A RANDOM INTERVAL FILLING PROBLEM[1]

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0. Summary. Consider an initial real interval $[0, x]$, $x > 0$, and place in it a random subinterval $I(x)$ defined by a pair of random variables (U_x, V_x); the former being the length of $I(x)$ and the latter its lower boundary point. The set $[0, x] - I(x)$ consists of two intervals of lengths x_1 and x_2, in which there are in turn placed random subintervals $I(x_1)$ and $I(x_2)$ defined by pairs of random variables (U_{x_1}, V_{x_1}) and (U_{x_2}, V_{x_2}). The process of placing random subintervals in $[0, x]$ is thus continued. Under the assumptions that the subintervals cannot overlap, and that their lengths are uniformly bounded away from zero, the procedure must terminate after a finite number of steps.

Let $N(b, x)$ denote the number of subintervals of $[0, x]$ of length at least b, in the terminal state. The asymptotic behaviour of the moments of $N(b, x)$ is here studied as $x \to \infty$. It is shown that under fairly general conditions the mean approaches a linear function of x at the rate x^{-n}, for any integer $n > 0$. Under the further condition that V_x is a family of uniform distributions the exact form of the linear relation is determined. In the last section it is indicated how this result can be extended to some more general distributions. A similar but less precise result is proved for the higher moments, the convergence rate x^{-n} not being established for this case.

1. Background. The setting for the problem to be discussed in this paper can most easily be introduced in terms of an example. Consider a street of length x, in which cars of fixed length U are to park. Represent the street by the interval $[0, x]$, and assume that cars park "at random" in the sense that the center of the first is uniformly distributed on $[\frac{1}{2}U, x - \frac{1}{2}U]$, that of the second is uniformly distributed on the space remaining available to it, and so on. Cars continue to arrive until there is no longer any space in which they can park. Let $N(x)$ be the number of cars which have successfully parked in the street.

The distribution of $N(x)$ has been investigated by A. Rényi [4], by A. Dvoretzky and H. Robbins [1], and by the author [3]. Let $\mu(x)$ be the mean of $N(x)$. Rényi has shown that if $U = 1$, then for any integer n

$$\mu(x) = cx - (1 - c) + O(1/x^n) \qquad \text{as } x \to \infty, \tag{1.1}$$

where the constant c is expressed as an integral which can be evaluated numerically. (It equals approximately 0.75.) The rate of convergence in (1.1) was

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independently demonstrated by Dvoretzky and Robbins, who have furthermore proved a central limit theorem for $N(x)$. The constant in (1.1) was also derived by the author in [3].

The problem can obviously be stated in purely geometrical language as the random filling of an interval by intervals of length U, the latter to be referred to as I-intervals. This paper treats a generalization of the model to the case where U is a random variable. It also examines the situation when the location of the I-intervals in $[0, x]$ is not necessarily uniformly distributed.

The generalization was initially prompted by an equivalent problem in the theory of binary cascades [3], for which the case when U is of fixed length is of little interest. An initial particle of energy x suffers a collision, and splits into particles of energies x_1 and x_2 respectively. These in turn collide and split, and so on. One may equate the cascade and interval filling problems by drawing correspondences between the following: (i) the initial energy x, and the length of the interval $[0, x]$, (ii) the energy loss upon collision, and the length of the initial I-interval, (iii) the energies x_1 and x_2 of the resultant particles, and the lengths of the intervals which remain when the first I-interval is subtracted from $[0, x]$, (iv) the energy of each particle created in the evolution of the cascade, and the length of an appropriate corresponding subinterval of $[0, x]$.

It is clear that if the random variable U is bounded away from zero, then the interval $[0, x]$ can accommodate at most finitely many I-intervals. The actual number accommodated and their lengths will be random variables. In particular, the random variable to be studied here is the number of these I-intervals of length at least b, to be denoted by $N(b, x)$. We shall obtain results of a character similar to (1.1) for the mean of $N(b, x)$, and for its higher moments.

