A MARKOVIAN MODEL FOR THE ANALYSIS OF THE EFFECTS OF MARGINAL TESTING ON SYSTEM RELIABILITY[1]

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Summary. In [3], a general model for the reliability analysis of systems under various preventive maintenance policies is postulated and analyzed. The integral equations that determine the expected number of failures, the expected number of preventive removals, and the survival probability function are developed. In the present paper, a particular model of a system subject to marginal testing is considered and explicit values of these performance measures are obtained.

Under a marginal testing policy, the system is maintained in operating condition by replacing all failed components as soon as they fail and, at regular intervals, conducting a test to locate those components which are still operating satisfactorily but which are expected to fail in the near future. All components located by this test are replaced.

In this model, it is assumed that a component may be in any one of $n + 1$ states, $0, 1, \cdots, n$, and, during normal operation, these states constitute a continuous-parameter Markov process in which state n is the failed state. When a component enters state n, it is immediately replaced by one in state 0. The marginal test detects the state and states $k, k + 1, \cdots, n - 1$ are considered marginal. The test is performed at fixed intervals, and, if a component is found in the marginal state, it is replaced by one in state 0.

Since this model provides for transitions from any operative state to any other state, recovery from the marginal state to the good state is permitted, a characteristic which was not allowed in the model of [3]. In addition, a choice of the level at which the component is considered marginal is permitted. The loss of generality lies in the assumption that the process is Markovian. As in [3], it is assumed that there is no dependence between transitions in different component positions and that every system failure is corrected by the replacement of one component, so that the problem of determining system performance measures is reduced to the problem of determining the corresponding quantities for a single component position.

In the analysis of this model, we shall first analyze the Markov process in the absence of marginal tests and determine the matrix of probabilities $H_{ij}(t)$ that a component is in state j at t given that it is in state i at 0, the distribution function $F_i(t)$ of time to failure for a component initially in state i, and $\mathcal{B}_i(t)$, the expected number of failures in a component position by t, given that the component is in state i at 0. It is then seen that, with the marginal test performed

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at intervals of length T and replacement of marginal components, the states of the component just before successive test points constitute a discrete-parameter Markov chain. The matrix of transition probabilities p_{ij} for this chain and the probability $P_i(r)$ that the component is found in state i at the rth test point are determined. Then the expected number of preventive removals in a component position, $U_p(t)$, may be expressed as the sum of the probabilities of finding the component marginal over all test points before t while the expected number of failures, $U_f(t)$, is expressible in terms of the P_r's and the $\mathcal{B}_i$'s. The asymptotic values of $U_p(t)$ and $U_f(t)$ are determined by the stationary properties of Markov chains. Finally, the Markovian nature of this model makes it relatively easy to express $R(t; x)$, the probability of no failure in a component position in an interval t following system age x, in terms of the matrices discussed previously.

1. Assumptions and analysis.

(1) The system is an assemblage of a finite number of components which performs some function.

(2) Associated with a given component, there are a finite number of states, $0, 1, \cdots, n$. (The value of n may vary with different component types.) During normal system operation, these states constitute a continuous-time Markov process. If the component is in state i at time t, the probability that it enters state j, $j \neq i$, by $t + dt$ is $Q_{ij}dt + o(dt)$, with $Q_{ii} = 0$ for all i.

(3) A component is initially in state 0.

(4) The component performs its function in the system in states $0, \cdots, n - 1$. When it enters state n, it immediately induces a system failure and is replaced.

(5) There exists a test which identifies the state of a component. States $k, k + 1, \cdots, n - 1$ are considered marginal and are collectively labeled state B. At fixed intervals of length T, the test is performed and, if the component is found in state B, it is replaced by a new one.

Now we proceed with the analysis which determines exact and asymptotic expressions for $U_p(t)$, $U_f(t)$, and $R(t; x)$. We shall make use of the following $n \times n$ matrices, related to the Markov process defined in Assumption (2).

Definitions. $\mathbf{H}(t)$ is the matrix of transition probabilities with elements $H_{ij}(t)$, the probability that a component is in state j at t, given that it is in state i at 0, $i, j = 0, \cdots, n - 1$, for $t \geqq 0$. $\mathbf{H}(t) = (0)$ for $t < 0$. $\mathbf{Q}^*$ is the matrix of probabilities of transition into state j when the component leaves state i.

$$Q^*_{ij} = Q_{ij}/Q_i, \quad \text{where } Q_i = \sum_{k=0}^{n} Q_{ik}, \quad i, j = 0, \cdots, n - 1.$$

$\mathbf{K}(t)$ is a diagonal matrix of probabilities that a component in state i at 0 makes a transition out of state i by t.

$$\begin{aligned} K_{ij}(t) &= \delta_{ij}[1 - \exp(-Q_i t)], \qquad i, j = 0, \cdots, n - 1, \qquad && t \geqq 0. \\ &= 0, && t < 0. \end{aligned}$$

Corresponding to each time-dependent matrix, there is a matrix of Laplace-Stieltjes transforms. In each case, the transform will be represented by the same symbol as the original matrix with a bar. Thus, we have, for example,

$$\bar{K}_{ij}(s) = \int_{0-}^{\infty} e^{-st}\, dK_{ij}(t) = \delta_{ij} Q_i (s + Q_i)^{-1}.$$

With these definitions, one obtains, as a special case of Pyke's Theorem 4.1 [6]

THEOREM 1.1. $\mathbf{H}(t)$ *is determined by the matrix integral equation,*

$$\begin{aligned} \mathbf{H}(t) &= \mathbf{I} - \mathbf{K}(t) + [\mathbf{K}(t)\mathbf{Q}^*] * \mathbf{H}(t), \qquad & t \geqq 0 \\ &= (0) & t < 0 \end{aligned} \tag{1.1}$$

where $*$ *denotes matrix convolution, and its transform is expressed*

$$\bar{\mathbf{H}}(s) = (\mathbf{I} - \bar{\mathbf{K}}(s)\mathbf{Q}^*)^{-1}(\mathbf{I} - \bar{\mathbf{K}}(s)), \tag{1.2}$$

where $(\mathbf{I} - \bar{\mathbf{K}}(s)\mathbf{Q}^*)$ *is non-singular when* $s > 0$.

