coefficients can also be found; e.g.,

$$\rho(x_i, y_i) = \tfrac{1}{2}(1 - \rho)^{\frac{1}{2}}, \qquad \rho(x_i, y_{i-2}) = -\rho/[2(1 - \rho)]^{\frac{1}{2}},$$

$$\rho(x_1, \bar{x}) = \frac{1 + \rho}{[n + 2(n - 1)\rho]^{\frac{1}{2}}}, \qquad \rho(y_i, y_{i+1}) = \frac{1 - 2\rho}{2(1 - \rho)},$$

and so on. Hence all these expected values can be worked out; e.g.,

$$E(x_1^4) = 3, \qquad E(\bar{x}_1^4) = 3n^{-1}(1 + 2\rho)^2 + O(n^{-2}),$$

$$E(y_1^4) = 12(1 - \rho)^2, \qquad E(\bar{x}^2 y_1^2) = 2n^{-1}(1 - \rho)(1 + 2\rho) + O(n^{-2}),$$

etc. Evaluating the expectations in (5), after considerable simplification, we get the following results for large n.

$$E(s^2) = 1 - [2\rho/n] \cong 1, \tag{6}$$

$$\text{Var}(s^2) = [2(1 + 2\rho^2)/n] + O(n^{-2}), \tag{7}$$

$$E(\delta^2) = 2(1 - \rho), \tag{8}$$

$$\text{Var}(\delta^2) = [4(3 - 8\rho + 7\rho^2)/n] + O(n^{-2}), \tag{9}$$

$$\text{Cov}(\delta^2, s^2) = [4(1 - 2\rho + 2\rho^2)/n] + O(n^{-2}). \tag{10}$$

Finally, by Corollary 1, for large n we have

$$E_{\to}(w^2) \cong [E(\delta^2)/E(s^2)] = 2(1 - \rho) + O(n^{-1}), \tag{11}$$

$$\begin{aligned} \text{Var}_{\to}(w^2) &\cong [\text{Var}(\delta^2) - 4(1 - \rho)\,\text{Cov}(\delta^2, s^2) + 4(1 - \rho)^2\,\text{Var}(s^2)] \\ &= [4(1 - 3\rho^2 + 4\rho^4)/n] + o(n^{-1}). \end{aligned} \tag{12}$$

3.3. Limit distribution of W. The evaluation of the mean and variance of the limiting distribution of W, under the alternative (4), differs, in some respects, from the procedure followed in 3.2. We shall first evaluate the mean and variance of d, for which we require $E(d)$ and $E(d^2)$. These expectations can be found by the use of the formulae for bivariate absolute moments, given by Kamat [7], by substituting therein the appropriate values of the variances and the correlation coefficients. For instance the following expectations are used in determining $E(d)$ and $E(d^2)$.

$$
\begin{aligned}
E(|y_1|) &= 2\pi^{-1/2}(1-\rho)^{\frac{1}{2}}, \qquad E(y_1^2) = 2(1-\rho),\\
E(|y_1|\,|y_2|) &= 2\pi^{-1}[(3-4\rho)^{\frac{1}{2}} + (1-2\rho)\phi_1],\\
E(|y_1|\,|y_3|) &= 2\pi^{-1}\{[(2-\rho)(2-3\rho)]^{\frac{1}{2}} + \rho\phi_2\},\\
\phi_1 &= \sin^{-1}\left(\frac{1-2\rho}{2(1-\rho)}\right), \qquad \phi_2 = \sin^{-1}\left(\frac{\rho}{2(1-\rho)}\right).
\end{aligned} \tag{13}
$$

After substitution and some simplification we have

$$
\begin{aligned}
E(d) &= 2\pi^{-\frac{1}{2}}(1-\rho),\\
\text{Var}(d) &= 2(\pi n)^{-1}\{(\pi-10)(1-\rho) + 2[3-4\rho]^{\frac{1}{2}}\\
&\quad + 2[(2-\rho)(2-3\rho)]^{\frac{1}{2}} + 2(1-2\rho)\phi_1 + 2\rho\phi_2\} + O(n^{-2}),
\end{aligned} \tag{14}
$$

where ϕ_1, ϕ_2 are defined in (13).

