ABSTRACTS OF PAPERS

(Abstracts of papers presented at the Eastern Regional Meeting of the Institute, Chapel Hill, April 12–14. Additional abstracts appeared in the March, 1962 issue.)

4. A Method for the Derivation of Expected Mean Squares in the Analysis of Variance. Klaus Abt, U. S. Naval Weapons Lab, Dahlgren, Virginia.

A simplified method is presented for the derivation of expected mean squares in the analysis of variance when sampling is either exhaustive (fixed effects) or from infinite populations (random effects) or of both types combined (mixed effects). So far it has been successfully applied to balanced cross-way and partially nested classifications assuming the "classical" model with unit error, treatment-unit interaction and technical error combined into experimental error and with equal error variance for all cells (treatment combinations).

The method is based upon defining the model components in terms of expectations of the cell responses. The expectation operator is generalized for taking care of either the fixed, mixed or random character, respectively, of the cell responses. The method is very easy and straightforward even for cases with unequal but proportional cell numbers. The results for some of these cases with proportional cell numbers, however, are slightly different from those given in the literature. The reason for this seems to be that the latter are obtained by the approach of sampling from finite populations and, in case of fixed effects, by a subsequent transition to exhaustive sampling, whereas the method presented does not suffer from this somewhat artificial procedure.

5. A Modified Bayes Stopping Rule. Sigmund J. Amster, University of North Carolina.

An easily computed sequential stopping rule is described and shown to possess certain interesting properties when used with a Bayes terminal decision rule. The rule requires that a preposterior analysis (Raiffa-Schlaiffer terminology) be performed at each stage and a *single* additional observation be taken if this shows that a terminal decision is not to be made.

The sample size is not larger than that of the Bayes sequential stopping rule. The average risk equals the limit of a non-increasing sequence whose maximum is the average risk for the non-sequential Bayes procedure. For the estimation problem, it is frequently sufficient to look ahead at only one potential observation, rather than carry out the preposterior analysis for each possible sample size. For the problem of testing two simple hypotheses, the procedure is a sequential probability ratio test.

6. On Successive Inferences and their Multivariate Applications in Biometry. Chooichiro Asano, Catholic University of America.

The problem of successive inferences is related to the validity tests as used in U. S. Pharmacopoeia. Like many authors, we apply previous information to improve the statistical inferences. In this connection, we discuss several inference procedures for a mean vector, for a generalized variance and for a variance-covariance matrix. This might be regarded as an extension of some methods of pooling data.

7. Experimental Study of the Power Associated with the Kolmogorov-Smirnov Test (Preliminary report). Clair J. Becker, Beverly E. Codding, and

BENJAMIN F. CRON, U.S. Navy Underwater Sound Lab, New London, Connecticut. (Introduced by Richard C. Taeuber.)

The Kolmogorov-Smirnov test is coming into greater use as a test for goodness of fit of a theoretical distribution to a given sample. In an experimental study, samples of different sizes were randomly drawn from a normal population. The Kolmogorov-Smirnov test was then used to test the hypothetical assumptions that the samples were drawn from normal, uniform, Rayleigh, logarithmic normal, and gamma distributions, respectively. The purpose of these tests was to determine how the probability of rejecting a false hypothesis (i.e., the power of the test) varies as a function of the size of the samples taken for a fixed type I error. Experimental graphs of the power function for the Kolmogorov-Smirnov test versus sample size are given for the various hypothetical assumptions. These results are compared with the lower bound for the power function of the Kolmogorov-Smirnov statistic derived by Massey ((1951). *J. Amer. Statist. Assn.* **46** 68–78).

8. On a Sequential Process Inspection Scheme with an Application to a Detection Problem (Preliminary report).

NAOMI BOGRAD AND JACK NADLER, Bell Telephone Labs, Whippany, N. J.

The use of a sequential process inspection scheme proposed by Page (*Biometrika*, 1954) is discussed when a finite sequence of random variables is observed. In this case, a control limit can be determined so that a pre-assigned probability of Type I error is approximately attained. An empirical rule for setting the limit is presented, as well as some methods of obtaining an upper bound on the actual probability. This procedure is applied to a problem in which a sequence $\{X_i\,;\, i = 1, \cdots, n\}$ of independent random variables with a common known variance is observed at equally spaced instances of time $t_0 + ih$. Up until some unknown instant t, the mean value structure of these random variables satisfies $E[X_i] = \alpha + \beta(t + ih)$. For $i > (t - t_0)/h$, however, $E[X_i] = \alpha + \beta(t + ih) + \gamma(t + ih)^2$, where the sign of γ is known and α and β are not. It is desired to detect an occurrence of t in the interval $[t_0 + h, t_0 + nh]$.

9. A Comparison of Three Different Procedures for Estimating Variance Components (Preliminary report).

NORMAN BUSH, North Carolina State College.

An analytic procedure for obtaining variances of estimates of variance components in a general multiway classification is described. As an application, three methods for estimating variance components are compared for a two way classification. Since the variances of the estimates are affected by the magnitude of the true variance components (parameters) as well as the arrangement of the unequal subclass numbers (n_{ij}'s), a numerical tabulation is necessary in order to make a comparison. Using a UNIVAC 1105, the variances of the estimates of variance components are evaluated over a substantial range of parameters and n_{ij}'s. These results are presented in tabular form for each of the estimating procedures.

10. Inventory Control Problem with Regular and Emergency Demand.

ROSHAN L. CHADDHA, Kansas State University.

Occasionally an inventory of an item is used to satisfy regular and emergency demands. For given holding costs, penalty costs for unfulfilled regular and emergency demands, and ordering costs the total inventory may be split into two parts, each satisfying separately the two types of demands. This paper is concerned with the problem of determining the "optimum" split of the inventory. The inventory content variable Z_t at time t, under certain assumption about demand, is "Markovian" in nature. For the four replenishment policies (a) M-policy (Gani; J. (1957). *J. Roy. Statist. Soc., Ser. B* **19,** 181–206), (b) Modified M-policy, (c) Ordinary Cycle Policy and (d) S-policy, the stationary distribution of Z_t may be obtained when the demand in successive periods is independent and geometric or negative exponential. It can be shown that for the purpose of the optimum split the policies (a) and (b) are two decision policies, whereas the policies (c) and (d) are single decision policies. Further, the latter two policies respectively are special cases of the former two policies.

11. **Binomial Distribution: When both Parameters are Random Variables** (Preliminary report). ROSHAN L. CHADDHA, Kansas State University. (By title)

Consider a population of individuals for which the desire $p (0 < p < 1)$ to attend meetings is fixed. Further, if n, the number of opportunities each individual gets during a period, is the same for all individuals then the number of meetings Y attended by a randomly drawn individual is a binomial variate. In practice, however, neither p nor n is fixed throughout the population. Assume both of them to be random variables so that $\mu_r'(p)$ and $\mu_r'(n)$ denote respectively the rth moment of p and rth factorial moment of n about origin. The rth factorial moment of the random variable Y is then simply the product $\mu_r'(p)\mu_r'(n)$. Further the probability $\Pr(Y = r)$ is given by

$$\sum_{m=r}^{\infty} (-1)^{m-t}\mu_m'(p)\mu_{(m)}'(n)/r!(m - r)!, \qquad r = 0, 1, 2 \cdots .$$

The results for the special cases when both n and p have one and two parameter distributions are obtained. The distribution of the variable $Z = \sum_{i=1}^{k} Y_i$ is also obtained under certain conditions. The problem of estimation of the parameters and of the probabilities $\Pr[Y = r]$ is being considered.

12. **Asymptotic Expansions for Tests of Goodness of Fit for Linear Autoregressive Schemes.** K. C. CHANDA, Washington State University.

Asymptotic expansions for the joint distributions of Bartlett-Diananda statistics have been investigated. Thus if $\{Y_t\}$ is a pure white noise process with $V(Y_t) = 1$ $(1 \leqq t \leqq n + k)$ where n and k are positive integers and $k(<n)$ is fixed, the statistics U_s $(0 \leqq s \leqq k)$ are defined by $U_s = \sum_{t=1}^{n} Y_t Y_{t+s}/n$ $(s \geqq 1)$, and $U_0 = \sum_{t=1}^{n} (Y_t^2 - 1)/n$. It has been proved that the rth order cumulant of U_s is of order $n^{-(r-1)}$ and that the $(r_0, r_1, \cdots, r_k)$ the order joint cumulant of $U_0, U_1, \cdots, U_k$ is of order $n^{-(\sum_{j=0}^{k} r_j - 1)}$. Two different types of asymptotic expansions have been discussed viz., (i) the one obtained by the method of saddle point approximations (Daniels (1954). *Ann. Math. Statist.* **25** 631–650), and (ii) the one obtained by using the classical Edgeworth expansions. Finally, the asymptotic expansions to order n^{-1} for the distributions of (i) $\chi_1^2 = n\sum_{s=1}^{k} U_s^2$ and $\chi_2^2 = n\sum_{s=1}^{k} U_s^2/(1 + U_0)^2$ have been derived, using the second method. χ_1^2 is employed as a test-criterion for testing whether a linear autoregressive scheme is of a particular type with regression parameters known and the order specified, the residuals $\{Y_t\}$ being assumed to have positive and known variance, whereas χ_2^2 is used for the same hypothesis except that the quantity $V(Y_t)$ is unknown but assumed positive, in each case the residuals being standardized to have unit variance. The results discussed in this paper are extensions of those already derived by Bartlett and Rajalakshman (*J. Roy. Statist. Soc. Ser. B* **15** (1953) 107–124) and Chanda (to be published in *J. Roy. Statist. Soc. Ser. B* **24** (1962)).