Now we define $F_i(t)$ as the distribution function of time to failure for a component in state i at time zero, $i = 0, \cdots, n - 1$. Since $F_i(t)$ is the probability that the component is in none of the states $0, \cdots, n - 1$ at time t, it follows that

$$F_i(t) = 1 - \sum_{j=0}^{n-1} H_{ij}(t) \qquad \text{and} \qquad \bar{F}_i(s) = 1 - \sum_{j=0}^{n-1} \bar{H}_{ij}(s). \tag{1.3}$$

Define $\mathcal{B}_i(t)$ as the expected number of successive failures by t, where the component in position at time 0 is in state i and no marginal test is performed in $(0, t]$, $i = 0, \cdots, n - 1$. One then obtains

THEOREM 1.2.

$$\mathcal{B}_i(t) = F_i(t) + \mathcal{B}_i(t) * F_0(t) \tag{1.4}$$

and hence

$$\bar{\mathcal{B}}_i(s) = \frac{\bar{F}_i(s)}{1 - \bar{F}_0(s)}. \tag{1.5}$$

PROOF. $\mathcal{B}_i(t) = \sum_{k=1}^{\infty} F_i(k; t)$, where $F_i(k; t)$ is the probability of at least k failures in $(0, t]$, given state i at zero. Now

$$F_i(1; t) = F_i(t) \qquad \text{and} \qquad F_i(k + 1; t) = F_i(k; t) * F_0(t),$$

since each replacement is initially in state 0, and the theorem follows.

At this point we focus our attention on Assumption (5) and define p_{ij} as the probability that the component in position at $rT-$ is in state j, given that the component in position at $rT-T-$ was in state i ($i = 0, \cdots, k - 1, B$, $j = 0, \cdots, k - 1, B$). It is clear that this probability is independent of r and that the sequence of states at time points $T-, 2T-, \cdots$ constitutes a discrete-parameter Markov chain with $k + 1$ states and transition probability matrix $\mathbf{p}$.

We can express p_{ij} in terms of quantities previously derived, as follows:

THEOREM 1.3.

$$p_{ij} = H_{ij}(T) + H_{0j}(T) * \mathcal{B}_i(T), \qquad i = 0, \cdots, k-1, j = 0, \cdots, k-1. \tag{1.6}$$

$$p_{iB} = \sum_{j=k}^{n-1} [H_{ij}(T) + H_{0j}(T) * \mathcal{B}_i(T)], \qquad i = 0, \cdots, k-1. \tag{1.7}$$

$$p_{Bj} = p_{0j}, \qquad j = 0, \cdots, k-1, B. \tag{1.8}$$

PROOF. For $i, j < k$, p_{ij} is the probability that the component in state i at the beginning of an interval of length T is in state j at the end of the interval or that it fails during the interval, one or more replacements occur, and some replacement is in state j at the end of the interval. The usual renewal argument then leads to (1.6). (1.7) arises from the fact that state B is the union of states $k, \cdots, n-1$, and (1.8) is based on the idea that a component in state B just before a test point is replaced by one in state 0.

Let $\mathbf{P}(r) = (P_0(r), \cdots, P_{k-1}(r), P_B(r))$, where $P_i(r)$ is the probability of state i at $rT-$, with $\mathbf{P}(0) = (1, \cdots, 0, 0)$. Clearly,

$$\mathbf{P}(r) = \mathbf{P}(0)\mathbf{p}^r \quad \text{or} \quad P_i(r) = (\mathbf{p}^r)_{0i}. \tag{1.9}$$

This completes the preliminaries necessary to the exact expressions for $U_p(t)$ and $U_f(t)$

THEOREM 1.4.

$$U_p(t) = \sum_{m=1}^{r} P_B(m) = \left(\sum_{m=1}^{r} \mathbf{p}^m\right)_{0B}, \qquad rT \leqq t < (r+1)T. \tag{1.10}$$

$$\begin{aligned} U_f(t) &= \sum_{m=0}^{r-1} \left[\sum_{i=0}^{k-1} P_i(m)\mathcal{B}_i(T) + P_B(m)\mathcal{B}_0(T)\right] \\ &\quad + \sum_{i=0}^{k-1} P_i(r)\mathcal{B}_i(t-rT) + P_B(r)\mathcal{B}_0(t-rT), \\ &\qquad rT \leqq t < (r+1)T. \end{aligned} \tag{1.11}$$

PROOF. Since the probability of a preventive removal at mT is the probability of state B at $mT-$, (1.10) expresses the expected number of preventive removals after r test points. The expected number of failures in an interval following mT is obtained by summing the expected number of failures conditional on initial state i over the probability of initial state i. Summation over all the intervals up to t yields (1.11).