The derivation of the other three results $E(s)$, Var (s) and Cov (d, s) requires the use of the following

THEOREM 3. *Let $H(v, s^2)$ be a function of two arguments, v and s^2, where v is a statistic (e.g., d or d_2) based on the first or second differences. Suppose the following two conditions are satisfied:*

(1) *In some neighborhood of the point $v = E(v)$, $s^2 = E(s^2)$, the function H is continuous and has continuous first and second order derivatives with respect to v and s^2.*

(2) *The third order moments of v and s^2 are of order $O(n^{-2})$.*

Let H_0, H_{11}, H_{12} and H_{22} be the values of H and its partial derivatives of second order in the point $v = E(v)$, $s^2 = E(s^2)$. Then

$$
\begin{aligned}
E(H) &= H_0 + (\tfrac{1}{2})H_{11}\,\text{Var}(v) + H_{12}\,\text{Cov}(v, s^2)\\
&\qquad + (\tfrac{1}{2})H_{22}\,\text{Var}(s^2) + O(n^{-2}).
\end{aligned} \tag{15}
$$

(A proof can be given on the same lines as for Cramer's Theorem in [2], p. 353 and is therefore omitted).

Now it can be easily verified that the third order moments of d and s^2, under the alternative (4), are of order $O(n^{-2})$. Taking $H(v, s^2) = (s^2)^{\frac{1}{2}}$, and applying Theorem 3, we have

$$E(s) \cong [E(s^2)]^{\frac{1}{2}} - (\tfrac{1}{8})(E(s^2))^{-\frac{3}{2}}\,\text{Var}(s^2),$$

and substituting from (6) and (7), for large n, we have

$$E(s) = 1 - (\tfrac{1}{4})n^{-1}(1 + 4\rho + 2\rho^2) + O(n^{-2}) \cong 1. \tag{16}$$

Then, from (6) and (16), we have

$$\text{Var}(s) = E(s^2) - E^2(s) = [(1 + 2\rho^2)/2n] + O(n^{-2}). \tag{17}$$

To determine Cov (d, s) we must find $E(d\,s)$ which is done by taking $H(d, s^2) = d(s^2)^{\frac{1}{2}}$ and using Theorem 3. We then have

$$E(ds) \cong E(d)[E(s^2)]^{\frac{1}{2}} + (\tfrac{1}{2})(E(s^2))^{-\frac{1}{2}} \text{Cov}(d, s^2) - (\tfrac{1}{8})E(d)(E(s^2))^{-\frac{3}{2}} \text{Var}(s^2), \tag{18}$$

and it is now necessary to evaluate Cov $(d, s^2) = E(ds^2) - E(d)E(s^2)$. For finding $E(ds^2)$ we use bivariate absolute moments with the appropriate values of the variances and the correlation coefficients. This enables us to evaluate terms such as

$$E(x_1^2 \sum |y_i|) = \frac{2n(1-\rho)^{\frac{1}{2}}}{\pi^{\frac{1}{2}}} - \frac{(1 - 2\rho^2)}{[\pi(1-\rho)]^{\frac{1}{2}}},$$

and others, which occur in $E(ds^2)$. Then, after considerable simplification, we have

$$\text{Cov}(d, s^2) = \frac{2(1 - 2\rho + 2\rho^2)}{n[\pi(1-\rho)]^{\frac{1}{2}}} + O(n^{-2}), \tag{19}$$

and using the results (6), (7), (14) and (19), in (18), after further simplification, we get

$$\text{Cov}(d, s) = \frac{1 - 2\rho + 2\rho^2}{n[\pi(1-\rho)]^{\frac{1}{2}}} + O(n^{-2}). \tag{20}$$

Finally, applying Corollary 1, we have, for large n,

$$E_*(W) \cong \frac{E(d)}{E(s)} = \frac{2(1-\rho)^{\frac{1}{2}}}{\pi^{\frac{1}{2}}} + O(n^{-1}), \tag{21}$$

$$\begin{aligned} \text{Var}_*(W) &\cong [E^4(s)]^{-1}[E^2(s)\,\text{Var}(d) - 2E(s)E(d)\,\text{Cov}(d, s) + E^2(d)\,\text{Var}(s)] \\ &= 2(n\pi)^{-1}\{\pi(1-\rho) - (11 - 13\rho + 2\rho^2 + 2\rho^3) + 2[3 - 4\rho]^{\frac{1}{2}} \\ &\quad + 2[(2-\rho)(2-3\rho)]^{\frac{1}{2}} + 2(1-2\rho)\phi_1 + 2\rho\phi_2\} + o(n^{-1}), \end{aligned} \tag{22}$$

where ϕ_1, ϕ_2 have the values given in (13).