13. **The Effect of Cross-Class and Cross-Characteristic Errors in 2 × 2 Tables.** EARL L. DIAMOND, Johns Hopkins School of Hygiene and Public Health.

Suppose we have two classes in which the proportions of individuals with a specified characteristic are P_1 and P_2. Let A indicate that an individual has the characteristic and $\bar{A}$ that he does not. Let B indicate that an individual is observed to have the characteristic and $\bar{B}$ that he is not. Then let $a_i = \Pr_i(B \mid A)$ and $b_i = \Pr_i(B \mid \bar{A})$ where $i = 1, 2$ denotes the class. Also, suppose that c_i is the probability that an individual placed in class i is in fact a member of that class. Then $1 - c_i$ is the probability that an individual placed in class i is in fact a member of the other class.

It is shown that the expectations of the proportion observed to have the characteristic are

$$p_1 = c_1[a_1P_1 + b_1(1 - P_1)] + (1 - c_1)[a_2P_2 + b_2(1 - P_2)],$$

$$p_2 = (1 - c_2)[a_1P_1 + b_1(1 - P_1)] + c_2[a_2P_2 + b_2(1 - P_2)].$$

It is shown that, depending upon the values of a_i, b_i, and c_i, an estimate of P_i may be biased either upward or downward and its variance either increased or decreased. In considering comparisons between P_1 and P_2 (assuming P_1 to be the larger) using the difference $P_1 - P_2$, the ratio P_1/P_2 or the relative risk $P_1(1 - P_2)/P_2(1 - P_2)$, it is shown that these are reduced in value in most realistic situations but in some cases may be increased.

14. **Speed versus Accuracy in Measurement.** E. A. FAY AND ROY LEIPNIK, University of Florida and USNOTS, China Lake, California.

The effect of speed, i.e., sampling rate n, and accuracy, i.e., number q of quantizing levels, on the variance of the minimum variance linear estimator of the mean of a stationary process is studied under various assumptions on the covariance between the measurement and the process, and the type of quantizing used. Explicit results are obtained, by matrix methods, for the case of a stationary correlated Markov process. Optimization of various loss functions, such as time, information per unit time, and cost is carried out. The main result is that optimal values of q are rather less, and optimal values of n greater, than usually assumed by operating personnel in the analog-digital conversion field. For a realistic range of cases, the optimal value of q is two, the minimum possible number of distinct levels.

15. **Sequential Analysis of Hierarchical Classifications.** BHASKAR K. GHOSH, Lehigh University.

Consider a pth-order hierarchical classification in which there are p hierarchies of groups, starting from the highest or the 1st-order main-group to the lowest or the pth-order sub-group. Let the mathematical model of the set-up be that of random-effects with σ_i^2 as the variance between the ith-order groups ($i = 1, 2, \cdots, p$) and σ_0^2 as the error variance within the pth-order groups. The purpose of the present paper is to develop sequential methods for testing the hypothesis $H(\theta_i = \theta_i^0)$, $i = 1, 2, \cdots, p$, where $\theta_i = \sigma_i^2/\sigma_0^2$. Johnson, (*Ann. Math. Statist.* **25** (1954), constructed a number of sequential tests for the special case where $p = 1$, that is, in a one-way classification by groups. The author here extends these notions to the general situation with particular reference to the case of $p = 2$, that is, a second-order classification by main-groups and sub-groups inside each main-group. A number of alternative sequential procedures are described, their properties, average sample sizes and cost functions are critically examined and compared. An optimum sequential method is suggested for each value of p.

16. Post Cluster Sampling. S. P. GHOSH, University of California, Berkeley. (By title)

In order to reduce the cost of the survey, cluster sampling is introduced, after building the clusters on the basis of a relatively large random sample from the population, in a situation where the boundaries of the clusters are not known before hand. A stochastic model is provided to analyze such sampling procedures. An unbiased estimate and a ratio estimate for the mean of the population, together with their variance is given. It is shown that ratio estimate can be better than the unbiased estimate for post cluster sampling in some situations where unbiased estimate is better than ratio estimate for ordinary cluster sampling. An estimate together with its variance is also provided when the post clusters are selected with varying probabilities. Finally the optimum design is worked out by minimizing the variance of the unbiased estimate when subjected to cost restriction.

17. Some Results concerning the Utility of an Active Standard in Screening Tests (Preliminary report). ALAN J. GROSS, National Institutes of Health, Bethesda.

Currently, there are two methods for evaluating the effectiveness of a test material in the cancer screening program. The first considers the quantity $u_1 = y - z$ where y is the log mean tumor weight or log mean survival time of test animals and z is the corresponding quantity for control animals. Alternatively, we can introduce an active standard known to produce tumor inhibition and consider the quantity $u_2 = y - \alpha_0 x - \beta_0 y$ where x is the analogue of y and z for animals on the active standard, and α_0 and β_0 are chosen to minimize var u_2. Thus to exclude an active standard in the screening procedure, on the basis of minimum variance, we must have $\alpha_0 = 0$. Assuming x, y, z possess the trivariate normal with means μ_x, μ_y, μ_z, variances σ_x^2, σ_y^2, σ_z^2, and correlations ρ_{xy}, ρ_{xz}, ρ_{yz}, the following tests have been constructed using the likelihood ratio criterion (1) $\rho_{xy} = \rho_{xz} = 0$; $\rho_{yz}^2 = \sigma_z^2/\sigma_y^2$ and (2) $\rho_{xy} = \rho_{xz}\rho_{yz}$. If (1) is accepted, conclude $\alpha_0 = 0, \beta_0 = 1$. If (1) is rejected and (2) accepted, conclude $\alpha_0 = 0$. If both (1) and (2) are rejected, conclude $\alpha_0 \neq 0, \beta_0 \neq 1$.

18. Smallest and Minimum Lifes in Fatigue. E. J. GUMBEL AND L. H. HERBACH, Columbia University and New York University.

For the analysis of fatigue life N the third asymptotic distribution of smallest values commonly called the Weibull distribution is used. It contains three parameters namely the characteristic life, the minimum life and a third parameter α_s. It is shown that the latter may be estimated from $\bar{t} = (\bar{N} - N_1)/s$ where $\bar{N}$ and s stand for the sample mean life and standard deviation and N_1 for the smallest observed life. The estimation of the two other parameters by the classical method of moments leads to linear combinations of N_1 and $\bar{N}$. The method guarantees that the estimate of the minimum life is smaller than N_1. The characteristic life, the minimum life and $1/\alpha_s$ increase with decreasing stress. The estimate of the lower limit is not confined to fatigue life. It can be used for any variable subject to this distribution. It is generally believed that tensile strengths are normally distributed. However, the use of this distribution leads to a very good fit.

19. Minimal Sufficient Statistics for the Partially Balanced Incomplete Block Design with Two Associate Classes under an Eisenhart Model II. C. H. KAPADIA, University of Georgia.

Weeks and Graybill have exhibited a minimal sufficient statistics for the Balanced Incomplete Block Designs under an Eisenhart Model II, (*Sankhya, Ser. A* **23**, Part 3, (1961) 261–268). In this paper similar results have been derived for the PBIB Design with two associate classes.

20. Approximation to Hypergeometrics by the Binomial and to the Binomial by Poisson. SHRINIWAS KESHAV KATTI, Florida State University.

Sandiford has discussed the approximation to a hypergeometric distribution by a binomial whose first two moments agree with the first two moments of the hypergeometric distribution in *J. Amer. Statist. Assn.*, 1960. Apparently the first two moments were selected because of the simplicity in estimating the parameters of the binomial distributions for this purpose. The general interest in such approximations invariably lies in obtaining a binomial distribution as "close to the hypergeometric as possible". In this context, a method of estimating the binomial parameters so as to minimize the Euclidean distance between the frequency functions of the distributions was employed. The procedure is applied to obtain Poisson approximations to the binomial. The equations for estimation being considerably complex, graphs and tables are provided from which the estimates can be quickly read out.