In order to find asymptotic values of $U_p(t)/t$ and $U_f(t)/t$, we shall use the limiting values of $\mathbf{P}(r)$ as r becomes large. These limiting values were expressed by Mihoc [4, pp. 114–116][2] in terms of the matrix $\mathbf{I} - \mathbf{p}$ as follows:

$$\pi_i = P_i(\infty) = D_i \left(\sum_{j=0}^{k-1} D_j + D_B \right)^{-1} \tag{1.12}$$

where D_i is the cofactor of $(\mathbf{I} - \mathbf{p})_{ii}$, i.e., the determinant obtained by striking out the ith row and ith column from $\mathbf{I} - \mathbf{p}$, and $D_i \neq 0$ for some i.

THEOREM 1.5.

$$\lim_{r\to\infty} [U_p(rT)/rT] = \pi_B/T, \tag{1.13}$$

$$\lim_{t\to\infty} \frac{U_f(t)}{t} = \frac{1}{T}\left[\sum_{i=0}^{k-1} \pi_i \, \mathcal{B}_i(T) + \pi_B \, \mathcal{B}_0(T) \right]. \tag{1.14}$$

PROOF. In this limiting case, the expected number of removals of either type per unit time is equal to the expected number of removals per maintenance interval divided by the length of the maintenance interval.

In order to formulate an expression for the reliability function, $R(t; x)$, it is necessary to define the following.

DEFINITIONS.

(1) $P_i(r, y)$ is the probability that the component in position at time $rT + y$ is in state i, where $i = 0, \cdots, n - 1$ and $y < T$. Thus, $\mathbf{P}(r, y)$ is an $n \times 1$ matrix.

(2) $p_{ij}(y)$ is the probability of state j at $rT + y$, given state i at $rT-$, where $i = 0, \cdots, k - 1, B, j = 0, \cdots, n - 1$ and $y < T$. $\mathbf{p}(y)$ is an $n \times k + 1$ matrix.

(3) $S_i(m, r, y)$ is the probability of state i at $(r + m)T+$ and no failure in the interval $(rT + y, rT + mT]$, where $i = 0, \cdots, k - 1$. $\mathbf{S}$ is a $k \times 1$ vector.

(4) q_{ij} is the probability of state j at $rT + T+$ and no failure in $(rT, rT + T]$, given state i at $rT+$, where $i, j = 0, \cdots k - 1$. $\mathbf{q}$ is a $k \times k$ matrix. $q_{ij}(y)$ is the probability of state j at $rT + T+$ and no failure in $(rT + y, rT + T]$, given state i at $rT + y$, where $i = 0, \cdots, n - 1, j = 0, \cdots, k - 1$. $\mathbf{q}(y)$ is a $k \times n$ matrix.

Now we shall express these matrices in terms of quantities previously derived and then proceed to formulate $R(t; x)$.

By the reasoning of Theorem (1.3), it may be seen that

$$p_{ij}(y) = H_{ij}(y) + H_{0j}(y) * \mathcal{B}_i(y), \quad i = 0, \cdots, k - 1, j = 0, \cdots, n - 1, \tag{1.15}$$

$$p_{Bj}(y) = p_{0j}(y) \tag{1.16}$$

and, as in (1.9),

$$\mathbf{P}(r, y) = \mathbf{P}(r)\mathbf{p}(y) \quad \text{or} \quad P_i(r, y) = (\mathbf{p}^r\mathbf{p}(y))_{0i}. \tag{1.17}$$

[2] This reference was obtained from Richard Barlow [1] who credits Professor J. Gani with bringing it to his attention.

From the definitions of q_{ij} and $q_{ij}(y)$, and the fact that, at $rT + T$, any component in state $k, \cdots, n-1$ is replaced by one in state 0, it is evident that

$$(1.18) \qquad q_{i0} = H_{i0}(T) + \sum_{j=k}^{n-1} H_{ij}(T), \qquad i = 0, \cdots, k-1,$$

$$(1.19) \qquad q_{ij} = H_{ij}(T), \qquad i = 0, \cdots, k-1, j = 1, \cdots, k-1,$$

$$(1.20) \qquad q_{i0}(y) = H_{i0}(T-y) + \sum_{j=k}^{n-1} H_{ij}(T-y), \qquad i = 0, \cdots, n-1,$$

$$(1.21) \qquad q_{ij}(y) = H_{ij}(T-y), \qquad i = 0, \cdots, n-1, j = 1, \cdots, k-1.$$

Finally, it is clear from the definitions that

$$(1.22) \qquad \mathbf{S}(1, r, y) = \mathbf{P}(r, y)\mathbf{q}(y)$$

and

$$(1.23) \qquad \mathbf{S}(m, r, y) = \mathbf{S}(m-1, r, y)\mathbf{q} = \mathbf{P}(r, y)\mathbf{q}(y)\mathbf{q}^{m-1}, \qquad m > 1.$$

Taking into account this preliminary analysis, we can now express $R(T; x)$ as follows:

THEOREM 1.7. *If* $y + z < T$,

$$(1.24) \qquad R(z; rT + y) = \sum_{i=0}^{n-1} P_i(r, y)[1 - F_i(z)],$$

$$(1.25) \qquad R(mT + z; rT + y) = \sum_{i=0}^{k-1} S_i(m, r, y)[1 - F_i(z + y)], \qquad m \geqq 1.$$

If $y + z \geqq T, y < T, z < T$,

$$(1.26) \qquad R(mT + z; rT + y) = \sum_{i=0}^{k-1} S_i(m + 1, r, y)[1 - F_i(z + y - T)].$$

PROOF. If there is no test point in an interval, then the probability of no failure in that interval is the summation over i of the probability of no failure conditional on state i at the beginning of the interval multiplied by the probability of state i at the beginning of the interval, which yields (1.24). If one or more test points occur during an interval, then the probability of no failure is obtained by summing over i the probability of no failure up to the last test point and state i at this point, times the probability of no failure between this point and the end of the interval, conditional on state i at the test point. Noting that, if $y + z < T$, there are m test points in the interval $mT + z$, while, if $y + z \geqq T$, there are $m + 1$ test points, we obtain (1.25) and (1.26).