3.4. Limit distribution of w_2^2. As in the case of w^2, here we have first to find $E(\delta_2^2)$, Var (δ_2^2) and Cov (δ_2^2, s^2) under the alternative defined in (4). This is facilitated if we take $z_i = x_i - 2x_{i+1} + x_{i+2}$. Then, x_i, $\bar{x}$ and z_i are normal variates with zero means and variances 1, $n^{-2}[n + 2(n-1)\rho]$ and $2(3 - 4\rho)$ respectively, and their mutual correlations can be found; e.g.,

$$\rho(x_i, z_i) = \frac{1 - 2\rho}{[2(3 - 4\rho)]^{\frac{1}{2}}}, \qquad \rho(z_i, z_{i+1}) = \frac{4 - 7\rho}{2(3 - 4\rho)},$$

$$\rho(\bar{x}, z_1) = \frac{\rho}{\{2(3 - 4\rho)[n + 2(n-1)\rho]\}^{\frac{1}{2}}}$$

etc. We then evaluate the expectations $E(z_i^2)$, $E(z_i^4)$, $E(z_i^2z_j^2)$, $E(x_i^2z_j^2)$, $E(\bar{x}^2z_j^2)$, etc. and following the same procedure as in 3.2, after considerable simplification, we have

$$E(\delta_2^2) = 2(3 - 4\rho), \tag{23}$$

$$\text{Var}(\delta_2^2) = \frac{4(35 - 112\rho + 98\rho^2)}{n} + O(n^{-2}), \tag{24}$$

$$\text{Cov}(\delta_2^2, s^2) = \frac{4(3 - 8\rho + 7\rho^2)}{n} + O(n^{-2}). \tag{25}$$

Finally, by using Corollary 1, we have, for large n,

$$E_{\rightarrow}(w_2^2) \cong \frac{E(\delta_2^2)}{E(s^2)} = 2(3 - 4\rho) + O(n^{-1}), \tag{26}$$

$$\begin{aligned} \text{Var}_{\rightarrow}(w_2^2) &\cong [\text{Var}(\delta_2^2) - 4(3 - 4\rho)\,\text{Cov}(\delta_2^2, s^2) + 4(3 - 4\rho)^2\,\text{Var}(s^2)] \\ &= \frac{4(17 - 16\rho - 46\rho^2 + 16\rho^3 + 64\rho^4)}{n} + o(n^{-1}). \end{aligned} \tag{27}$$

3.5. Limit distribution of W_2. We proceed in the same manner as in the case of W and first find $E(d_2)$ and $\text{Var}(d_2)$, under the alternative defined in (4). For this purpose, we need the following bivariate absolute moments.

$$\begin{aligned} E(|z_1|) &= 2\pi^{-\frac{1}{2}}(3 - 4\rho)^{\frac{1}{2}}, \\ E(|z_1||z_2|) &= 2\pi^{-1}[(20 - 40\rho + 15\rho^2)^{\frac{1}{2}} + (4 - 7\rho)\phi_3], \\ E(|z_1||z_3|) &= 2\pi^{-1}[(35 - 88\rho + 48\rho^2)^{\frac{1}{2}} + (1 - 4\rho)\phi_4], \\ E(|z_1||z_4|) &= 2\pi^{-1}[(36 - 96\rho + 63\rho^2)^{\frac{1}{2}} + \rho\phi_5], \end{aligned}$$

where

$$\phi_3 = \sin^{-1}\left(\frac{4 - 7\rho}{2(3 - 4\rho)}\right), \qquad \phi_4 = \sin^{-1}\left(\frac{1 - 4\rho}{2(3 - 4\rho)}\right), \qquad \phi_5 = \sin^{-1}\left(\frac{\rho}{2(3 - 4\rho)}\right). \tag{28}$$