21. Some Theory of Queues. TATSUO KAWATA, Catholic University of America. (Invited paper)

In the queue problem with a single server and with the first-come, first-served discipline, let us suppose that $0 < t_0 < t_1 < \cdots$ are moments when successive customers arrive at the counter. There are i persons at $t = 0$, waiting or being served and $X_n = t_n - t_{n-1}$, Y_n, $n = 1, 2, \cdots$ are respectively the interarrival times and the service time which the nth customer demands. Suppose that each of X_n and Y_n has the respective identical distributions $A(x)$ and $B(x)$, and $(X_1, X_2, \cdots, Y_1, Y_2, \cdots)$ forms a sequence of independent random variables. Define $G(x) = \sum_{n=1}^{\infty} P(S_1 > 0, \cdots, S_{n-1} > 0, S_n \leqq y)$ ($P(S_1 \leqq y)$ for $n = 1$) which is uniformly convergent over $-\infty < x < \infty$, provided that $a > b$, $a = EX_n$, $b = EY_n$, where $S_n = \sum_1^n (Y_k - X_k)$. Denoting the number of customers waiting or being served at t by $X(t)$, our main result can be written as $\lim_{m\to\infty} P(X(t_n - 0) = n) = e^{-K}\int_0^{\infty} (e^K - G(x))\, d(A_{m-1}(x) - A_m(x))$ for $m \geqq 1$, $= e^{-K}$ for $m = 0$, provided only that $b < a$, while $\lim_{m\to\infty} P(X(t) = m) = b^{-1}e^{-K}\int_0^{\infty} (e^K - G(x))(A_{m-1}(x) - A_m(x))\, dx$ for $m \geqq 1$, $=1 - b/a$ for $m = 0$, where $A_m(x)$ is the mth convolution of $A(x)$ with itself, $A_0(x)$ is the unit distribution and $K = \sum_1^{\infty} P(S_n > 0)/n$. In case $A(x)$ and $B(x)$ are absolutely continuous, P. D. Finch (*Acta Math. Hung.* **10** (1959)) obtained the relation between $\lim_{n\to\infty} P(X(t_n - 0) = m)$ and $\lim_{t\to\infty} P(X(t) = m)$ even in many server problems. Our result gives a more explicit formula as far as the single server case is concerned within the more general conditions on $A(x)$ and $B(x)$.

22. On the Simultaneous Tests for Equality of Variances Against Certain Alternatives when the Samples are Drawn from a Multivariate Normal Population (Preliminary report). P. R. KRISHNAIAH, Remington Rand Univac, Blue Bell, Pennsylvania.

Consider a multivariate normal population with known correlation matrix and unknown variances $\sigma_1^2, \cdots, \sigma_p^2$. In the present paper, the distribution problems associated with the following tests are investigated:

(i) Testing the hypotheses H_{ij} ($i \neq j = 1, 2, \cdots, p$) and $H = \bigcap_{i \neq j=1}^{p} H_{ij}$ simultaneously against the respective alternatives A_{ij} and $A = \bigcup_{i \neq j=1}^{p} A_{ij}$ where $H_{ij} : \sigma_i^2 = \sigma_j^2$ and $A_{ij} : \sigma_i^2 \neq \sigma_j^2$.

(ii) Testing the hypotheses H_i ($i = 1, 2, \cdots, (p-1)$) and $H = \bigcap_{i=1}^{p-1} H_i$ simultaneously against the respective alternatives A_i and $A = \bigcup_{i=1}^{p-1} A_i$ where $H_i : \sigma_i^2 = \sigma_p^2$ and $A_i : \sigma_i^2 \neq \sigma_p^2$.

(iii) Testing the hypotheses H_i^* ($i = 1, 2, \cdots (p-1)$) and $H = \bigcap_{i=1}^{p-1} H_i^*$ simultaneously against the respective alternatives A_i^* and $A^* = \bigcup_{i=1}^{p-1} A_i^*$ where $H_i^* : \sigma_i^2 = \sigma_{i+1}^2$ and $A_i^* : \sigma_i^2 < \sigma_{i+1}^2$.

The tests discussed in this paper are based on Roy's union-intersection principle.

23. **Mutual Determination of Random Variables by Correlation.** H. O. LANCASTER, University of Sydney. (By title)

Two random variables, X and Y, may be said to be mutually determined if there is a pair of transformations so that $X = x(Y)$, $Y = y(X)$, a.e. There have been attempts to use a unit value of the maximum correlation to characterize such mutual determination. This breaks down in the general case. In fact, it is possible to have an infinity of such maximal correlations equal to unity without the variables being mutually determined. For example, take a mixture of a uniform distribution along the open interval, $(-1, -1)$ to $(+1, +1)$ a distribution consisting of weights $1 + \rho XY$ at the four points, $(\pm 1, \pm 1)$, $|\rho| < 1$. Defining suitably standardized polynomials on the open intervals $(-1, +1)$ of each marginal distribution, each polynomial in X has unit correlation with the same polynomial in Y, but X and Y are not independent. Without limiting the form of distribution, the following can be proved.

Theorem. A necessary and sufficient condition that random variables, $X_1, X_2, X_3, \cdots$, should be mutually determined is that there should exist complete sets of orthonormal functions (or orthonormal bases), $\{x_1^{(i)}\}$, $\{x_2^{(i)}\}$, $\cdots$ defined on the marginal distributions such that the matrices of correlations, $\mathbf{H}_{k_1k_2}$, $\mathbf{H}_{k_2k_3}$, $\mathbf{H}_{k_3k_4} \cdots \mathbf{H}_{k_nk_1}$ between the sets, $\{x_{k_1}^{(i)}\}$ and $\{x_{k_2}^{(i)}\}$, should be orthogonal.

24. **Student's t in an n-Way Classification with Unequal Variances** (Preliminary report). KAY KNIGHT MAZUY AND W. S. CONNOR, Research Triangle Institute.

This study investigates the n-way classification with m observations per cell, where each cell is determined by one member from each class. It assumes that an observation is a realization of a random variable which is expressed as a sum of fixed effects, the overall mean and effects for the members which determine the cell, and a random error. The random error is assumed to be normally distributed with mean zero. A variance is associated with each member of each class, and the variance of the random error is the sum of the variances associated with the members which determine the cell. In order to use Student's t for testing contrasts among effects it is necessary to find error contrasts which have the same variance as the contrast to be tested, are independent of it and of each other. The maximum number of such contrasts is determined, and a method for finding them is presented.

25. **Expectations and Covariances of Serial and Cross Correlation Coefficients in Complex Stationary Time Series.** DONALD F. MORRISON, National Institute of Mental Health, Bethesda.

Let $X(t) = c(t) + is(t)$ be a stationary Gaussian stochastic process characterized by the expectations $Ec(t) = Es(t) = 0$, $Ec(t)s(t) = 0$, $Ec(t)c(t + \tau) = Es(t)s(t + \tau) = \sigma^2\rho_c(\tau)$,

$Ec(t)s(t + \tau) = -Ec(t + \tau)s(t) = \sigma^2\rho_s(\tau)$, for all $t \varepsilon T$. Asymptotic expectations, variances, and covariances of the estimates

$$\hat{\rho}_c(m) = N(N - m)^{-1}\left(\sum_{t=1}^{N-m} c_t c_{t+m} + \sum_{t=1}^{N-m} s_t s_{t+m}\right) \bigg/ \left(\sum_{t=1}^{N} c_t^2 + \sum_{t=1}^{N} s_t^2\right) \quad \text{and}$$

$$\hat{\rho}_s(m) = N(N - m)^{-1}\left(\sum_{t=1}^{N-m} c_t s_{t+m} - \sum_{t=1}^{N-m} s_t c_{t+m}\right) \bigg/ \left(\sum_{t=1}^{N} c_t^2 + \sum_{t=1}^{N} s_t^2\right)$$

of the correlation functions have been computed for general $\rho_c(\tau)$, $\rho_s(\tau)$. These expressions have been evaluated for $X(t)$ a Markov process, i.e., with $\rho_c(\tau) = \rho^{|\tau|} \cos \beta\tau$, $\rho_s(\tau) = \rho^{|\tau|} \sin \beta\tau$.

26. **A Generalization of Ballot Theorems.** T. V. NARAYANA, National Institutes of Health, Bethesda.

Problems of arrangements called "ballot problems" have attracted the interest of students of combinatorial analysis. (Cf., W. Feller, *An Introduction to Probability Theory and its Applications*, **1**, 66, Wiley, New York where further details and references can be found.) The author has proved a combinatorial theorem (to appear in *Can. Math. Bull.* **5**) which suggests a unified approach to ballot problems as well as enumeration problems connected with simple sampling plans. We extend our approach to yield a partition of voting records in the ballot theorem consisting of equivalence classes based on number-theoretic properties. This further partition of voting records generalizes the ballot theorem and yields as a corollary: Let $m = 2n+1$, $k = 2$, in the ballot theorem. To every voting record in which P's vote always exceed twice Q's vote corresponds a simple sampling plan of size n and conversely.

27. **Confidence Regions Bounded by Generalized Order Statistics.** PETER NEMENYI, New York University Medical Center.

From any significance test for translation, a confidence interval can be derived—by a tedious process of trial and error. In the case of a sign test, the trial and error element eliminates itself automatically, leading to Nair's interval bounded by two order statistics. Walsh has achieved the same thing in the case of the signed-rank statistic by expressing it as a count of positive signs of pairwise averages. Imitating Walsh, the two-sample rank statistic is converted to a count of positive (or of negative) signs among the n_1n_2 possible differences between values in the respective samples, (= Mann-Whitney statistic), and a confidence interval results which is bounded by two order statistics of this sample of differences.

In all three cases—signs, signed-ranks, ranks—the argument can be extended very easily to the problem of multiple comparisons, leading to trial-and-error-free confidence boxes.