COROLLARY 1.7.1. *If* $y + z < T$,

$$(1.28) \qquad \lim_{r \to \infty} R(z; rT + y) = \sum_{i=0}^{n-1} (\pi \mathbf{p}(y))_i[1 - F_i(z)],$$

$$\lim_{r\to\infty} R(mT+z; rT+y) = \sum_{i=0}^{k-1} (\pi\mathbf{p}(y)\mathbf{q}(y)\mathbf{q}^{m-1})_i[1 - F_i(z+y)], \tag{1.29}$$

$$m \geqq 1.$$

If $y + z \geqq T$,

$$\lim_{r\to\infty} R(mT+z; rT+y) = \sum_{i=0}^{k-1} (\pi\mathbf{p}(y)\mathbf{q}(y)\mathbf{q}^{m})_i[1 - F_i(z+y-T)]. \tag{1.30}$$

These relations follow immediately when r goes to infinity in (1.17, 23, 24, 25, 26).

2. Examples. As a simple example of the explicit computations involved in analyzing this model, consider a component with which three states, 0, 1, and 2, are associated and suppose transitions always occur from one state to a higher one, i.e. 0 to 1, 0 to 2, or 1 to 2. A new component is in state 0, state 1 is marginal (i.e., $k = 1$) and state 2 is failed. Let $Q_{01} = \lambda_1$, $Q_{02} = \lambda_2$, and $Q_{12} = \lambda_3$, and all other $Q_{ij} = 0$. Then

$$\mathbf{Q}^* = \begin{pmatrix} 0 & \dfrac{\lambda_1}{\lambda_1+\lambda_2} \\ 0 & 0 \end{pmatrix},$$

$$\mathbf{K}(t) = \begin{pmatrix} 1 - e^{-(\lambda_1+\lambda_2)t} & 0 \\ 0 & 1 - e^{-\lambda_3 t} \end{pmatrix}, \qquad t \geqq 0$$

$$\bar{\mathbf{H}}(s) = \begin{pmatrix} \dfrac{s}{s+\lambda_1+\lambda_2} & \dfrac{\lambda_1 s}{(s+\lambda_1+\lambda_2)(s+\lambda_3)} \\ 0 & \dfrac{s}{s+\lambda_3} \end{pmatrix} \text{ by (1.2),}$$

and

$$\mathbf{H}(t) = \begin{pmatrix} e^{-(\lambda_1+\lambda_2)t} & \dfrac{\lambda_1}{\lambda_1+\lambda_2-\lambda_3}[e^{-\lambda_3 t} - e^{-(\lambda_1+\lambda_2)t}] \\ 0 & e^{-\lambda_3 t} \end{pmatrix}, \qquad t \geqq 0$$

$$F_0(t) = 1 - e^{-(\lambda_1+\lambda_2)t} - \frac{\lambda_1}{\lambda_1+\lambda_2-\lambda_3}[e^{-\lambda_3 t} - e^{-(\lambda_1+\lambda_2)t}],$$

and $F_1(t) = 1 - e^{-\lambda_3 t}$ by (1.3).

$$\mathcal{B}_0(t) = \frac{\lambda_3(\lambda_1+\lambda_2)}{\lambda_1+\lambda_3}t - \frac{\lambda_1(\lambda_3-\lambda_2)}{(\lambda_1+\lambda_3)^2}[1 - e^{-(\lambda_1+\lambda_3)t}] \text{ by (2.4),}$$

$$\mathbf{P} = \begin{pmatrix} 1 - \dfrac{\lambda_1}{\lambda_1+\lambda_3}[1 - e^{-(\lambda_1+\lambda_3)T}] & \dfrac{\lambda_1}{\lambda_1+\lambda_3}[1 - e^{-(\lambda_1+\lambda_3)T}] \\ 1 - \dfrac{\lambda_1}{\lambda_1+\lambda_3}[1 - e^{-(\lambda_1+\lambda_3)T}] & \dfrac{\lambda_1}{\lambda_1+\lambda_3}[1 - e^{-(\lambda_1+\lambda_3)T}] \end{pmatrix}$$

by (1.6), (1.7), and (1.8). In this case $\mathbf{p}^r = \mathbf{p}$, and $P_i(r) = p_{0i}$, so that, by (1.10),

$$(2.1)\quad U_p(t) = \frac{r\lambda_1}{\lambda_1+\lambda_3}[1 - e^{-(\lambda_1+\lambda_3)T}], \qquad rT \leqq t < (r+1)T.$$

$$U_f(t) = r\mathcal{B}_0(T) + \mathcal{B}_0(t - rT)$$

$$(2.2)\quad = \frac{\lambda_3(\lambda_1+\lambda_2)}{\lambda_1+\lambda_3}t - \frac{\lambda_1(\lambda_3-\lambda_2)}{(\lambda_1+\lambda_3)^2} \cdot \{r[1 - e^{-(\lambda_1+\lambda_3)T}] + [1 - e^{-(\lambda_1+\lambda_3)(t-rT)}]\},$$

$$rT \leqq t < (r+1)T \text{ by (1.11)}.$$

Furthermore, from Theorem 1.5,

$$(2.3)\quad \lim_{r\to\infty} \frac{U_p(rT)}{rT} = \frac{\lambda_1}{\lambda_1+\lambda_3}\left[\frac{1 - e^{-(\lambda_1+\lambda_3)T}}{T}\right],$$

$$(2.4)\quad \lim_{t\to\infty} \frac{U_f(t)}{t} = \frac{\lambda_3(\lambda_1+\lambda_2)}{\lambda_1+\lambda_3} - \frac{\lambda_1(\lambda_3-\lambda_2)}{(\lambda_1+\lambda_3)^2}\left[\frac{1 - e^{-(\lambda_1+\lambda_3)T}}{T}\right].$$