Then by the usual procedure, after much simplification, we have

$$(29) \qquad E(d_2) = 2\pi^{-\frac{1}{2}}(3 - 4\rho)^{\frac{1}{2}},$$

$$(30) \qquad \begin{aligned} \text{Var}\,(d_2) = {} & 2(\pi n)^{-1}[(\pi - 14)(3 - 4\rho) + 2(20 - 40\rho + 15\rho^2)^{\frac{1}{2}} \\ & + 2(35 - 88\rho + 48\rho^2)^{\frac{1}{2}} + 2(36 - 96\rho + 63\rho^2)^{\frac{1}{2}} \\ & + 2(4 - 7\rho)\phi_3 + 2(1 - 4\rho)\phi_4 + 2\rho\phi_5] + O(n^{-2}). \end{aligned}$$

To find Cov (d_2, s) we need $E(d_2 s)$, which, by Theorem 3, can be written in the form

$$(31) \qquad \begin{aligned} E(d_2 s) \cong {} & E(d_2)[E(s^2)]^{\frac{1}{2}} + (\tfrac{1}{2})(E(s^2))^{-\frac{1}{2}} \text{Cov}\,(d_2, s^2) \\ & - (\tfrac{1}{4})E(d_2)(E(s^2))^{-\frac{3}{2}} \text{Var}\,(s^2). \end{aligned}$$

Proceeding as in 3.3, using the appropriate bivariate absolute moments, it can be shown that

$$(32) \qquad \text{Cov}\,(d_2, s^2) = \frac{2(3 - 8\rho + 7\rho^2)}{n[\pi(3 - 4\rho)]^{\frac{1}{2}}} + O(n^{-2}).$$

Finally, using (6), (7), (29) and (32), in (31), after much simplification, we have

$$(33) \qquad \text{Cov}\,(d_2, s) = \frac{3 - 8\rho + 7\rho^2}{n[\pi(3 - 4\rho)]^{\frac{1}{2}}} + O(n^{-2}).$$

Then, by Corollary 1, we get, for large n,

$$(34) \qquad E_{\rightarrow}(W_2) \cong \frac{E(d_2)}{E(s)} = \frac{2(3 - 4\rho)^{\frac{1}{2}}}{\pi^{\frac{1}{2}}} + O(n^{-1}),$$

$$(35) \qquad \begin{aligned} \text{Var}_{\rightarrow}\,(W_2) \cong {} & [E^4(s)]^{-1}[E^2(s)\,\text{Var}\,(d_2) - 2E(s)E(d_2)\,\text{Cov}\,(d_2, S) \\ & + E^2(d_2)\,\text{Var}\,(s)] \\ = {} & 2(n\pi)^{-1}[\pi(3 - 4\rho) - (45 - 68\rho + 8\rho^2 + 8\rho^3) \\ & + 2(20 - 40\rho + 15\rho^2)^{\frac{1}{2}} + 2(35 - 88\rho + 48\rho^2)^{\frac{1}{2}} \\ & + 2(36 - 96\rho + 63\rho^2)^{\frac{1}{2}} + 2(4 - 7\rho)\phi_3 \\ & + 2(1 - 4\rho)\phi_4 + 2\rho\phi_5] + o(n^{-1}), \end{aligned}$$

where ϕ_3, ϕ_4, ϕ_5 have the values given in (28).

3.6. Limit distribution of u^2. In Section 3.2 it was shown that δ^2 is asymptotically normal with mean $2(1 - \rho)$. Therefore, by Corollary 1, $u^2 = \delta_2^2/\delta^2$ is also asymptotically normal, provided $\rho \neq 1$, and further, the mean and the variance of the distribution for large n, are given by

$$(36) \qquad E_{\rightarrow}(u^2) \cong \frac{E(\delta_2^2)}{E(\delta^2)} = \frac{3 - 4\rho}{1 - \rho},$$

$$\text{(37)}\quad \begin{aligned} \operatorname{Var}_{\rightarrow}(u^2) \cong 4(1-\rho)^{-4}[&(1-\rho)^2 \operatorname{Var}(\delta_2^2) \\ &- 2(1-\rho)(3-4\rho)\operatorname{Cov}(\delta_2^2, \delta^2) + (3-4\rho)^2 \operatorname{Var}(\delta^2)]. \end{aligned}$$

In (37) the only term which is not so far evaluated is Cov (δ_2^2, δ^2). Since y_i, z_i are normally distributed and $\rho(y_i, z_j)$ can be determined, e.g.,

$$\rho(y_i, z_i) = \frac{3-4\rho}{2[(3-4\rho)(1-\rho)]^{\frac{1}{2}}},$$

all expectations in the expression for Cov (δ_2^2, δ^2) can be evaluated. Substitution and simplification leads us to

$$\text{(38)}\qquad \operatorname{Cov}(\delta_2^2, \delta^2) = [8(5 - 15\rho + 13\rho^2)/n] + O(n^{-2}).$$