28. **A Bayesian Indifference Rule** (Preliminary report). MELVIN R. NOVICK, University of North Carolina.

A Bayesian indifference rule and a mode of estimating the density of a random variable are proposed. If $f_\theta(x)$ is the density of a random variable X dependent on a parameter θ and $\xi \varepsilon \Xi$ is a prior distributive for θ, the prior marginal density of X may de defined as $f\xi(x) = \int_\Theta f_\theta(x)\, d\xi(\theta)$. In cases where no prior information is available it is proposed that a prior density $\xi \varepsilon \Xi$ be chosen which minimizes the Shannon information measure of $f\xi(x)$. Sufficient conditions are given under which the prior density will be uniquely specified. In common examples ξ agrees with that specified by the Bayes Postulate. The proposed postulate, however, is invariant under one-to-one transformation of the parameter space. The posterior marginal density of X, is considered as an estimator of the true density of X. Under general conditions, the sequence of posterior marginals is consistent, and has certain interesting information theoretic properties.

29. A Generalized MANOVA Model Useful especially for Growth Curve Problems. RICHARD F. POTTHOFF AND S. N. ROY, University of North Carolina.

The usual multivariate analysis of variance (MANOVA) model is

$$E[\mathbf{X}(n \times p)] = \mathbf{A}(n \times m)\xi(m \times p), \tag{1}$$

where $\mathbf{A}$ contains known constants, ξ contains unknown parameters, and each row of $\mathbf{X}$ follows a p-variate normal distribution. We may consider a MANOVA model more general than (1), of the form

$$E[\mathbf{X}_0(n \times q)] = \mathbf{A}(n \times m)\xi(m \times p)\mathbf{P}(p \times q), \tag{2}$$

where $\mathbf{A}$ and $\mathbf{P}$ contain known constants, ξ contains unknown parameters, and each row of $\mathbf{X}_0$ follows a q-variate normal distribution. A means of testing a hypothesis of the form

$$\mathbf{C}(s \times m)\xi(m \times p)\mathbf{V}(p \times u) = \mathbf{O}(s \times u) \tag{3}$$

(where $\mathbf{C}$ and $\mathbf{V}$ consist of specified constants and $\mathbf{O}$ is the null matrix) under the model (2) is obtained. This problem of testing the hypothesis (3) under the model (2) is attacked by reducing it to the problem of testing the hypothesis (3) under the model (1). Some growth curve applications of the generalized MANOVA model (2) are pointed out.

30. A Simple Procedure of Unequal Probability Sampling without Replacement. J. N. K. RAO AND H. O. HARTLEY, Iowa State University, AND W. G. COCHRAN, Harvard University.

To draw a sample of size $n(\geqq 2)$, the following sampling procedure is considered: Split the population of N units at random into n groups. Select one unit independently from each group with probabilities proportional to the initial probabilities p_i. It is shown that the variance of the estimator of the population total for this procedure is smaller than the variance of the usual estimator of the population total in sampling with replacement with probabilities p_i. Also, the estimated variance is always positive. The optimum group size is N/n, so that if N is not a multiple of n (say $N = nr + k$) we choose k groups each of size $(r + 1)$ and $(n - k)$ groups each of size r. The results are extended to multistage sampling.

31. Some Properties of Random Allocation Designs. B. V. SHAH AND O. KEMPTHORNE, Iowa State University.

Some properties of estimators in random allocation designs (Dempster (1960). *Ann. Math. Statist.* **31,** 885–905, and *ibid.* (1961) **32,** 387–405) and some approximate tests of hypothesis based on those estimators are examined. It is shown that the estimators, are in general, not invariant under change of origin and that the F-tests derived from them would give a misleading picture of the true hypothesis.

32. Randomization in Fractional Factorials. B. V. Shah and O. Kempthorne, Iowa State University.

Fractional replication of the complete factorial design is widely used to estimate main effects and lower order interactions neglecting higher order interactions. However, a fractional replicate fails to provide a test for these effects, particularly if higher order effects are present. The properties of randomized fractions are explored and it is shown that properly randomized fractions (more than one) of a basic fractional factorial design, will provide tests for main effects and lower order interactions even when higher order interactions are present. They would also yield some tests of significance of higher order interactions.

33. Queueing Times for an M | G | 1 Queue with Poisson Type Breakdowns. R. Emerson Thomas, University of North Carolina.

The service mechanism of an $M\ |G|\ 1$ queue is subject to breakdowns which occur according to a Poisson process. Durations of breakdowns are independent and identically distributed. Active breakdowns occur only during service times of customers and Independent breakdowns occur at any time. Re-entry of a customer to the service mechanism after a breakdown can be governed by one of the preemptive disciplines; resume, repeat identical, and repeat independent, (Gaver, to appear in *J. Roy. Statist. Soc. Ser. B*). These disciplines involve different completion time distribution functions which were characterized by Gaver. The transient behavior of the queueing time is characterized by a functional equation involving breakdown duration and completion time distribution functions. We investigate conditions for the existence of a limiting queueing time distribution function. This distribution function is shown to satisfy certain functional equations from wihch are obtained the first two moments for each of the three preemptive disciplines and the two types of breakdowns. The queueing time for Independent breakdowns is mathematically equivalent to the queueing time for a non-priority customer in a 2-level preemptive priority queue where a breakdown duration is considered as a busy period of the priority customers in isolation.

34. The M | G | 1 Queue with Poisson Type Breakdowns of the Service Mechanism. R. Emerson Thomas, University of North Carolina. (By title)

Consider the $M\ |G|\ 1$ queue where breakdowns of the service mechanism occur according to a Poisson process. Repair times for breakdowns are independent and identically distributed. In this paper results are obtained for breakdowns which occur in each of the following four ways: (i) only during service times of customers, or (ii) during service times of customers and repair times of components of the service mechanism, or (iii) only during periods when service mechanism is not being repaired, or (iv) at any time. A pre-emptive resume discipline (Miller (1960). *Ann. Math. Statist.*) governs re-entry to the service mechanism after breakdowns. Let us define the Stochastic process $Z_t = (X_t, Y_t)$ of unexpired repair and service time. Assume no arrivals or breakdowns occur after time t. Then at instant $(t + X_t)$ the service mechanism becomes repaired and commences servicing customers. At instant $(t + X_t + Y_t)$ the service mechanism completes service for all customers who were in the system at time t. We obtain a characterization of the transient behavior by employing a system of integro-differential equations. Under certain conditions the limiting distribution of $Z_t = (X_t, Y_t)$ exists. This distribution is characterized by, at most, four equations. An algorithm is developed which leads to any desired joint moment of the limiting bivariate distribution.

35. **Simultaneous Confidence Intervals for the Dispersion Parameters** (Preliminary report). W. A. Thompson, Jr., National Bureau of Standards.

This paper deals with a topic in Multivariate Analysis. Consider that a sample of size $n + 1$ has been collected from a p-variate normal distribution having dispersion matrix $(\sigma_{jj'})$. Let $a_{jj'}/n$ denote the usual unbiased estimate of $\sigma_{jj'}$. Further, let $0 < l < u$ be constants such that all characteristic roots of a matrix having the Wishart distribution lie in the interval $[l, u]$ with probability $1 - \alpha$. A theorem of Roy, Bose and Gnanadesikan (*Am. Math. Statist.* **24**; *Biometrika* **44**) may be stated as follows: The probability is $1 - \alpha$ that every principal minor determinant of $l^{-1}(a_{jj'}) - (\sigma_{jj'}) - u^{-1}(a_{jj'})$ is non-negative. The previous result may be used to prove the main theorem of the present paper.

Theorem. The probability is at least $1 - \alpha$ that the following system of relations hold simultaneously $u^{-1}a_{jj} \leqq \sigma_{jj} \leqq l^{-1}a_{jj}$; $j = 1, \cdots, p$ and $|\sigma_{jj'} - \frac{1}{2}(u^{-1} + l^{-1})a_{jj'}| \leqq \frac{1}{2}(l^{-1} - u^{-1})(a_{jj}a_{j'j'})^{\frac{1}{2}}$, $j \neq j'$.

36. **On the Characteristic Functions of Absolutely Continuous Distribution Functions.** H. Robert van der Vaart, North Carolina State College and Institute for Theoretical Biology, Leiden, Netherlands.