In order to derive an expression for $R(mT + z; rT + y)$, we require the matrices $\mathbf{p}(y)$, $\mathbf{q}$, and $\mathbf{q}(y)$.

$$\mathbf{p}(y) = \begin{pmatrix} 1 - \frac{\lambda_1}{\lambda_1+\lambda_3}[1 - e^{-(\lambda_1+\lambda_3)y}] & \frac{\lambda_1}{\lambda_1+\lambda_3}[1 - e^{-(\lambda_1+\lambda_3)y}] \\ 1 - \frac{\lambda_1}{\lambda_1+\lambda_3}[1 - e^{-(\lambda_1+\lambda_3)y}] & \frac{\lambda_1}{\lambda_1+\lambda_3}[1 - e^{-(\lambda_1+\lambda_3)y}] \end{pmatrix}$$

by (1.15) and (1.16); by (1.18),

$$\mathbf{q} = (q_{00}) = \left(\frac{\lambda_1}{\lambda_1+\lambda_2-\lambda_3}e^{-\lambda_3 T} - \frac{\lambda_3-\lambda_2}{\lambda_1+\lambda_2-\lambda_3}e^{-(\lambda_1+\lambda_2)T}\right)$$

and, by (1.20),

$$\mathbf{q}(y) = \begin{pmatrix} q_{00}(y) \\ q_{10}(y) \end{pmatrix} = \begin{pmatrix} \frac{\lambda_1}{\lambda_1+\lambda_2-\lambda_3}e^{-\lambda_3(T-y)} - \frac{\lambda_3-\lambda_2}{\lambda_1+\lambda_2-\lambda_3}e^{-(\lambda_1+\lambda_2)(T-y)} \\ e^{-\lambda_3(T-y)} \end{pmatrix}$$

For this case, we have, from (1.17),

$$\mathbf{P}(r, y) = \mathbf{P}(0, y)$$

$$(2.5)\quad = \left(1 - \frac{\lambda_1}{\lambda_1+\lambda_3}[1 - e^{-(\lambda_1+\lambda_3)y}], \frac{\lambda_1}{\lambda_1+\lambda_3}[1 - e^{-(\lambda_1+\lambda_3)y}]\right),$$

and, from (1.23),

$$S_0(m, r, y) = \left\{\left(1 - \frac{\lambda_1}{\lambda_1 + \lambda_3}[1 - e^{-(\lambda_1+\lambda_3)y}]\right)\right.$$

$$(2.6)\qquad \cdot\left(\frac{\lambda_1}{\lambda_1 + \lambda_2 - \lambda_3} e^{-\lambda_3(T-y)} - \frac{\lambda_3 - \lambda_2}{\lambda_1 + \lambda_2 - \lambda_3} e^{-(\lambda_1+\lambda_2)(T-y)}\right)$$

$$\left. + \left(\frac{\lambda_1}{\lambda_1 + \lambda_3}[1 - e^{-(\lambda_1+\lambda_3)y}]\right)(e^{-\lambda_3(T-y)})\right\}$$

$$\cdot\left\{\frac{\lambda_1}{\lambda_1 + \lambda_2 - \lambda_3} e^{-\lambda_3 T} - \frac{\lambda_3 - \lambda_2}{\lambda_1 + \lambda_2 - \lambda_3} e^{-(\lambda_1+\lambda_2)T}\right\}^{m-1}.$$

Finally, if $y + z < T$, we have from Theorem (1.6)

$$R(z; rT + y) = \left\{1 - \frac{\lambda_1}{\lambda_1 + \lambda_3}[1 - e^{-(\lambda_1+\lambda_3)y}]\right\}$$

$$(2.7)\qquad \cdot\left\{\frac{\lambda_1}{\lambda_1 + \lambda_2 - \lambda_3} e^{-\lambda_3 z} - \frac{\lambda_3 - \lambda_2}{\lambda_1 + \lambda_2 - \lambda_3} e^{-(\lambda_1+\lambda_2)z}\right\}$$

$$+ \left\{\frac{\lambda_1}{\lambda_1 + \lambda_3}[1 - e^{-(\lambda_1+\lambda_3)y}]\right\}\{e^{-\lambda_3 z}\},$$

$$R(mT + z; rT + y) = S_0(m, r, y)$$

$$(2.8)\qquad \cdot\left\{\frac{\lambda_1}{\lambda_1 + \lambda_2 - \lambda_3} e^{-\lambda_3(z+y)} - \frac{\lambda_3 - \lambda_2}{\lambda_1 + \lambda_2 - \lambda_3} e^{-(\lambda_1+\lambda_2)(z+y)}\right\}, \qquad m \geqq 1,$$

and if $y + z \geqq T$, $y < T$, $z < T$

$$R(mT + z; rT + y) = S_0(m + 1, r, y)$$

$$(2.9)\qquad \cdot\left\{\frac{\lambda_1}{\lambda_1 + \lambda_2 - \lambda_3} e^{-\lambda_3(z+y-T)} - \frac{\lambda_3 - \lambda_2}{\lambda_1 + \lambda_2 - \lambda_3} e^{-(\lambda_1+\lambda_2)(z+y-T)}\right\}$$

where $S_0(m, r, y)$ is given in (2.6).