Finally, using results (9), (24) and (38), in (37), we have, for large n,

$$\text{(39)}\qquad \operatorname{Var}_{\rightarrow}(u^2) = \frac{2(1 - 3\rho + 2\rho^2 + \rho^4)}{n(1-\rho)^4} + o(n^{-1}).$$

3.7. Limit distribution of U. In 3.3 it has been shown that d is asymptotically normally distributed with mean $2(1-\rho)^{\frac{1}{2}}/\pi^{\frac{1}{2}}$. Hence, applying Corollary 1, it follows that $U = d_2/d$ is also asymptotically normally distributed, provided $\rho \neq 1$. Further, the mean and the variance of the distribution, for large n, are given by

$$\text{(40)}\qquad E_{\rightarrow}(U) \cong \frac{E(d_2)}{E(d)} = \left(\frac{3-4\rho}{1-\rho}\right)^{\frac{1}{2}},$$

$$\text{(41)}\quad \begin{aligned} \operatorname{Var}_{\rightarrow}(U) \cong \frac{\pi}{4(1-\rho)^2}[&(1-\rho)\operatorname{Var}(d_2) \\ &- 2[(3-4\rho)(1-\rho)]^{\frac{1}{2}} \operatorname{Cov}(d_2, d) + (3-4\rho)\operatorname{Var}(d)]. \end{aligned}$$

In (41) the only term which is still to be determined is Cov (d_2, d). This requires the evaluation of $E(d_2\, d)$, under the alternative (4), which in turn depends on the evaluation of bivariate absolute moments of the type $E(|y_i||z_j|)$. These can be determined as before from the formulae for absolute moments; e.g.,

$$E(|y_i||z_i|) = 2\pi^{-1}[(3-4\rho)^{\frac{1}{2}} + (3-4\rho)\phi_8],$$

where ϕ_8 is defined in (43) below. Then, following the usual procedure, after considerable simplification, we have

$$\text{(42)}\quad \begin{aligned} \operatorname{Cov}(d_2, d) = 4(n\pi)^{-1}[&(12 - 28\rho + 15\rho^2)^{\frac{1}{2}} + (11 - 22\rho + 7\rho^2)^{\frac{1}{2}} \\ &+ (3-4\rho)^{\frac{1}{2}} - 6[(1-\rho)(3-4\rho)]^{\frac{1}{2}} + \rho\phi_6 + (1-3\rho)\phi_7 \\ &+ (3-4\rho)\phi_8] + O(n^{-2}), \end{aligned}$$

where

$$(43)\quad \phi_6 = \sin^{-1}\left(\frac{\rho}{2[(1-\rho)(3-4\rho)]^{\frac{1}{2}}}\right), \quad \phi_7 = \sin^{-1}\left(\frac{1-3\rho}{2[(1-\rho)(3-4\rho)]^{\frac{1}{2}}}\right),$$
$$\phi_8 = \sin^{-1}\left(\frac{3-4\rho}{2[(1-\rho)(3-4\rho)]^{\frac{1}{2}}}\right).$$

And, finally, substituting the values given by (14), (30) and (42), in (41), we obtain, for large n,

$$\begin{aligned}(44)\quad \mathrm{Var}_{\rightarrow}(U) = {} & [n(1-\rho)]^{-1}\{\pi(1-\rho)(3-4\rho) + (3-4\rho)[(3-4\rho)^{\frac{1}{2}} \\ & + ((2-\rho)(2-3\rho))^{\frac{1}{2}} + (1-2\rho)\phi_1 + \rho\phi_2] \\ & + (1-\rho)[(20-40\rho+15\rho^2)^{\frac{1}{2}} + (35-88\rho+48\rho^2)^{\frac{1}{2}} \\ & + (36-96\rho+63\rho^2)^{\frac{1}{2}} + (4-7\rho)\phi_3 + (1-4\rho)\phi_4 + \rho\phi_5] \\ & - 2((1-\rho)(3-4\rho))^{\frac{1}{2}}[(12-28\rho+15\rho^2)^{\frac{1}{2}} \\ & + (11-22\rho+7\rho^2)^{\frac{1}{2}} + (3-4\rho)^{\frac{1}{2}} + \rho\phi_6 + (1-3\rho)\phi_7 \\ & + (3-4\rho)\phi_8]\} + o(n^{-1}),\end{aligned}$$

where ϕ_1 to ϕ_8 are defined in (13), (28) and (43).