A standard textbook remark is concerned with the fact that, if a characteristic function $\varphi(t)\varepsilon L_1(-\infty, \infty)$, then the corresponding distribution function $F(x)$ is absolutely continuous and has the density function $(2\pi)^{-1} \int_{-\infty}^{\infty} \varphi(t) \exp(-ixt)\, dt$. In most applications of this remark the density function is actually calculated as the *Cauchy principal value*, $f(x)$ say, of this integral. Therefore, the following stronger result, besides being interesting in itself, is better adapted to practical needs: if $f(x)$ is defined almost everywhere, it can be proved that (1) $f(x)$ is measurable, (2) $f(x)$ is non-negative almost everywhere, (3) $f(x)\varepsilon L_1(-\infty, \infty)$, (4) $\int_{-\infty}^{\infty} f(x)\, dx = 1$, and (5) (most interesting of all) $F(x)$ is absolutely continuous and has density function $f(x)$. Thus, the most basic of the conditions that are formally necessary in order that a distribution function $F(x)$ may be determined as the indefinite integral of the above Cauchy principal value $f(x)$ (viz., $f(x)$ is defined a.e.), is at the same time sufficient for the procedure to be valid. Moreover, all these statements remain true if $f(x)$ is replaced throughout by $f^*(x)$, the $(C, 1)$-value of $(2\pi)^{-1} \int_{-\infty}^{\infty} \varphi(t) \exp(-ixt)\, dt$. If $f(x)$ is defined, so is $f^*(x)$. and $f^*(x) = f(x)$. So it suffices to prove our results under the assumption that $f^*(x)$ is defined a.e.; the proof combines Fatou's lemma with certain tricks used in some proofs of the well known representation theorem for non-negative definite functions.

(Abstracts of papers presented at the Western Regional Meeting of the Institute, Albuquerque, New Mexico, April 19–20, 1962. Additional abstracts appeared in the March, 1962 issue.)

3. **A Contribution to Counter Theory.** Donald L. Bentley, Colorado State University. (By title)

Consider a counting process with independent arrivals which are distributed identically with cumulative distribution $F(x)$. Instead of a dead time there is a probability function $p(t)$ that an arrival will be counted where t is the length of time between the arrival and the last count prior to the arrival. $H(z)$, the cumulative distribution of time between counts, and $P(u)$, the limiting probability that an arrival at time $(t + u)$ will be counted given no arrivals between times t and $(t + u)$, are derived. In particular explicit results are obtained for the case where $F(x) = 1 - e^{-\lambda x}$ and $p(t)$ has a continuous derivative. If the further restriction is made that $p(t) = 1 - e^{-\mu t}$, then $H(z) = 1 - \exp\{-\lambda z + (\lambda/\mu)(1 - e^{-\mu z})\}$, and $P(u) = 1 - I(\lambda/\mu)e^{-\mu u}/I(\lambda/\mu, \lambda/\mu - 1)$ where $I(x, r) = \int_0^x y^r e^{-y}\, dy/\Gamma(r + 1)$.

4. Pairwise Comparison and Ranking. HANS BÜHLMANN AND PETER HUBER, University of California, Berkeley.

n items compared pairwise are to be ranked in their order of preference. Consider the items as players and a comparison as a game between two players. The "tournament" then establishes a preference matrix $A = (a_{ij})$ where $a_{ij} = 0, \frac{1}{2}$ or 1 according to whether player i has won over, drawn with or lost to player j. The problem is to estimate the correct ranking (defined in terms of an underlying probability matrix $p_{ij} = P[a_{ij} = 1]$) from the observed preference matrix A.

There is no uniformly best ranking procedure which remains invariant under permutations of the players. The Ordinary Ranking Procedure defined as: "Rank in descending order of the scores $s_i = \sum_j a_{ij}$" has the following property: For a wide class of loss functions it uniformly minimizes the risk among all permutation-invariant procedures if the class of permissible probability matrices is restricted to those of the form $p_{ij} = F(\theta_i - \theta_j)$ where $\{\theta_i\}$ are parameters associated with the different players and $F(t)$ is the logistic distribution function.

5. Optimum Stratification with Multiple Characters. S. P. GHOSH, University of California, Berkeley.

Dalenius (1950) has defined optimum stratification for the univariate case, and the same concept is extended for the multivariate case taking the generalized variance as a measure of dispersion. Optimum stratification is Stratified sampling, when stratification is based on multiple characters, and is defined to be the set of points which minimize the generalized variance of the sample means. An exact solution for optimum stratification points has been worked out for the bivariate case when proportional allocation is used. Dalenius' solution for the univariate situation follows as a special case. The optimum double dichotomy for any symmetric bivariate distribution is its center of gravity. For a bivariate normal distribution rotation of axes, passing through the center of gravity, does not reduce the genralized variance significantly. When the axes are rotated to coincide with the principal axes of the ellipsoid of concentration and the stratification is made on the transformed variables, the saving ratio of the stratified sampling to random sampling is the product of the univariate saving ratios.

6. On the Minimax Property of Hotelling's T^2-Test. N. C. GIRI, University of Arizona.

Let $X = (X_1, X_2, \cdots, X_p)'$ be normally distributed with mean $\xi = (\xi_1, \xi_2, \cdots, \xi_p)'$ and non-singular covariance matrix $\Sigma = E(X - \xi)(X - \xi)'$. On the basis of N random observations $X^1, X^2, \cdots, X^N$ on X, Hotelling's T^2-Test of the hypothesis $H_0:\xi = 0$, against the alternative $N\, \xi^1\Sigma^{-1}\xi = \delta > 0$ is given by, reject H_0, if $N\bar{X}S^{-1}\bar{X} \geqq K$ and accept otherwise, where $\bar{X}$, S are the sample mean, sample covariance matrix based on $X^1, X^2, \cdots, X^N$ respectively and K is constant chosen in such a way that the test has size α. The strongest known optimum property of this test is that of Semika, which, states that of all tests whose power function depends on $N\, \xi^1\Sigma^{-1}\xi$, Hotelling's T^2-Test is uniformly most powerful. The admissibility of this has been discussed by Stein, who has shown that this test is admissible for testing H_0 against unrestricted alternatives.

In this paper the minimax property of this test has been discussed. It has been shown that for $p \geqq 2$, and $N = p + 1$, and $\alpha = .01, .05, .10, .25, .50$, Hotelling's T^2-Test is not minimax.

7. An Inequality on the Concentration of Sums of Random Variables. LUCIEN LECAM, University of California, Berkeley. (By title)

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two linear spaces placed in proper duality by a bilinear form $\langle x, y\rangle$. A linear process X over $\mathfrak{Y}$ is a family $\{X(y);\ y \varepsilon \mathfrak{Y}\}$ of random variables such that $X(\alpha_1 y_1 + \alpha_2 y_2) = \alpha_1 X(y_1) + \alpha_2 X(y_2)$ for α_j real and $y_j \varepsilon \mathfrak{Y}$. Let Φ be the real part of the uniformly closed algebra generated by the functions $x \rightarrow \exp\{i\langle x, y\rangle\}$ for $y \varepsilon \mathfrak{Y}$. The characteristic function $\phi(y) = E \exp[iX(y)]$ extends to a positive linear functional on Φ. The process X will be called tight if for every $\epsilon > 0$ there is a $\delta > 0$ and a $w(\mathfrak{X}, \mathfrak{Y})$ compact convex subset K of $\mathfrak{X}$ such that $f \varepsilon \Phi$, $|f| \leqq 1$ and $|f(x)| < \delta$ for $x \varepsilon K$ implies $|EF| < \epsilon$. Such a tight expectation can be extended in the manner of Radon to a domain which includes in particular all indicators of closed convex subsets of $\mathfrak{X}$.

Theorem. Let $\{X_j;\ j = 1, 2, \cdots, n\}$ be n independent tight linear processes over $\mathfrak{Y}$. Let $S = \sum_{i=1}^{n} X_j$ and let K be a $w(\mathfrak{X}, \mathfrak{Y})$ closed convex symmetric subset of $\mathfrak{X}$. Take probabilities for the Radon extensions and let $C[2K;\ X_j] = \sup\{P[X_j \varepsilon x + 2K];\ x \varepsilon \mathfrak{X}\}$, $s = \sum_j \{1 - C[2K; X_j]\}$. Then for every $x \varepsilon \mathfrak{X}$ we have $P[S \varepsilon x + K] \leqq (1.3)(s + 1)^{-1/6}$. If the X_j have symmetric distributions, then $P[S \varepsilon x + K] \leqq \frac{3}{2}(t + 1)^{-\frac{1}{3}}$ with $t = \sum_j P[X_j \varepsilon K^c]$

8. A Note on the Central Limit Theorem. LUCIEN LECAM, University of California, Berkeley. (By title).

Let $\{X_n;\ n = 1, 2, \cdots\}$ be independent random variables having a sum $S = \sum_n X_n$. Let F be the cumulative distribution of S and let G be the cumulative distribution of the infinitely divisible distribution which has a characteristic function ϕ such that $\log \phi(u) = \sum_n E[e^{iuX_n} - 1]$. The following variations on results of Kolmogorov are obtained.

Theorem 1. Let $\alpha = \sup_n P[|X_n| > 0]$, then

$$\sup_x |F(x) - G(x)| \leqq (10)\alpha^{\frac{1}{3}}.$$

Let ξ_n, η_n, U_n, V_n be independent random variables such that $P[\xi_n = 1] = P[\eta_n = 1] = \alpha_n = 1 - P[\xi_n = 0] = 1 - P[\eta_n = 0]$. Let $X_n = (1 - \xi_n)U_n + \xi_n V_n$ and $Y_n = (1 - \eta_n)U_n + \xi_n V_n$. Let $T = \sum_n Y_n$ have cumulative distribution H.