Since these expressions are independent of r, nothing new is introduced by taking the limit as r goes to infinity.

As a second example of the applicability of this model, suppose there are $n + 1$ states, $0, \cdots, n$, associated with a component and transitions take place from one state to the next higher. Let $Q_{i,i+1} = \lambda$ for $i = 0, \cdots, n - 1$, all other $Q_{ij} = 0$. This implies that the distribution function of time from state 0 to state k, i.e., to the marginal state, is $\Gamma_k(\lambda t)$, while the distribution function of time from entering the marginal state to state n, failure, is $\Gamma_{n-k}(\lambda t)$. This seems to be a reasonable model for monotonic component parameter drift, with a choice of k corresponding to a choice of the parameter value at which the margin is set. We have

$$Q_{ij}^* = \delta_{i,j-1}, \qquad K_{ij}(t) = \delta_{ij}(1 - e^{-\lambda t}), \qquad t \geqq 0$$

$$\bar{H}_{ij}(s) = \begin{cases}\lambda^{j-i}s(s + \lambda)^{-(j-i+1)}, & i \leqq j \\ 0, & i > j\end{cases} \qquad \text{by (1.2)},$$

and

$$H_{ij}(t) = \begin{cases} (\lambda t)^{j-i} e^{-\lambda t} & i \leqq j,\ t \geqq 0. \\ 0, & i > j. \end{cases}$$

By (2.3), $F_i(t) = 1 - e^{-\lambda t} \sum_{j=0}^{n-i-1} (\lambda t)^j = \Gamma_{n-i}(\lambda t)$, so that

$$\bar{F}_i(s) = (\lambda/(s+\lambda))^{n-i},$$

and by (1.5),

$$\mathcal{B}_i(s) = \frac{\left(\dfrac{\lambda}{s+\lambda}\right)^{n-i}}{1 - \left(\dfrac{\lambda}{s+\lambda}\right)^n} = \sum_{m=1}^{\infty} \left(\frac{\lambda}{s+\lambda}\right)^{mn-i}.$$

Thus, $\mathcal{B}_i(t) = \sum_{n=1}^{\infty} \Gamma_{mn-i}(\lambda t)$.

To reduce $\mathcal{B}_i(t)$ to a finite sum, note that

$$\frac{d\mathcal{B}_i(t)}{dt} = \lambda e^{-\lambda t} \sum_{m=1}^{\infty} \frac{(\lambda t)^{mn-i-1}}{(mn-i-1)!} = \frac{\lambda e^{-\lambda t}}{n} \sum_{r=0}^{n-1} \alpha^{(i+1)r} \exp\{\alpha^r \lambda t\},$$

where $\alpha = e^{2\pi i/n}$. Integrating,

$$\mathcal{B}_i(t) = \frac{\lambda t}{n} + \frac{1}{n} \sum_{r=1}^{n-1} \frac{\alpha^{(i+1)r}}{1-\alpha^r} [1 - \exp\{-(1-\alpha^r)\lambda t\}].$$

By (1.6),

$$p_{ij} = [\Gamma_{j-i}(\lambda T) - \Gamma_{j-i+1}(\lambda T)] + [\Gamma_j(\lambda T) - \Gamma_{j+1}(\lambda T)] * \sum_{m=1}^{\infty} \Gamma_{mn-i}(\lambda T),$$
$$i = 0, \cdots, k-1, \quad j = i, \cdots, k-1.$$

$$= [\Gamma_j(\lambda T) - \Gamma_{j+1}(\lambda T)] * \sum_{m=1}^{\infty} \Gamma_{mn-i}(\lambda T),$$
$$i = 0, \cdots, k-1, j = 0, \cdots, i-1,$$

and by (1.7)

$$p_{iB} = \sum_{j=k}^{n-1} \left\{ [\Gamma_{j-i}(\lambda T) - \Gamma_{j-i+1}(\lambda T)] + [\Gamma_j(\lambda T) - \Gamma_{j+1}(\lambda T)] * \sum_{m=1}^{\infty} \Gamma_{mn-i}(\lambda T) \right\},$$
$$i = 0, \cdots, k-1,$$

while, by (1.8), $p_{Bj} = p_{0j}$. Explicitly,

$$p_{ij} = \frac{e^{-\lambda T}}{n} \sum_{r=0}^{n-1} \alpha^{(i-j)r} \exp\{\alpha^r \lambda T\}, \qquad i, j = 0, \cdots, k-1,$$

$$p_{Bj} = \frac{e^{-\lambda T}}{n} \sum_{r=0}^{n-1} \alpha^{-jr} \exp\{\alpha^r \lambda T\}, \qquad j = 0, \cdots, k-1,$$

$$p_{iB} = 1 - \sum_{j=0}^{k-1} p_{ij}, \qquad i = 0, \cdots, k-1, B.$$

Now the exact and asymptotic values of $U_p(t)$ and $U_f(t)$ may be determined by direct application of (1.9–1.14) and the reliability function $R(t; x)$ from (1.15–1.30).