TABLE 1

Power of ratio statistics against serial correlation (ρ = 0.000 (0.025) 0.300) *for sample sizes* n = 100, 200 *and* 400
(Table gives 1000 × power)

ρ \ n	w^2			W			w_2^2			W_2			u^2			U		
	100	200	400	100	200	400	100	200	400	100	200	400	100	200	400	100	200	400
0.000	050	050	050	050	050	050	050	050	050	050	050	050	050	050	050	050	050	050
0.025	081	098	126	078	092	115	078	094	120	078	092	116	074	085	103	069	077	090
0.050	125	173	259	117	157	226	117	162	244	116	157	229	107	137	190	094	115	150
0.075	183	278	442	168	245	381	169	260	421	168	248	391	151	210	315	126	166	236
0.100	256	407	641	231	356	559	236	386	623	232	365	578	206	305	472	166	232	346
0.125	343	550	809	306	482	728	319	530	802	312	499	754	274	419	639	214	312	475
0.150	440	689	919	392	611	858	416	678	921	405	638	884	355	545	787	271	405	611
0.175	544	808	974	484	731	939	524	807	978	506	764	957	446	671	896	337	508	739
0.200	647	896	994	579	830	978	636	903	996	611	865	988	545	785	959	412	614	845
0.225	743	951	999	672	904	994	744	961	999	712	934	998	646	876	988	493	716	920
0.250	826	981	—	753	952	999	838	988	—	803	973	—	742	939	997	578	807	965
0.275	892	994	—	830	979	—	911	997	—	878	991	—	828	975	—	664	881	988
0.300	940	998	—	889	992	—	959	—	—	933	998	—	897	992	—	746	935	997

(Dashes indicate that the corresponding power is almost unity or greater than 0.999).

4. Comparison of power for large samples. Since the six ratio criteria, defined in Section 1, are asymptotically normally distributed for the null hypothesis of randomness, ($\rho = 0$), as well as the alternative of serial dependence defined in (4), we may compare their power with the help of the means and variances of their asymptotic distributions obtained in Section 3 above. This is done in Table 1, where the power of the six ratio criteria is given for $\rho = 0.000$ (0.025) 0.300, and for sample sizes $n = 100$, 200 and 400. (It is assumed that the ratios are almost normally distributed for these sample sizes and for these values of ρ.) Power curves are also drawn for $n = 200$ and 400.

The following conclusions seem to emerge from the study of Table 1.

1. The ratio criteria w^2, w_2^2 and u^2, which are based on the squares of successive differences, appear to be more powerful, in general, than the corresponding criteria W, W_2, and U based on the *absolute values* of successive differences.

2. For alternatives very near the null hypothesis, ($\rho = 0$), w^2 seems to be the most powerful amongst the six criteria; but for alternatives which are further away, w_2^2 seems to take the lead over w^2 and it is actually the most powerful for certain values of ρ.

3. An equally surprising fact is the superiority of W_2 over W, for alternatives further away from $\rho = 0$. A comparison of the corresponding columns, for $n = 400$, suggests that W_2 may be more powerful than W for *all* alternatives, provided n is sufficiently large. It looks as though W is the least powerful among the four criteria, w^2, W, w_2^2 and W_2. (See Section 5 below.)

4. u^2 appears to be more powerful than U. It even seems to catch up with W for alternatives further removed from $\rho = 0$. In general, however, both u^2 and U are inferior to the other four.

5. Asymptotic relative efficiencies. It is clear from these investigations that when the sample size becomes sufficiently large the powers of all the six ratio criteria, defined in Section 1, will be practically unity. In other words all of them are then equally efficient or powerful. In such cases, where the power function does not provide a satisfactory approach for comparison, one may use the approach of *asymptotic relative efficiency*, suggested by Pitman and extended by Noether [10]. It should be remembered, however, that in this approach the alternative hypothesis is formulated in such a manner that the parameter defining the alternative tends to its value for the null hypothesis, as the sample size tends to infinity. Briefly, Noether's extension which is used here, may be stated as follows:

Suppose we want to test a null hypothesis, $\theta = \theta_0$, against the alternative, $\theta = \theta_n > \theta_0$, where $\theta_n \to \theta_0$ as $n \to \infty$. Let the two tests which are to be compared be based on the two statistics, T_1 and T_2, which are asymptotically normally distributed under both the null and the alternative hypotheses. Let $E(T_i) = \psi_i(\theta)$ and Var $(T_i) = \sigma_i^2(\theta)$, $i = 1, 2$; and further let m_i be the least integer for which $\psi_i^{(m_i)}(\theta_0) = [\partial^{m_i}\psi_i(\theta)/\partial\theta^{m_i}]_{\theta=\theta_0} \neq 0$, for $i = 1, 2$. Defining

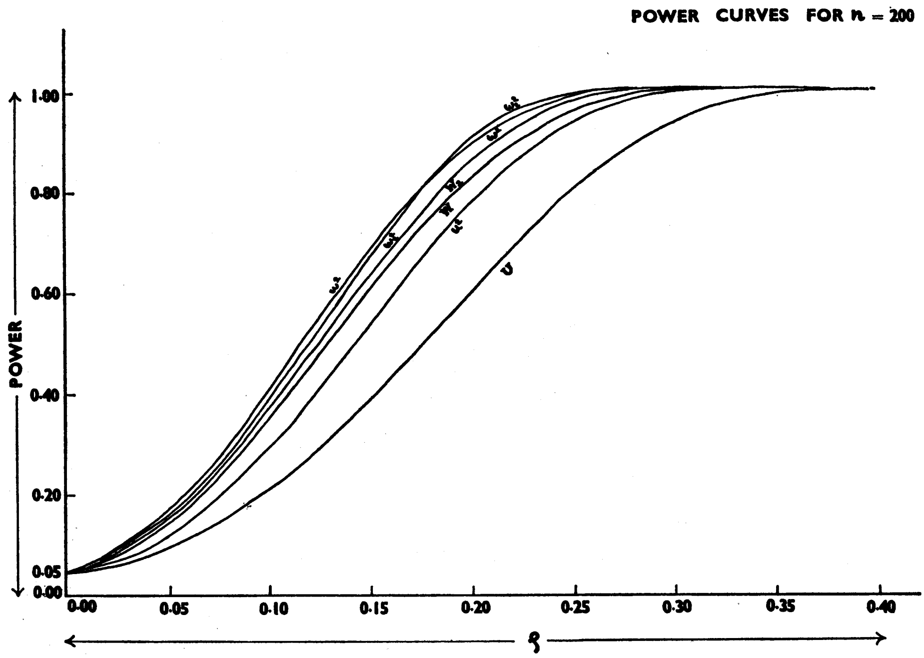
POWER CURVES FOR n = 200
POWER
1.00
0.80
0.60
0.40
0.20
0.05
0.00
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0.35
0.40
U

Fig. 1.

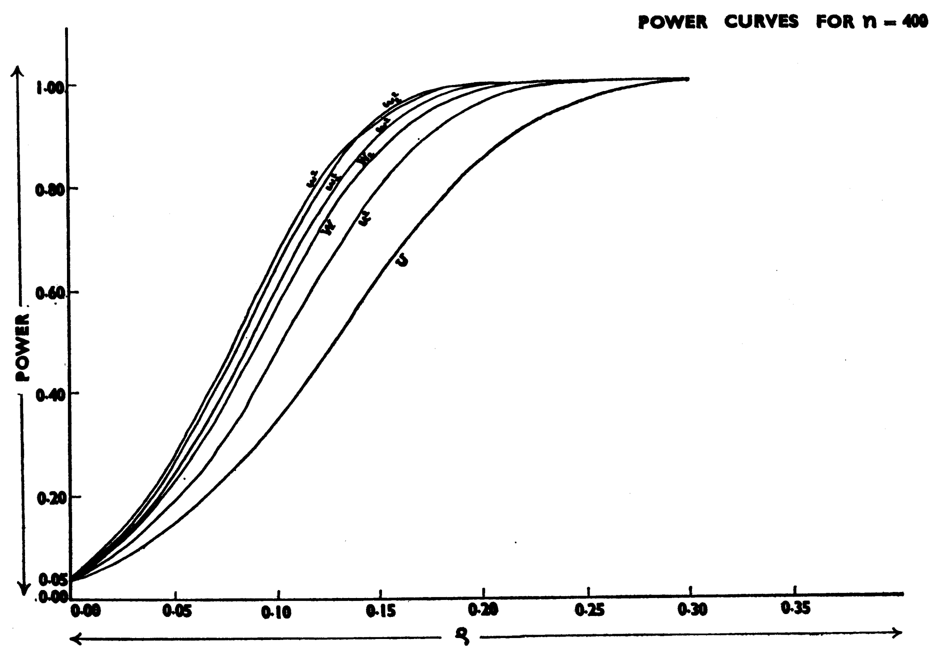
POWER CURVES FOR n = 400
POWER
1.00
0.80
0.60
0.40
0.20
0.05
0.00
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0.35
U