Theorem 2. Let C_F, C_G and C_H be the concentration functions of F, G and H respectively. (For instance $C_F(\tau) = \sup_x P[x \leqq S \leqq x + \tau]$). Let $C' = \min(C_F, C_H)$ and let $\alpha = \sup \alpha_n$. Then $\sup_x |F(x) - H(x)| \leqq \delta + (10\alpha^{\frac{1}{3}}$ with $\delta = \sup_\tau [C'(\tau) - C_G(\tau)]$.

9. A Theorem on Sequences of Independent Time Variable Games. JAMES S. MACQUEEN, University of California, Los Angeles. (Introduced by Jacob Marschak.)

Each game $g \varepsilon G$ is defined by a family of distribution functions $\{F_g(v, t; s);\ s \varepsilon S_g\}$ for v, the reward received at the end of the game, and t, the duration of play, S_g being the strategy space of g. The games $g_1, g_2, \cdots$, with $g_i \varepsilon G$ are played one after another using a policy r which determines an element $s_i \varepsilon S_{g_i}$, $i = 1, 2, \cdots$, at the beginning of g_i, and then, for the outcome (v_{i+1}, t_{i+1}) in g_{i+1}, $\Pr\{v_{i+1} \leqq v, t_{i+1} \leqq t \mid s_{i+1}, (v_1, t_1), \cdots (v_1, t_i)\} = F_{g_{i+1}}(v, t; s_{i+1})$; i.e., the games are independent. The object is to do well in terms of the total reward received at the end of large fixed time period T; that is, $\sum_{i=1}^{n(T)} v_i$, where $n(T)$ is the largest integer n such that $t_1 + t_2 + \cdots t_n \leqq T$.

Let $\mathcal{S}$ be a class of policies for which the strong law of large numbers holds for both rewards and time of play. Conditions are given under which $P[\liminf T^{-1} \sum_{i=1}^{n(T)} v_i \geq \limsup T^{-1} \sum_{i=1}^{n'(T)} v_i'] = 1$ for $r' \varepsilon \mathcal{S}$ if, and only if, $r \varepsilon \mathcal{S}$ has the characterization $Ev_i - L^*Et_i \geq Ev_i' - L^*Et_i' - \epsilon_i$, where $L^* = \lim \sum_{i=1}^{n(T)} v_i/T$, $(v_1', t_1'), (v_2', t_2'), \cdots$ is the process associated with r', $n'(T)$ is the largest integer n such that $t_1' + t_2' + \cdots t_n' \leq T$, and $\epsilon_1, \epsilon_2, \cdots$ is a non-negative sequence with $\sum_{i=1}^{n} \epsilon_i/n \to 0$. This characterization is also necessary and sufficient for $P[\liminf (U_T(\sum_{i=1}^{n(T)} v_i) - U_T(\sum_{i=1}^{n'(T)} v_i'))T^{-1} \geq 0] = 1$, where the long run utility functions $\{U_T ; T > 0\}$ satisfy for $h > 0$, $\delta h \leq U_T(x + h) - U_T(x) < kh$, for some fixed $\delta, k > 0$.

10. On the Optimality of the Sequential Probability Ratio Test. TED K. MATTHES, RAND Corporation.

Consider the problem of testing sequentially a simple hypothesis H_0 against a single alternative H_1. In proving the optimality of any given sequential probability ratio test S_o it is customary to introduce a related Bayes problem and show that for each *a priori* probability ξ that H_0 is true, $0 < \xi < 1$, losses w_0 and w_1 corresponding to the two possible types of error can be chosen in such a way that S_o is Bayes relative to ξ, w_0, w_1. (There is assumed to be a unit cost per observation.) It is shown that the possibility of so choosing w_0 and w_1 follows easily from a simple theorem on the continuous mapping of certain subsets in the plane to the plane. In applying this theorem one need only verify that the mapping behaves properly on the boundary.

11. Rank Tests for Paired-Comparison Experiments Involving K Treatments. K. L. MEHRA, University of California, Berkeley.

Consider a paired-comparison experiment involving K treatments in which each of the N_{ij} comparisons of a pair (i, j) of treatments provides an observed comparison difference Z_{ijk} $(i \leq i < j \leq K; k = 1, 2, \cdots N_{ij})$. Assuming continuity of the underlying distributions $F_i(x)$ $(i = 1, 2, \cdots K)$ of the treatment effects, suitable rank tests for the hypothesis of no difference among them are proposed. For most situations, these tests provide considerable efficiency improvements over the available non-parametric tests, namely, the Bradley-Terry and Durbin tests. When all comparisons are made under the same experimental conditions the proposed test statistic is

$$L_N = 6 \sum_{i=1}^{K} \{\sum_{j \neq i} [R_N^{(i,j)} - S_N^{(i,j)}/(N_{ij})^{\frac{1}{2}}]\}^2/(N + 1)(2N + 1)K$$

where $N = \sum_{i<j} N_{ij}$ and $R_N^{(i,j)}$, $S_N^{(i,j)}$ are the sums of ranks of the positive and the negative Z_{ij}'s in a combined ranking of the $|Z_{ijk}|$ $(1 \leq i < j \leq K; k = 1, 2, \cdots N_{ij})$. For $K = 2$, the test reduces to the two-sided Wilcoxon paired-comparison test. Under suitable regularity conditions, the statistic L_N has, as $N \to \infty$, s limiting non-central χ^2 distribution against shift alternatives. The asymptotic efficiencies of this test relative to the Bradley-Terry and Durbin tests and the corresponding F-test are independent of K. The asymptotic efficiency relative to the F-test is $3/\pi$ in case of normality whereas that of Bradley-Terry (or Durbin) test is $2/\pi$. These results are extended to the case of nonuniformity of experimental conditions, in which case the test is based on rankings of the absolute Z's in each "block" separately.

12. Multi-Sample Analogues of some One-Sample Tests. M. L. PURI, University of California, Berkeley.

Consider t treatments in an experiment involving paired comparisons and suppose that the observations $(X_1^{(i)}, X_1^{(j)}), \cdots, (X_{mij}^{(i)}, X_{mij}^{(j)})$ constitute a sample corresponding to the

(i, j)th treatment; $i < j$; $i, j = 1, \cdots, t$. Then for testing the hypothesis of no difference between these treatments on the basis of $\binom{t}{2} = c$ samples, test statistics of the form

$$V = \sum_{i=1}^{t} \left[\sum_{j \neq i} \left\{ \sum_{\nu=1}^{N} \xi_{N,\nu}^{(i,j)} a_{N,\nu} \right\} \Big/ (B_{N,ij})^{\frac{1}{2}} \right]^2$$

are considered. Here the $a_{N,\nu}$ are given numbers; the $B_{N,ij}$ are certain constants; and $\xi_{N,\nu}^{(i,j)} = +1(-1)$ if the νth smallest of $N = \sum_{i<j} m_{ij}$ ordered absolute values $|X_r^{(i)} - X_r^{(j)}|$, $r = 1, \cdots, m_{ij}$; $1 \leqq i < i \leqq t$ comes from the (i, j)th pair and is from a positive (negative) difference and $\xi_{N,\nu}^{(i,j)} = 0$ otherwise. Extending the results of Puri (abstract, *Ann. Math. Statist.* **32** (1961) 1350) and Govindarajulu (Thesis, University of Minnesota (1960)), sufficient conditions are given for the joint asymptotic normality of $R_N^{(i,j)} = \sum_{\nu=1}^{N} \xi_{N,\nu}^{(i,j)} a_{N,\nu}$. Assuming Pitman's shift alternatives and Lehmann's distribution free alternatives, the limiting distributions of V are derived. Finally, the asymptotic relative efficiencies of the V test relative to some of its competitors viz. the Bradley-Terry Test (*Biometrika* **39** (1952) 324–345), Durbin's test (*British J. Psychology* **4** (1951) 85–90) and Mehra's test which is a particular case of the V test (abstract on p. 827 of this issue) are obtained.

13. **Determination of Estimates for a Time Series** (Preliminary report). PAUL H. RANDOLPH, Purdue University.

On the basis of the measurements $\{[t_1, X(t_1)], \cdots, [t_m, X(t_m)]\}$ one sometimes wishes to estimate a true value $L(x)$, (t), where L is a given operator. For example, we may specify that $L(x)(t) = (d \times (t)/dt)$, (t) the velocity of a vehicle at time t. If we denote the estimator by $L^*\{t_1, X(t_1), \cdots, t_m, X(t_m)\}(t)$, then we can define a risk function g of the form $g(x_d, L^*) = P_{x_d}\{|L^*(\cdot)(t) - L(x)(t)| > h(L(x)(t))\}$ where x_d is the deterministic part of the signal and h is some function. We choose L^* so as to minimize $\sup_{x_d} g(x_d, L^*)$.