2. Definitions and assumptions. The process described in Section 1 will now be formally defined. Consider an interval $J(x)$, of length $x < \infty$. Define the distribution of a random subinterval $I(x)$ of $J(x)$ in terms of a pair of random variables (U_x, V_x), where U_x denotes the length of $I(x)$, and V_x is its lower boundary point. Assume that the essential infimum of U_x, say d, is positive. The set $J(x) - I(x)$ will be composed of two intervals, say $J(x_1)$ and $J(x_2)$, of lengths x_1 and x_2 respectively. Subintervals $I(x_1)$ and $I(x_2)$ of x_1 and x_2 are then determined by random variables (U_{x_i}, V_{x_i}), $i = 1, 2$. The set

$$J(x) - (I(x) \cup I(x_1) \cup I(x_2))$$

is a union of four intervals, in each of which there is in turn chosen a random subinterval. The process of choosing I-subintervals of J-intervals is thus continued until there remain only J-intervals of length less than d. It will then be impossible to extract further I-intervals, and the procedure terminates, having partitioned $J(x)$ into terminal J-intervals $(J_1(x), \cdots, J_{n+1}(x))$ of length less than d, and I-intervals $(I_1(x), \cdots, I_n(x))$ of length greater than d. Define $N(b, x)$ to be the number of $I_j(x), j = 1, \cdots, n$, of length at least b.

To completely define the process, it is sufficient to specify the distributions of (U_x, V_x). Let $P\{-\}$ denote the probability of the set in brackets. We assume that there exist $0 < d < a < \infty$ such that $P\{d \leqq U_x \leqq a\} = 1$. We further assume that the distribution of U_x is independent of x, i.e., that the distribution of the lengths of parking cars does not depend on the length of the street. This assumption must clearly be modified when $x < a$ (it is impossible to park a car of length greater than x in a street of length x). To be precise, let $G(u)$ and $G_x(u)$ be any distribution functions satisfying $G(a) - G(d) = 1$ and $G_x(x) - G_x(d) = 1$. We assume that

$$P\{U_x \leqq u\} = \begin{cases} G_x(u) & \text{when } x < a \\ G(u) & \text{when } x \geqq a. \end{cases} \tag{2.1}$$

It is no loss of generality to take $d = 1$, and hence we do so.

It remains only to specify the distribution of V_x. The latter depends on U_x; in fact the range of V_x, given that $U_x = u$, is $x - u$. It is sufficient to specify the conditional distribution $F_x(v \mid u)$ of V_x, given $U_x = u$, for all $1 \leqq u \leqq a$.

If we are to expect asymptotic behavior of the kind displayed in (1.1), then clearly as $x \to \infty$, the distributions $F_x(v \mid u)$ will have to become close to each other in an appropriate sense. Consider a simple illustration. Suppose that $U_x \equiv 1$, and that for $2n \leqq x < 2n + 1$, $n = 0, 1, \cdots$

$$F_x(v \mid u) = \begin{cases} 0 & \text{for } v < 0 \\ v/(x-u) & \text{for } 0 \leqq v < x - u \\ 1 & \text{for } v \geqq x - u, \end{cases}$$

while for $2n + 1 \leqq x < 2n + 2$

$$F_x(v \mid u) = \begin{cases} 0 & \text{for } v < 0 \\ 1 & \text{for } v \geqq 0. \end{cases}$$

Then as $x \to \infty$, $\mu(x)/x$ will alternately be close to 0.75 or to 1.00, depending on whether x is in $[2n, 2n + 1]$ or in $[2n + 1, 2n + 2]$. Thus the assertion $\mu(x) \sim cx$ is false for any c.

The regularity conditions we shall require below are designed to extend the kind of convergence of (1.1) to the present more general situation. We shall find it convenient to assume that there is associated with $F_x(v \mid u)$ a density function $f_x(v \mid u)$. For any real number α, denote the conditional expectation of V_x^α, given that $U_x = u$, by $E(V_x^\alpha \mid u)$, and define the truncated expectation

$$E_\delta(V_x^\alpha \mid u) = \int_{x-u-\delta}^{x-u} v^\alpha f_x(v \mid u)\, dv.$$

We shall assume that

$$E(V_x^\alpha \mid u) = O(x^\alpha), \tag{2.2}$$

$$E_\delta(V_x^\alpha \mid u) = O(x^{\alpha-1}) \qquad \text{for all } 0 < \delta < a, \tag{2.3}$$

and

$$E(V_x \mid u) = \tfrac{1}{2}(x - u). \tag{2.4}$$

We further require that the densities $f_x(v \mid u)$ be uniformly bounded in a neighborhood of $x - u$, namely that there exist numbers $d_0 > 0$ and $D < \infty$ such that for all $x > 2a$ and $x - u - d_0 \leqq v \leqq x - u$,

$$f_x(v \mid u) \leqq D. \tag{2.5}$$

Among the family of distributions satisfying (2.2)–(2.5) will be all those whose densities $f_x(v \mid u)$ are such that $xf_x(v \mid u)$ is uniformly bounded, and whose means are the midpoints of their ranges. This condition on the means, (2.4), is very reasonable for the models discussed in Section 1. In fact, in the cascade model it would be customary to assume the stronger condition that $f_x(v \mid u)$ is symmetric about $\frac{1}{2}(x - u)$.