Fig. 2.

$\delta_i > 0$, such that

$$\lim_{n\to\infty} \frac{n^{-m_i\delta_i}\psi_i^{(m_i)}(\theta_0)}{\sigma_i(\theta_0)} = C_i, \qquad i = 1, 2, \tag{45}$$

Noether has shown that, in the particular case when $m_i = 1$ and $\delta_i = \frac{1}{2}$, $i = 1, 2$, the asymptotic relative efficiency of the test based on T_1, as compared to the test based on T_2, is given by the formula

$$E_{21} = C_2^2/C_1^2. \tag{46}$$

(It should be mentioned that the C_i should be both positive or both negative.)

We now apply this technique to compare the criteria w, W, w_2^2, W_2, u^2 and U with one another. Taking

$$H_0 : \rho = \rho_0 = 0, \qquad H_1 : \rho = \rho_n > 0, \tag{47}$$

and assuming that $\rho_n \to 0$ as $n \to \infty$, it is easy to verify that all these criteria satisfy the conditions mentioned above, and that $m = 1$ and $\delta = \frac{1}{2}$ for all of them. It is fairly straight-forward to calculate C^2 in each case. For instance, taking $T = w^2$, we have

$$E(T) = \psi(\rho) = 2(1 - \rho), \qquad (\partial\psi/\partial\rho)_0 = -2, \qquad \sigma^2(\rho_0) = 4/n. \tag{48}$$

Hence $m = 1$ and $\delta = \frac{1}{2}$, and, using (45), we get $C^2 = 1$.

The values of C^{-2}, (which are obtained more conveniently than those of C^2), for the six ratio criteria are given below.

(49)

w^2	1
W	$8\pi/3 - 2(7 - 2\sqrt{3}) = 1.3058$
w_2^2	$17/16 = 1.0625$
W_2	$9\pi/8 - 99/8 + (\frac{3}{4})(2\sqrt{5} + \sqrt{35} + 4\sin^{-1}(\frac{2}{3}) + \sin^{-1}(\frac{1}{6})) = 1.2656$
u^2	2
U	$6(7 - 4\sqrt{3}) + 12(2\sqrt{5} + \sqrt{35} + 3\sqrt{3} - 2\sqrt{33} - 6) + 12(4\sin^{-1}(\frac{2}{3}) + \sin^{-1}(\frac{1}{6}) - 2\sqrt{3}\sin^{-1}(2\sqrt{3})^{-1}) = 3.3595$

Using (46) we calculate the asymptotic relative efficiencies of the ratio criteria with respect to w^2. They are as follows:

(50)

w^2	1.000,	W	0.766,
w_2^2	0.941,	W_2	0.790,
u^2	0.500,	U	0.298.

From these values it follows that the three criteria w^2, w_2^2, u^2, based on the *squares* of successive differences, are more efficient than the corresponding three criteria, W, W_2, U, based on the *absolute* successive differences, which confirms a similar conclusion, drawn above, from Table 1. It is also interesting to note that w_2^2, which is based on second order differences, is more efficient than W, which is based on first order differences. But perhaps the most surprising result is that W_2 is more efficient, although only slightly so, than W. We had noted, in our comments on Table 1, that the power functions of W and W_2, for the sample size $n = 400$, almost indicated this result. It may be observed in this connection that the alternative hypothesis, (47), assumed in the Noether procedure, applied above, corresponds to those alternatives defined in (4) which are very near to the null hypothesis, ($\rho = 0$), together with large sample sizes, in the power-function approach. In such situations the six statistics, as arranged in the descending order of asymptotic relative efficiencies, will be

$$w^2, w_2^2, W_2, W, u^2, U.$$

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