14. **The Cumulative Hazard Rate in Reliability Studies.** ROBERT B. RUTLEDGE AND WALDO A. VEZEAU, Saint Louis University. (By title)

The cumulative hazard rate is defined as a function of time that satisfies certain general properties. For distribution functions that have a hazard rate, the cumulative hazard rate is the integral of the hazard rate, although if no hazard rate exists then the cumulative hazard rate is well defined. A one to one correspondence between the set of cumulative hazard rates and the set of failure distribution functions is set up. Logical operators are defined in the set of all failure distribution functions and the algebraic properties of the system are developed in terms of the cumulative hazard rate. Modes of failure of various redundant structures are related to the algebraic properties of the system. A generalized model of a component having two modes of failure is presented, and a discussion of optimal redundant structures of such components is given. The cumulative hazard rate is related to the Poisson process and certain multi-mode failure models presented. Finally the cumulative hazard rate is used to generate families of distribution functions having special properties that facilitate the analysis of the preceding systems. The results of the paper are general in that no restrictive assumptions need be made on the form of the failure distribution functions, such as exponential or Weibull.

15. **Least Squares and Restraints.** MARVIN ZELEN, University of Wisconsin.

Theorem. Let $E(Y) = X'\beta$, var $Y = \sigma^2 I$ where $Y(n \times 1), \beta(p \times 1), X(p \times n)$ with rank $q(q \leqq p)$. Furthermore the β satisfy the $k = k_1 + k_2$ restraints $K_i'\beta = m_i(i = 1, 2)$ where

$K_i(p \times k_i)$ has rank k_i such that the $K_1'\beta$ are *not* estimable functions and $K_2'\beta$ *are* estimable functions. Let $A = XX'$, $r = p - q$ and define $C(p \times p)$, $H_1(p \times k_1)$, $H_2(p \times r - k_1)$, $L(p \times r - k_1)$ such that (i) $CAC = C$, (ii) $H_i'A = 0$, (iii) $H_1'K_1$, $H_2'L$, and $(K_2'CK_2)$ are nonsingular. Then the minimum variance unbiased estimate (among the class of estimators linear in Y) of any estimable function $\theta = a'\beta$, $a = (a_1, a_2, \cdots, a_p)$, is $\hat{\theta} = a'\hat{\beta}$ where $\hat{\beta} = CXY + CK_2(K_2'CK)^{-1}(m_2 - K_2'CXY) + H_1(K_1'H_1)^{-1}m_1 + H_2(L'H_2)^{-1}m$ and m is an arbitrary $r - k_1 \times 1$ vector.

(Abstracts of papers to be presented at the European Regional Meeting of the Institute, Dublin, Ireland, September 3–5, 1962. Additional abstracts will appear in the September, 1962 issue.)

1. On the Combination of Independent Two-Sample Tests of a General Class. M. L. Puri, University of California, Berkeley.

Let $X_{i1}, \cdots, X_{im_i}$ and $Y_{i1}, \cdots, Y_{in_i}$, $i = 1, \cdots, k$, be k pairs of samples of mutually independent observations from continuous distribution functions $F_i(x)$ and $G_i(y)$ respectively; $i = 1, \cdots, k$. Then, for testing the hypothesis $F_i = G_i$, $i = 1, \cdots, k$ the test statistics of the form (i) $T = \sum_{i=1}^k c_i t_i$ and, (ii) $Q = \sum_{i=1}^k c_i Q_i$ are considered. Here c_i are the weights which may depend upon the sample sizes, t_i is the student's statistic for testing the equality of means between two normal populations with the same variance corresponding to the ith pair of samples and Q_i is the Chernoff-Savage statistic (*Ann. Math. Statist.* **29** (1958) 972–994). Under suitable assumptions, the weights c_i which maximize the local asymptotic powers of the tests (i) and (ii) are obtained. These results are specialized to (a) Pitman's shift alternatives and (b) Lehmann's distribution free alternatives.

2. On the Elteren W-Test and Non-Null Hypotheses. M. L. Puri, University of California, Berkeley. (By title)

Consider $\sum_{i=1}^k (m_i + n_i)$ independent random variables, of which m_i have the common cumulative distribution function $F_i(x)$ and n_i have the $G_i(y)$. The paper investigates the behavior of W and T tests (Elteren (1960). *Bull, Inst. Internat. Statist.* **37** No. 3 351–361; Puri, "On the combination of a class of independent two sample tests;" preceding abstract) when $F_i(x)$ and $G_i(y)$ are normal populations with slightly different variances. Under suitable assumptions the approximate power functions of both W and T tests are derived and their relative sensitivities are measured by comparing their coefficients of robustness.

3. On the Combination of Independent One-Sample Tests of a General Class. M. L. Puri, University of California, Berkeley. (By title)

Consider k samples of paired observations and let $(X_{i1}, Y_{i1}), \cdots, (X_{iN_i}, T_{iN_i})$ constitute the ith sample; $i = 1, \cdots, k$. X_{ij} and Y_{ij} have the continuous cumulative distribution functions $F^{(i)}(x)$ and $F^{(i)}(x; \theta)$ respectively. Then for testing the hypothesis that each of the distributions F is symmetric with respect to zero, test statistics of the form $\tau = \sum_{i=1}^K C_i\tau_i$ are considered. Here C_i are real positive numbers which may depend upon the sample sizes and $\tau_i = \sum_{j=1}^{N_i} E_{N_i,j}^{(i)} Z_{N_i,j}^{(i)}$ where $Z_{N_i,j}^{(i)} = 1$, if the jth smallest of the ordered absolute values $|Y_{i1} - X_{i1}|, \cdots, |Y_{iN_i} - X_{iN_i}|$ comes from a positive difference, and $Z_{N_i,j}^{(i)} = 0$, otherwise: and $E_{N_i,j}^{(i)}$ are given numbers. Under suitable assumptions the coefficients C_i are found for which the class of tests τ are L.A.M.P. (locally asymptotically most powerful). In particular it is shown that both against (a) Lehmann's alternatives and (b) Pitman's alternatives, the L.A.M.P. τ test is design free in the sense that the weights $C_i(N_i)$ are independent of sample sizes N_i. The asymptotic efficiency of the τ test relative to some of its parametric competitors is discussed.

4. **Distributions of Certain Statistics in Samples from a Triangular Population.** PAUL R. RIDER, Aeronautical Research Lab, Wright-Patterson AFB, Ohio.

Distributions of means, midranges, medians, and geometric means in samples from a triangular population are derived.

(Abstracts of papers to be presented at the Annual Meeting of the Institute, Minneapolis, Minnesota, September 7–10, 1962. Additional abstracts will appear in the September, 1962 issue.)

1. **Optimal Classification Rules** (Preliminary report). SOMESH DAS GUPTA, University of North Carolina. (Introduced by George E. Nicholson, Jr.)

Let π_i's ($i = 0, 1, 2$) be three populations in which the p-dimensional response vector $\mathbf{X}(p \times 1)$ is distributed as $N(\boldsymbol{\mu}_i, \Sigma)$ respectively; $\boldsymbol{\mu}$'s and Σ are unknown. The problem is to take one of the decisions $d_i : \boldsymbol{\mu}_0 = \boldsymbol{\mu}_i$ ($i = 1, 2$). Let $\mathbf{x}_i$'s be sample mean vectors from random samples of sizes n_i from π_i ($i = 0, 1, 2$); S is the pooled unbiased estimator of Σ. The loss due to wrong classification in π_i is assumed to be a monotonic increasing function of $(\boldsymbol{\mu}_0 - \boldsymbol{\mu}_i)'\Sigma^{-1}(\boldsymbol{\mu}_0 - \boldsymbol{\mu}_i)k_i$, where $k_i = (1/n_0 + 1/n_i)^{-1}$ for unequal n_i's and equal to 1 if $n_1 = n_2$. It is proved that the rule, to take decision d_j if minimum of $(\mathbf{x}_0 - \mathbf{x}_i)'S^{-1}(\mathbf{x}_0 - \mathbf{x}_i)k_i$ is obtained for $i = j (j = 1, 2)$, is minimax and admissible in the class of all rules. Also among the class of all rules for which the probabilities of correct classifications, depending only on the distance between π_1 and π_2, are equal, the above rule gives the maximum probability of correct classification. This rule coincides with the maximum-likelihood rule. Some other problems in classification in $K(>2)$ populations and structured populations and classification of N different individuals are under investigation.

2. **Approximate Distribution of Extremes for Nonsample Cases.** JOHN E. WALSH, Systems Development Corp. (By title)

Random variable is $(X_1, \cdots, X_n)$, possibly discontinuous in one or more coordinates, and $X_{(i)} = i$th order statistic of X's. The cumulative distribution function (cdf) of X_i is $F_i(x)$ and $\bar{F}(x) = \sum_{i=1}^{n} F_i(x)/n$. Suppose that $X_1, \cdots, X_n$ are independent or that they have an m-dependence analogous to that considered in "The Poisson distribution as a limit for dependent binomial events with unequal probabilities," J. E. Walsh (1955). *J. Oper. Res. Soc. Amer.* **3** 198–209. Then, for n large, $m \ll n$, and $t \ll n$,

$$P[X_{(n-t)} \leqq x] \doteq \exp\{-n[1 - \bar{F}(x)]\} \textstyle\sum_{j=0}^{t} \{n[1 = \bar{F}(x)]\}^j/j!$$

and

$$P[X_{(1+t)} > x] \doteq \exp[-n\bar{F}(x)] \textstyle\sum_{j=0}^{t} [n\bar{F}(x)]^j/j!.$$

The proportion of X's $\leqq x$, denoted by $\bar{F}_n(x)$, estimates $\bar{F}(x)$, with the arithmetic average of the $\bar{F}_n(x)$'s used for more than one observation on $(X_1, \cdots, X_n)$. The extremes are known to have three types of asymptotic distributions when the X's are a sample from a continuous population with cdf $F(x)$, subject to $F(x)$ satisfying some conditions. These types of asymptotic distributions also occur for the extremes of $(X_1, \cdots, X_n)$ if the m-dependence requirements hold and $\bar{F}(x)$ satisfies conditions like those for $F(x)$. These results may explain why extreme value theory has approximated many practical situations. Also, a basis is furnished for deciding, from the characteristics of the situation, whether extreme value methods are usable.