Our final condition expresses the requirement that the $f_x(v \mid u)$ become "close" to each other as $x \to \infty$. We assume that there exist real numbers $\delta_0 > 0$, $\beta < 0$, and $A < \infty$, such that for all δ in $[-\delta_0, \delta_0]$, $x > 3a$, and $0 \leqq v \leqq x - u$ we have

$$\left| \frac{f_{x+\gamma}(v \mid u)}{f_x(v \mid u)} - 1 \right| \leqq Ax^{\beta}. \tag{2.6}$$

We now introduce the following notations and definitions:

$$p_n(b, x) = P\{N(b, x) = n\}, \tag{2.7}$$

$$\mu(b, x) = \sum_{n=0}^{\infty} np_n(b, x), \tag{2.8}$$

$$\mu_m(b, x) = \sum_{j=m}^{\infty} j(j - 1) \cdots (j - m + 1)p_j(b, x), \tag{2.9}$$

$$EU = \int_1^a u\, dG(u) = \nu, \tag{2.10}$$

$$A(b, x) = \begin{cases} 0 & \text{when } x < 2a \\ 3 - G(b) - 2G(x - 2a) - 2\int_1^a F_x(a \mid u)\, dG(u) & \\ \quad + 2\int_1^{x-2a} F_x(2a \mid u)\, dG(u) & \text{when } 2a \leqq x < 3a \\ 1 - G(b) + 2\int_1^a [F_x(2a \mid u) - F_x(a \mid u)]\, dG(u) & \\ & \text{when } 3a \leqq x. \end{cases} \tag{2.11}$$

Let $\alpha(b, s)$ denote the Laplace transform of $A(b, x)$, and $g(s)$ be the Laplace-Stieltjes transform of $G(u)$. Let primes denote derivatives with respect to s, and define

$$w_1(s) = \int_s^{\infty} \frac{g(t)}{t}\, dt, \qquad w_2(s) = \int_0^s \frac{g'(t)}{g(t)}\, dt, \tag{2.12}$$

(2.13) $$w(s) = \exp \{2w_1(s) + w_2(s)\},$$

(2.14) $$k = \exp \left\{2w(1) - 2 \int_0^1 \frac{1 - g(t)}{t}\, dt\right\},$$

(2.15) $$r(b, s) = \alpha(b, s) \frac{g'(s)}{g(s)} - \alpha'(b, s),$$

(2.16) $$C(b) = k \int_0^\infty \frac{r(b, t)}{w(t)}\, dt.$$

The existence of the moments (2.8)–(2.10) is trivial to verify, as is the existence of α, g, α' and g'. The convergence of the integrals w_1 and w_2 can be verified by substituting the definitions of g and g' in (2.12) and performing a simple calculation. From the fact that $s^{-1}\{1 - g(s)\} \to \nu$ as $s \to 0$, it follows that $\int_0^1 t^{-1}\{1 - g(t)\}\, dt$ converges, and hence that k is finite and well defined. The only non-trivial question raised by the above definitions is that of the convergence of $C(b)$, and this will be proved below.

3. Results.

LEMMA. *If the distribution of U_x is of the form* (2.1), *and $f_x(v \mid u)$ is the uniform density on* $[0, x - u]$, *then*

(3.1) $$\mu(b, x) \sim C(b)x \qquad \text{as} \quad x \to \infty,$$

where $C(b)$ is a convergent integral defined by (2.16).

PROOF. In Section 5.

THEOREM 1. *If the distribution of U_x is of the form* (2.1), *and $f_x(v \mid u)$ satisfies* (2.2)–(2.6), *then there exists a function of b, say $K(b)$, such that for any integer n*

(3.2) $$\mu(b, x) = (x + \nu)K(b) - 1 + G(b) + O(1/x^n) \qquad \text{as} \quad x \to \infty.$$

PROOF. In Section 5.

Note that while $C(b)$ is known and explicitly computable, the function $K(b)$ in the more general setting of Theorem 1, is as yet unknown. In the particular case when $f_x(v \mid u)$ is the uniform density on $[0, x - u]$, we can easily verify that (2.2)–(2.6) are satisfied. This implies that for this case $\mu(b, x)$ satisfies both (3.1) and (3.2), and hence that $K(b) = C(b)$. Thus we have

THEOREM 2. *If the distribution of U_x is of the form* (2.1), *and $f_x(v \mid u)$ is the uniform density on* $[0, x - u]$, *then*

(3.3) $$\mu(b, x) = (x + \nu)C(b) - 1 + G(b) + O(1/x^n) \qquad \text{as} \quad x \to \infty,$$

where $C(b)$ is the convergent integral defined by (2.16).