For example, suppose $n = 3$. This leaves us with the choice of setting k equal to 1 or 2. We shall determine $\lim_{r\to\infty}[U_p(rT)/rT]$ and $\lim_{t\to\infty}[U_f(t)/t]$ for both values of k. We find

$$\mathcal{B}_0(t) = \frac{\lambda t}{3} - \frac{1}{3}\left[1 - e^{-\frac{3}{2}\lambda t}\left(\cos\frac{3^{\frac{1}{2}}}{2}\lambda t + \frac{1}{3^{\frac{1}{2}}}\sin\frac{3^{\frac{1}{2}}}{2}\lambda t\right)\right],$$

$$\mathcal{B}_1(t) = \frac{\lambda t}{3} - \frac{2}{3(3)^{\frac{1}{2}}}e^{-\frac{3}{2}\lambda t}\sin\frac{3^{\frac{1}{2}}}{2}\lambda t,$$

and for $k = 1$,

$$p_{00} = p_{B0} = \frac{1}{3}\left[1 + 2e^{-\frac{3}{2}\lambda T}\cos\frac{3^{\frac{1}{2}}}{2}\lambda T\right],$$

$$p_{0B} = p_{BB} = \frac{2}{3}\left[1 - e^{-\frac{3}{2}\lambda T}\cos\frac{3^{\frac{1}{2}}}{2}\lambda T\right],$$

while for $k = 2$,

$$p_{00} = p_{B0} = \frac{1}{3}\left[1 + 2e^{-\frac{3}{2}\lambda T}\cos\frac{3^{\frac{1}{2}}}{2}\lambda T\right],$$

$$p_{01} = p_{B1} = \frac{1}{3}\left[1 - e^{-\frac{3}{2}\lambda T}\left(\cos\frac{3^{\frac{1}{2}}}{2}\lambda T - 3^{\frac{1}{2}}\sin\frac{3^{\frac{1}{2}}}{2}\lambda T\right)\right],$$

$$p_{0B} = p_{BB} = \frac{1}{3}\left[1 - e^{-\frac{3}{2}\lambda T}\left(\cos\frac{3^{\frac{1}{2}}}{2}\lambda T + 3^{\frac{1}{2}}\sin\frac{3^{\frac{1}{2}}}{2}\lambda T\right)\right],$$

$$p_{10} = \frac{1}{3}\left[1 - e^{-\frac{3}{2}\lambda T}\left(\cos\frac{3^{\frac{1}{2}}}{2}\lambda T + 3^{\frac{1}{2}}\sin\frac{3^{\frac{1}{2}}}{2}\lambda T\right)\right],$$

$$p_{11} = \frac{1}{3}\left[1 + 2e^{-\frac{3}{2}\lambda T}\cos\frac{3^{\frac{1}{2}}}{2}\lambda T\right],$$

$$p_{1B} = \frac{1}{3}\left[1 - e^{-\frac{3}{2}\lambda T}\left(\cos\frac{3^{\frac{1}{2}}}{2}\lambda T - 3^{\frac{1}{2}}\sin\frac{3^{\frac{1}{2}}}{2}\lambda T\right)\right].$$

Thus, for $k = 1$, we find from (1.12) that

$$\pi_0 = \frac{1}{3}\left[1 + 2e^{-\frac{3}{2}\lambda T}\cos\frac{3^{\frac{1}{2}}}{2}\lambda T\right], \qquad \pi_B = \frac{2}{3}\left[1 - e^{-\frac{3}{2}\lambda T}\cos\frac{3^{\frac{1}{2}}}{2}\lambda T\right],$$

while, for $k = 2$,

$$\pi_0 = \frac{1}{3}\,\frac{1 - e^{-3\lambda T}}{1 - e^{-\frac{3}{2}\lambda T}\left(\cos\frac{3^{\frac{1}{2}}}{2}\lambda T - \frac{1}{3^{\frac{1}{2}}}\sin\frac{3^{\frac{1}{2}}}{2}\lambda T\right)},$$

$$\pi_1 = \frac{1}{3}\,\frac{1 - e^{-\frac{3}{2}\lambda T}\left(\cos\frac{3^{\frac{1}{2}}}{2}\lambda T - 3^{\frac{1}{2}}\sin\frac{3^{\frac{1}{2}}}{2}\lambda T\right)}{1 - e^{-\frac{3}{2}\lambda T}\left(\cos\frac{3^{\frac{1}{2}}}{2}\lambda T - \frac{1}{3^{\frac{1}{2}}}\sin\frac{3^{\frac{1}{2}}}{2}\lambda T\right)},$$

$$\pi_B = \frac{1}{3}\,\frac{1 - 2e^{-\frac{3}{2}\lambda T}\cos\frac{3^{\frac{1}{2}}}{2}\lambda T + e^{-3\lambda T}}{1 - e^{-\frac{3}{2}\lambda T}\left(\cos\frac{3^{\frac{1}{2}}}{2}\lambda T - \frac{1}{3^{\frac{1}{2}}}\sin\frac{3^{\frac{1}{2}}}{2}\lambda T\right)}.$$