(Abstracts of papers not presented at any meeting of the Institute.)

1. **Further Results on Optimal Sequential Tests for the Mean of a Normal Distribution** (Preliminary report). H. Chernoff and J. V. Breakwell, Stanford University and Lockheed Space and Missile Co., Palo Alto. (By title)

In the *Proc. Fourth Berkeley Symp. Math. Stat.* Prob. **1**, 1960, Chernoff reduced the problem of sequentially testing whether the drift μ of a Wiener process is positive or negative to the solution of a free boundary problem. Normalizing with cost of one per unit time of observation, variance one per unit time and loss $r(\mu) = |\mu|$ for a wrong decision, the optimal procedure is characterized by the boundary $\tilde{x}(t)$. The discrete problem of testing whether the mean of a normal distribution with known variance is positive or negative is equivalent to the continuous problem where one is permitted to stop only at times t_0, $t_0 + \delta$, $t_0 + 2\delta \cdots$ and yields boundary $\tilde{x}_\delta(t)$. As $t \to \infty$, $\tilde{x}(t) \sim (1/4t)[1 - (1/12t^3) + (7/240t^6) - \cdots]$. As $t \to 0$, $\tilde{x}(t) = (t[-3 \log t + 0(1)])^{1/2}$ and a formal expansion gives $\tilde{x}(t) = (t[-3 \log t - \log 8\pi + (2/\log t) + \cdots])^{1/2}$. As $\delta \to 0$, $\tilde{x}(t) = \tilde{x}_\delta(t) + k(\delta^{1/2}) + O(\delta^{1/2})$, where k is approximately .6. Numerical calculations for $\tilde{x}_\delta$ and the risk have been carried out. Recently obtained and unpublished overlapping results by Moriguti and Robbins and by Bather have been brought to our attention.

2. **A Generalization of the Likelihood-Ratio Test for Guarantee Time Associated with the Exponential Failure Law** (Preliminary report). Satya D. Dubey, Procter and Gamble Co., Cincinnati, Ohio. (By title)

This is a continuation of two other papers of the author whose abstracts have appeared in the June, 1961 issue of *Ann. Math. Statist.* Here he considers a censoring scheme in which only middle $(b - a + 1)$ *ordered* observations in a random sample of size n with $1 \leqq a < b \leqq n$ are available. In a previous work, he derived the likelihood-ratio test of the hypotheses on guarantee time when $a = 1$ and $1 < b \leqq n$ besides considering several other problems. In the present case, he has defined a test function based on $(b - a + 1)$ *ordered* observations which includes the likelihood-ratio test, derived previously. This test function has the advantage of receiving applications in many varietal situations. Besides, considering test function for general alternatives, it has also been discussed for a one-sided alternative which is realistic for guarantee time in life testing situations. The exact distribution of the statistic has been derived. The lower and upper 1, 5 and 10 per cent critical values are computed up to the sample size 5. The exact kth moment expression of this statistic has also been obtained. Finally, the power functions of the tests for all cases have been derived and some properties have been investigated.

3. **Comparison of Randomized and Non-Randomized Factorial Experiments.** S. Ehrenfeld and S. Zacks, Columbia University and Israel Institute of Technology. (By title)

A comparison of randomized and non-randomized factorial experiments is made in terms of decision theory. In this context the statistician chooses a *strategy* which consists of a randomization procedure combined with an estimator. The class of randomization procedures considered includes, as special cases, the classical fixed fractional factorial experiments as well as the procedures previously studied by S. Ehrenfeld and S. Zacks in *Ann. Math. Statist.* 1961. A preference relationship between procedures is defined in terms of minimax risk using a mean square error loss function. Minimax strategies, for various states of information concerning nuisance parameters, are derived. It is shown that unless *all* the signs of the nuisance parameters are known, randomization procedure I, previously studied is optimal. When all the signs are known there exists a fractional factorial experiment, combined with a suitable *adjusted estimator*, which is optimal.

4. Minimax Closeness Strategies and Randomized Factorial Experiments. S. EHRENFELD AND S. ZACKS, Columbia University and Israel Institute of Technology. (By title)

Further investigation of the optimal properties of randomization procedures, studied previously, was carried out in a decision framework using the *closeness criterion* as the payoff function. This criterion evaluates estimators in terms of the probability of falling within a prescribed neighborhood of the true value. Complete class theorems are proven and minimax estimators derived. These estimators relate to various states of information about the nuisance parameters. The relative gain in information is measured in terms of minimax closeness risk. These estimators are compared with minimax variance estimators.

5. On the Adjoined State of a Markov Chain. DAVID A. FREEDMAN, University of California, Berkeley. (By title)

Let $\{x_t : t \geqq 0\}$ be a separable measurable Markov chain on (Ω, F, P) with a countable number of states, all stable, stationary transition probabilities, and satisfying the forward and backward equations. Let ∞ be the adjoined state, and $S_\infty(\omega) = \{t : x_t(\omega) = \infty\}$. Then for a.a. ω, $S_\infty(\omega)$ is a closed set of Lebesgue measure 0. Conversely, given any $D \subset [0, \infty)$ which is closed and of measure 0, there exists a Markov chain satisfying the requirements above, such that for a.a. ω, $S_\infty(\omega)$ is homeomorphic to D. There exists also a Markov chain satisfying those requirements, and the further requirements that each state is positive recurrent, and that it is possible to pass from each state to any other state in a finite number of jumps; such that, for a.a. ω, $S_\infty(\omega)$ contains a countable number of disjoint homeomorphic images of D. Finally, the topology of $S_\infty(\omega)$ does not depend on the infinitesimal generator. There exist processes having the same infinitesimal generators as the processes described above, and obeying the same regularity conditions, with $S_\infty(\omega)$ consisting a.s. of isolated points.

6. Fourier Integral Representations for Markov Transition Probabilities (Preliminary report). J. F. C. KINGMAN, Pembroke College, Oxford, England. (By title) (Introduced by D. G. Kendall.)

Let $p_{jk}(t)$ $(t \geqq 0)$ be the transition probabilities of a temporally homogeneous Markov process on a denumerable state space, and suppose that $p_{jk}(t) \to \delta_{jk}$ as $t \to 0$. Kendall (*Proc. Lond. Math. Soc.* **9** (1959) 417–31) has given an integral representation $p_{jk}(t) = \pi_{jk} + \int_{-\infty}^{\infty} e^{i\lambda t} m_{jk}(\lambda) d\lambda$, for $p_{jk}(t)$. Using a theorem of Neveu, it is shown that, for $j = k$, the function $m_{jk}(\lambda)$ is given by $m_{jj}(\lambda) = \pi^{-1} \operatorname{Re}\{\lim_{z \to i\lambda, \operatorname{Re}(z) > 0} \int_0^\infty e^{-zt} p_{jj}(t) dt\}$, when $\lambda \neq 0$, and that the integral can be expressed in terms of a function which reduces, when j is stable, to the recurrence time characteristic function of j. The function $m_{jj}(\lambda)$ is even, strictly positive (or identically zero), continuous, and integrable. The analysis is extended to deal with a class of recurrent events in continuous time.

7. Minimax Variance Strategies, Information and Randomized Factorial Experiments. S. Zacks, Israel Institute of Technology and Columbia University. (By title)

The problem of estimating a set of pre-assigned parameters by a randomly chosen fractional replicate of a full factorial system, previously studied by S. Ehrenfeld and S. Zacks, *Ann. Math. Statist.* 1961, is reduced into a decision framework. The loss function adopted is the sum of variances of the estimators. A strategy consists of a randomization procedure and an estimator. It is proven that the class of *conditional least squares* estimators is complete with respect to all linear unbiased estimators. The variances of these conditional least squares estimators depend only on the nuisance parameters and not on the pre-assigned ones. The class of conditional least squares estimators is characterized by *adjusting* the usual least squares estimator, according to treatments sampled and the available information concerning nuisance parameters. *Minimax* estimators are derived for various states of information. The types of information studied relate to the situations where: nuisance parameters are bounded, *some* or *all* of their signs are known, relative magnitudes are known within limits. The relative gain in minimax risk, in connection with these various states, being a measure of information, is studied.

CORRECTION TO ABSTRACT

The Editor regrets that the authorship of Abstract No. 2, these *Annals* **32** 1346, was incorrectly given. It should have read: 2. **Unbiased Estimation of Probability Densities** (Preliminary Report). S. G. Ghurye and Ingram Olkin, University of Minnesota.