We shall also establish

THEOREM 3. *If the distribution of U_x is of the form* (2.1), *and $f_x(v \mid u)$ is the uniform density on* $[0, x - u]$, *then*

(3.4) $$\mu_n(b, x) \sim [C(b)]^n x,$$

where $C(b)$ is the same function as in (3.1).

PROOF. In Section 5.

It is clear that relation (3.4) is true not only for the n'th factorial moment, but also for the ordinary moment. Similar results can also be derived for other moments, for example the variance of $N(b, x)$ (see [3]).

4. An example. Consider the case when

$$G(u) = \begin{cases} 0 & \text{when } u < 1 \\ 1 & \text{when } u \geqq 1, \end{cases}$$

$$f_x(v \mid 1) = \begin{cases} 1/(x-1) & \text{when } 0 \leqq v \leqq x-1 \\ 0 & \text{elsewhere.} \end{cases}$$

Write $\mu(0, x) = \mu(x)$, $C(0) = c$, and $A(0, x) = A(x)$. Clearly when $b \leqq 1$, $\mu(b, x) = \mu(x)$ and $C(b) = c$, while when $b > 1$, $\mu(b, x) = C(b) = 0$. Using the definitions of Section 2, we see that $g(s) = e^{-s}$, and hence that

$$r(0, s) = -\alpha(s) - \alpha'(s) = \int_0^\infty (x-1)A(x)\, e^{-sx}\, dx.$$

By (2.11) we calculate that

$$A(x) = \begin{cases} 0 & \text{when } x < 2, \\ \dfrac{3x-5}{x-1} & \text{when } 2 \leqq x < 3, \\ \dfrac{x+1}{x-1} & \text{when } 3 \leqq x. \end{cases}$$

Further calculation shows that

$$r(0, s) = e^{-s} \int_0^\infty xA(x+1)\, e^{-sx}\, dx,$$

and that

$$\int_0^\infty xA(x+1)\, e^{-sx}\, dx = \frac{e^{-s}}{s} + 3\frac{e^{-s}}{s^2} - 2\frac{e^{-2s}}{s^2}.$$

From (2.12) and (2.13) it follows that $w(s) = e^{2\epsilon(s)-s}$, where $\epsilon(s)$ denotes the exponential integral, i.e.,

$$\epsilon(s) = \int_s^\infty \frac{e^{-t}}{t}\, dt.$$

Thus to determine $c = k\int_0^\infty [r(0, t)/w(t)]\, dt$ as defined in (2.16), it remains only to determine k as defined in (2.14). To do so we make use of the well known series expansion of the exponential integral, namely

$$\epsilon(s) = -\gamma - \log s + s = \frac{s^2}{2.2!} + \frac{s^3}{3.3!} - \cdots,$$

where γ is Euler's constant and the logarithm is to the base e. From this we see that

$$\int_0^1 \frac{1 - e^{-t}}{t}\, dt = \epsilon(1) + \gamma,$$

and hence that

$$k = \exp\left\{w_1(1) - 2\int_0^\infty \frac{1 - e^{-t}}{t}\, dt\right\} = e^{-2\gamma}.$$

We thus conclude that

$$c = e^{-2\gamma} \int_0^\infty \left[\frac{e^{-t}}{t} + 3\,\frac{e^{-t}}{t^2} - 2\,\frac{e^{-2t}}{t^2}\right] e^{-2\epsilon(t)}\, dt.$$

Integrating by parts we see that

$$\int_0^\infty \frac{1}{t^2}\, e^{-2\epsilon(t)}\, dt = 2\int_0^\infty \frac{e^{-t}}{t^2}\, e^{-2\epsilon(t)}\, dt,$$

and that

$$\int_0^\infty \frac{e^{-t}}{t^2}\, e^{-2\epsilon(t)}\, dt = \int_0^\infty \left(2\,\frac{e^{-2t}}{t^2} - \frac{e^{-t}}{t}\right) e^{-2\epsilon(t)}\, dt.$$

With these two formulas we may simplify the expression for c to read

$$c = e^{-2\gamma} \int_0^\infty \frac{e^{-2\epsilon(t)}}{t^2}\, dt.$$

This agrees with the constant obtained directly for this special case in [3] and [4]. Numerical evaluation of the integral shows that to two decimal places $c \cong 0.75$. Theorem 1 thus takes the form

$$\mu(x) = cx - 1 + c + O(1/x^n) \qquad \text{for any } n,$$

which checks with (1.1). Theorem 3 says that the m'th factorial moment (and hence the m'th ordinary moment) satisfies $\mu_m(x) \sim c^m x$.