Using (1.13) and (1.14) we arrive at the results that, for $k = 1$,

$$\lim_{r\to\infty} \frac{U_p(rT)}{rT} = \frac{2}{3T}\left[1 - e^{-\frac{3}{2}\lambda T} \cos \frac{3^{\frac{1}{2}}}{2}\lambda T\right],$$

$$\lim_{t\to\infty} \frac{U_f(t)}{t} = \frac{\lambda}{3} - \frac{1}{3T}\left[1 - e^{-\frac{3}{2}\lambda T}\left(\cos \frac{3^{\frac{1}{2}}}{2}\lambda T + \frac{1}{3^{\frac{1}{2}}} \sin \frac{3^{\frac{1}{2}}}{2}\lambda T\right)\right],$$

and for $k = 2$,

$$\lim_{r\to\infty} \frac{U_p(rT)}{rT} = \frac{1}{3T} \frac{1 - 2e^{-\frac{3}{2}\lambda T} \cos \frac{3^{\frac{1}{2}}}{2}\lambda T + e^{-3\lambda T}}{1 - e^{-\frac{3}{2}\lambda T}\left(\cos \frac{3^{\frac{1}{2}}}{2}\lambda T - \frac{1}{3^{\frac{1}{2}}} \sin \frac{3^{\frac{1}{2}}}{2}\lambda T\right)},$$

$$\lim_{t\to\infty} \frac{U_f(t)}{t} = \frac{\lambda}{3} - \frac{2}{9T} \cdot \frac{1 - 2e^{-\frac{3}{2}\lambda T} \cos \frac{3^{\frac{1}{2}}}{2}\lambda T + e^{-3\lambda T}}{1 - e^{-\frac{3}{2}\lambda T}\left(\cos \frac{3^{\frac{1}{2}}}{2}\lambda T - \frac{1}{3^{\frac{1}{2}}} \sin \frac{3^{\frac{1}{2}}}{2}\lambda T\right)}.$$

3. Conclusions. A plausible model of a system maintained in operating condition by immediate replacement of failed components and periodic marginal tests has been formulated and analyzed. Explicit expressions for the expected number of failures, the expected number of preventive removals, and the survival probability, both exact and asymptotic values, have been obtained. These quantities may be used to evaluate the expected cost of maintaining a system and to optimize both the level (k) of a marginal test and the interval (T) between tests.

For example, suppose the cost of maintenance is a linear function of the number of failures, the number of preventive removals, and the number of tests, i.e. for each component position,

$$C(t) = C_f N_f(t) + C_p N_p(t) + C_m[t/T],$$

where $N_f(t)$ is the number of failures by t, $N_p(t)$ is the number of preventive removals by t, T is the maintenance interval, C_f, C_p, and C_m are the costs of a failure, a preventive removal, and a test, respectively. Then

$$E\{C(t)\} = C_f U_f(t) + C_p U_p(t) + C_m[t/T],$$

where, for any given value of k and T, $U_f(t)$ and $U_p(t)$ are expressed by (1.11) and (1.10).

In some cases, it is known in advance that a system will be used only for some specified length of time. Then it is desirable to find those values of k and T which minimize $E\{C(t)\}$ for that length of time. In other situations, it is intended that the system be used indefinitely and the figure to be minimized is the asymptotic expected cost per unit time, i.e.,

$$\lim_{t\to\infty} E\left\{\frac{C(t)}{t}\right\} = C_f \lim_{t\to\infty} \frac{U_f(t)}{t} + C_p \lim_{t\to\infty} \frac{U_p(t)}{t} + \frac{C_m}{T},$$

where, for given k and T, $\lim_{t\to\infty} [U_f(t)/t]$ and $\lim_{t\to\infty} [U_p(t)/t]$ are given by (1.14) and (1.13).

As a specific example, consider the Γ_3 distribution of time to failure discussed at the end of Section 2. For this particular model, we have a choice of setting k equal to 1 or 2 and setting T at any value from 0 to ∞. (The value ∞ corresponds to no marginal testing at all.) For $k = 1$, we have

$$\lim_{t\to\infty} E\left\{\frac{C(t)}{t}\right\} = \frac{C_f\lambda}{3} - \frac{C_f}{3T}\left[1 - e^{-\frac{3}{2}\lambda T}\left(\cos\frac{3^{\frac{1}{2}}}{2}\lambda T + \frac{1}{3^{\frac{1}{2}}}\sin\frac{3^{\frac{1}{2}}}{2}\lambda T\right)\right] + \frac{2}{3}\frac{C_p}{T}\left(1 - e^{-\frac{3}{2}\lambda T}\cos\frac{3^{\frac{1}{2}}}{2}\lambda T\right) + \frac{C_m}{T}.$$

For $k = 2$, we have

$$\lim_{t\to\infty} E\left\{\frac{C(t)}{t}\right\} = \frac{C_f\lambda}{3} - \frac{2C_f - 3C_p}{9T}\left[\frac{1 - 2e^{-\frac{3}{2}\lambda T}\cos\frac{3^{\frac{1}{2}}}{2}\lambda T + e^{-3\lambda T}}{1 - e^{-\frac{3}{2}\lambda T}\left(\cos\frac{3^{\frac{1}{2}}}{2}\lambda T - \frac{1}{3^{\frac{1}{2}}}\sin\frac{3^{\frac{1}{2}}}{2}\lambda T\right)}\right] + \frac{C_m}{T}.$$

Clearly, for $T = \infty$, i.e., no marginal testing, $\lim_{t\to\infty} E\{C(t)/t\} = C_f\lambda/3$.

The two functions of T above have minima less than $C_f\lambda/3$ whenever $C_p/C_f + 3C_m/2C_f < 1/2$ or $C_p/C_f + 3C_m/C_f < 2/3$, respectively. Thus, given the values of C_p, C_f, and C_m, and λ, it is possible to find those values of T which minimize the cost function for each value of k. Comparison of the two minimum costs then indicates the optimal choice of k. Similar computational methods could be used for other special cases of the general Markovian model.

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