5. Proofs. Let $p_n(b, x \mid u, v)$ be the conditional probability that $N(b, x) = n$, given that $(U_x, V_x) = (u, v)$, and let $Q(b, x, z) = \sum_{n=0}^\infty z^n p_n(b, x)$. Then clearly

$$(5.1)\qquad p_n(b, x \mid u, v) = \begin{cases} \sum_{i=0}^{n-1} p_i(b, v)\, p_{n-1-i}(b, x - v - u) & \text{when } u \geqq b \\ \sum_{i=0}^{n} p_i(b, v)\, p_{n-i}(b, x - v - u) & \text{when } u < b, \end{cases}$$

and

$$(5.2)\qquad p_n(b, x) = \int_1^a dG_{0x}(u) \int_0^{x-u} p_n(b, x \mid u, v) f_x(v \mid u)\, dv,$$

where

$$G_{0x}(u) = \begin{cases} G_x(u) & \text{when } x < a \\ G(u) & \text{when } x \geqq a. \end{cases}$$

Substituting (5.1) in (5.2) we obtain

$$p_n(b, x) = \int_1^b dG_{0x}(u) \int_0^{x-u} \sum_{i=0}^{n} p_i(b, v)\, p_{n-i}(b, x - v - u) f_x(v \mid u)\, dv$$
$$+ \int_b^a dG_{0x}(u) \int_0^{x-u} \sum_{i=0}^{n-1} p_i(b, v)\, p_{n-i-i}(b, x - v - u) f_x(v \mid u)\, dv,$$

and hence, after multiplying thru by z^n and adding over n,

$$(5.3)\qquad \begin{aligned} Q(b, x, z) &= \int_1^b dG_{0x}(u) \int_0^{x-u} Q(b, v, z) Q(b, x - v - u, z) f_x(v \mid u)\, dv \\ &\quad + z \int_b^a dG_{0x}(u) \int_0^{x-u} Q(b, v, z) Q(b, x - v - u, z) f_x(v \mid u)\, dv. \end{aligned}$$

Differentiating (5.3) m times with respect to z and setting $z = 1$, we get

$$(5.4)\qquad \begin{aligned} \mu_m(b, x) &= \int_1^b dG_{0x}(u) \int_0^{x-u} \sum_{i=0}^{m} \binom{m}{i} \mu_{m-i}(b,v) \mu_i(b, x - v - u) f_x(v \mid u)\, dv \\ &\quad + m \int_b^a dG_{0x}(u) \int_0^{x-u} \sum_{i=0}^{m-1} \binom{m-1}{i} \mu_{m-1-i}(b,v) \mu_i(b, x - v - u) f_x(v \mid u)\, dv. \end{aligned}$$

From this expression we proceed to proofs of the lemma and of Theorems 1 and 3.

Proof of Lemma. When $m = 1$ and $f_x(v \mid u)$ is the uniform density function on $[0, x - u]$, then (5.4) reads

$$(5.5)\qquad \mu(b, x) = \int_b^a dG_{0x}(u) + 2 \int_1^a \frac{dG_{0x}(u)}{x - u} \int_0^{x-u} \mu(b, v)\, dv.$$

In order to guarantee the sufficiently rapid convergence of certain Laplace transforms which play a role below, it is expedient to introduce the modified mean function

$$M(b, x) = \begin{cases} 0 & \text{when } x < 2a \\ \mu(b, x) & \text{when } x \geqq 2a. \end{cases}$$

It can be verified by a direct, but moderately lengthy calculation which we omit, that $M(b, x)$ satisfies the relation

$$(5.6)\qquad M(b, x) = A(b, x) + 2 \int_1^x \frac{dG(u)}{x - u} \int_0^{x-u} M(b, v)\, dv.$$

Note that the functions G_{0x} and G_x no longer play any role in (5.6). Taking the Laplace transform of both sides (5.6), and using the notation given in (2.12)–(2.16), we see that

$$(5.7)\qquad m(b, s) = \alpha(b, s) + 2g(s) \int_s^\infty m(b, y) \frac{dy}{y},$$

where $m(b, s)$ denotes the Laplace transform of $M(b, x)$. After some manipulation, (5.7) yields

$$m'(b, s) + \{2[g(s)/s] - [g'(s)/g(s)]\}m(b, s) + r(b, s) = 0. \tag{5.8}$$

This is a first order linear differential equation, and setting

$$2[g(s)/s] - [g'(s)/g(s)] = q(s),$$

we may write the solution in the form

$$m(b, s) = \exp\left\{-\int_{s_0}^{s} q(t)\, dt\right\} \cdot \left[m(b, s_0) - \int_{s_0}^{s} r(b, t) \exp\left\{\int_{s_0}^{t} q(u)\, du\right\} dt\right]$$

for any s_0. After noticing that

$$\exp\left\{-\int_{s_0}^{s} q(t)\, dt\right\} = \frac{w(s)}{w(s_0)},$$

we may rewrite the above solution in the form

$$m(b, s) = w(s)\left\{\frac{m(b, s_0)}{w(s_0)} + \int_{s}^{s_0} \frac{r(b, t)}{w(t)}\, dt\right\}. \tag{5.9}$$

But

$$m(b, s_0) \leqq \int_{2a}^{\infty} x e^{-s_0 x}\, dx = O(e^{-2as_0}) \qquad \text{as} \quad s_0 \to \infty,$$

and $w(s_0) \geqq e^{w_2(s_0)}$. Since

$$w_2(s_0) = \int_0^{s_0} \frac{g'(t)}{g(t)}\, dt \geqq -as_0,$$

we see that $w(s_0) \geqq e^{-as_0}$, and hence that

$$\frac{m(b, s_0)}{w(s_0)} = O(e^{-as_0}) \qquad \text{as} \quad s_0 \to \infty. \tag{5.10}$$

Furthermore, noting the relations

$$\alpha(b, s) = O(e^{-2as_0}), \qquad \alpha'(b, s) = O(e^{-2as_0}),$$

$$\left|\frac{g'(s_0)}{g(s_0)}\right| \leqq a,$$

and recalling the definition of r in (2.15), we see that $r(b, s_0) = O(e^{-2as_0})$. This, together with the already proved fact that $w(s_0) \geqq e^{-as_0}$, implies that

$$\int_{s}^{s_0} \frac{r(b, t)}{w(t)}\, dt \quad \text{converges} \qquad \text{as} \quad s_0 \to \infty. \tag{5.11}$$

Thus, letting $s_0 \to \infty$, and applying (5.10) and (5.11), we see that (5.9) yields

$$m(b, s) = w(s) \int_{s}^{\infty} \frac{r(b, t)}{w(t)}\, dt. \tag{5.12}$$

We proceed to look at (5.12) as $s \to 0$. From the fact that $t^{-1}\{1 - g(t)\} \to \nu$ as $t \to 0$, it clearly follows that

$$\lim_{s\to 0} \int_s^1 \frac{1 - g(t)}{t}\, dt = \int_0^1 \frac{1 - g(t)}{t}\, dt \quad \text{exists.}$$

Hence $w_1(s) + \log s = w_1(1) - \int_s^1 [(1 - g(t))/t]\, dt$ converges as $s \to 0$, and thus there is a $k > 0$ such that $w_1(s) + \log s \to \frac{1}{2} \log k$. Since furthermore $w_2(s) \to 0$ as $s \to 0$, we see that

$$\exp\{2w_1(s) + 2 \log s + w_2(s)\} \to k,$$

or equivalently that

$$w(s) \sim k/s^2 \qquad \text{as} \quad s \to 0. \tag{5.13}$$

it is also clear that

$$\tfrac{1}{2} \log k = w_1(1) + \int_0^1 \frac{1 - g(t)}{t}\, dt,$$

since each side of this equality is the limit as $s \to 0$ of $w_1(s) + \log s$. Hence k is as defined in (2.14).

We turn to the integral in (5.12). Since $A(b, x)$ is bounded, say by K, as can be seen directly from its definition, we have

$$|\alpha(b, s)| \leqq K/s, \qquad \text{and} \qquad |\alpha'(b, s)| \leqq K/s^2.$$

Recalling that $|g'(s)/g(s)| \leqq a$, we see that

$$|r(b, s)| \leqq (aK/s) + (K/s^2).$$

Together with (3.18), this implies that $\int_s^\infty [r(b, t)/w(t)]\, dt$ converges as $s \to 0$. Applying (5.13) again, we may thus conclude that

$$m(b, s) \sim \frac{k}{s^2} \int_0^\infty \frac{r(b, t)}{w(t)}\, dt \qquad \text{as} \quad s \to 0,$$

or, to use the notation given in (2.16),

$$m(b, s) \sim C(b)/s^2 \qquad \text{as} \quad s \to 0. \tag{5.14}$$

By the Karamata Tauberian theorem (see e.g., Widder [5], p. 192), this result implies that

$$\int_0^x M(b, t)\, dt \sim C(b)\tfrac{1}{2}x^2 \qquad \text{as} \quad x \to \infty. \tag{5.15}$$

From the definition of M it also follows that the same result holds if M is replaced by μ, i.e.,

$$\int_0^x \mu(b, t)\, dt \sim C(b)\tfrac{1}{2}x^2 \qquad \text{as} \quad x \to \infty. \tag{5.16}$$

Adapting a result of Hardy and Littlewood (see p. 93 of [5]), we see that we can differentiate both sides of the asymptotic equality (5.15) if we can show that $(\partial/\partial x)\mu(b, x)$ is bounded for $x > 2a$. But from (5.5) it follows that for $x > 2a$

$$\left|\frac{\partial\mu(b, x)}{\partial x}\right| \leqq \left|2\int_1^a \frac{-dG(u)}{(x-u)^2}\int_0^{x-u}\mu(b, t)\,dt\right| + \left|2\int_1^a \frac{dG(u)}{x-u}\mu(b, x-u)\,du\right| \leqq \int_1^a dG(u) + 2\int_1^a dG(u) = 3.$$

Thus we conclude that $\mu(b, x) \sim C(b)x$ as $x \to \infty$, where $C(b)$ is given by (2.16), which proves the lemma.

Proof of Theorem 1. The proof of this theorem is based on the modification of a technique developed by Dvoretzky and Robbins [1]. To provide the motivation for the proof we briefly sketch its main ideas before going on to the details. We first observe that the function $\mu(b, x) + 1 - G(b)$ satisfies an integral equation which for the present we refer to as I.E. We further observe that linear functions of the form $c(x + \nu)$ also satisfy I.E. The main step is then to show that if any two functions satisfy I.E. and are related by a particular inequality, say I^*, when their arguments are in an interval $(a_0, a_0 + a)$ for some sufficiently large a_0, then they are related by I^* for *all* values of their arguments greater than a_0. Using this idea we are able to conclude that $\mu(b, x) + 1 - G(b)$ lies between two lines, and thence that

$$\frac{\mu(b, x) + 1 - G(b)}{x + \nu}$$

converges to a constant depending on b only, say $K(b)$.

The remainder of the proof consists of looking at the function

$$\mu(b, x) + 1 - G(b) - (x + \nu)K(b),$$

and using a property of the zeros of this function to show that its order of magnitude as $x \to \infty$ is not greater than x^β. We then observe that this function of $\mu(b, x)$ itself satisfies I.E., and by substituting it successively into I.E. it follows that the order of magnitude of this function is less than x^{-n} for any n. This will then imply the theorem. We proceed now to the details.

It follows from (5.4) that for $m = 1$ and $x > a$, $\mu(b, x)$ satisfies

$$\mu(b, x) = 1 - G(b) + 2\int_1^a dG(u)\int_0^{x-u}\mu(b, t)f_x(t \mid u)\,dt. \tag{5.17}$$

Let $\theta(b, x) = \mu(b, x) + 1 - G(b)$. Then

$$\theta(b, x) = 2\int_1^a dG(u)\int_0^{x-u}\theta(b, t)f_x(t \mid u)\,dt. \tag{5.18}$$

Consider two functions $f_1(x)$ and $f_2(x)$ satisfying (5.18), and such that $|f_i(x)| \leqq Bx$ for some $0 < B < \infty$. Suppose also that for some $y > 3a$ and for $y - a \leqq x \leqq y$, we have $f_1(x) \geqq f_2(x)$. Then we shall show that there is a $c < \infty$ such that

$$f_1(x) \geqq f_2(x) - cx^{\beta+1} \qquad \text{for all} \quad x \geqq y, \tag{5.19}$$

where β is the constants specified in (2.6).

Let $\epsilon = \min(d_0, (4D)^{-1}, \gamma_0, 1)$, where d_0, D, and γ_0 are the constants specified in (2.5) and (2.6). Suppose that (5.19) holds for $y - a \leqq x \leqq y$. It is sufficient to show that this implies the truth of the statement for $y \leqq x \leqq y + \epsilon$, since we may then extend the result by steps of length ϵ to any x. Suppose that x is in $[y, y + \epsilon]$. Then $y - a \leqq x - u \leqq y$, and hence by hypothesis

$$\begin{aligned} f_1(x) &= 2\int_1^a dG(u)\left\{\int_0^{y-u} + \int_{y-u}^{x-u} f_1(t)f_x(t \mid u)\,dt\right\} \\ &\geqq 2\int_1^a dG(u)\int_0^{y-u} f_1(t)f_x(t \mid u)\,dt + 2\int_1^a dG(u)\int_{y-u}^{x-u} f_2(t)f_x(t \mid u)\,dt \\ &\qquad - 2\int_1^a dG(u)\int_{y-u}^{x-u} ct^{\beta+1}f_x(t \mid u)\,dt. \end{aligned} \tag{5.20}$$

Using (2.5) and the fact that $0 \leqq x - y \leqq \epsilon \leqq \min(d_0, (4D)^{-1})$ we note that

$$2\int_1^a dG(u)\int_{y-u}^{x-u} ct^{\beta+1}f_x(t \mid u)\,dt \leqq \tfrac{1}{2}cx^{\beta+1}. \tag{5.21}$$

Furthermore

$$\int_0^{y-u} [f_1(t) - f_2(t)]f_x(t \mid u)\,dt = \int_0^{y-u} [f_1(t) - f_2(t)]\frac{f_x(t \mid u)}{f_y(t \mid u)}f_y(t \mid u)\,dt$$

which, by (2.6)

$$\geqq \int_0^{y-u} [f_1(t) - f_2(t)]f_y(t \mid u)\,dt - Ax^\beta \int_0^{y-u} |f_1(t) - f_2(t)|\, f_y(t \mid u)\,dt,$$

and by the assumption that $f_i(x) \leqq Bx$, and (2.4),

$$\geqq \int_0^{y-u} [f_1(t) - f_2(t)]f_y(t \mid u)\,dt - ABx^{\beta+1}.$$

Thus

$$\begin{aligned} &\int_1^a dG(u)\int_0^{y-u} [f_1(t) - f_2(t)]f_x(t \mid u)\,dt \\ &\qquad \geqq \int_1^a dG(u)\int_0^{y-u} [f_1(t) - f_2(t)]f_y(t \mid u)\,dt - ABx^{\beta+1} \\ &\qquad = \tfrac{1}{2}f_1(y) - \tfrac{1}{2}f_2(y) - \tfrac{1}{2}ABx^{\beta+1} \geqq -ABx^{\beta+1}, \end{aligned}$$

since $f_1(y) \geqq f_2(y)$ by hypothesis. Substituting this result in (5.20) and using (5.21) then yields

$$f_1(x) \geqq 2 \int_1^a dG(u) \int_0^{y-u} f_2(t) f_x(t \mid u)\, dt - 2ABx^{\beta+1}$$

$$+ 2 \int_1^a dG(u) \int_{y-u}^{x-u} f_2(t) f_x(t \mid u)\, dt - \tfrac{1}{2}cx^{\beta+1}.$$

Thus if we take $c > 4AB$, we may conclude that

$$f_1(x) \geqq f_2(x) - (2AB + \tfrac{1}{2}c)x^{\beta+1} \geqq f_2(x) - cx^{\beta+1}.$$

This establishes (5.19).

Now define

$$\theta_0(b, x) = [\theta(b, x)]/(x + \nu),$$

$$h(b, y) = \inf \{\theta_0(b, x) : y - a \leqq x \leqq y\},$$

$$H(b, y) = \sup \{\theta_0(b, x) : y - a \leqq x \leqq y\}.$$

Since for $y - a \leqq x \leqq y$

$$h(b, y)[x + \nu] \leqq \theta(b, x) \leqq H(b, y)[x + \nu],$$

and since $c(x + \nu)$ satisfies (5.18) for any c, (5.19) implies that

$$(5.22)\quad h(b, y) - \frac{cx^{\beta+1}}{x + \nu} \leqq \theta_0(b, x) \leqq H(b, y) + \frac{cx^{\beta+1}}{x + \nu} \quad \text{for} \quad \textit{all } x \geqq y > 3a.$$

Now since h and H are bounded, and since from (5.18) one can see that the variation of $\theta(b, x)$ over intervals of fixed length is bounded, it follows that there is a $K < \infty$ such that for $y - a \leqq x \leqq y$, we have

$$|H(b, y)(x + \nu) - h(b, y)(x + \nu)| \leqq K.$$

Dividing through by $x + \nu$, and letting $y \to \infty$ (and hence *a fortiori* $x \to \infty$), it follows that

$$|H(b, y) - h(b, y)| \to 0.$$

This together with the fact that (5.22) holds for all $x \geqq y$ implies that $\theta_0(b, x)$ converges to a function of b, say $K(b)$, which satisfies

$$(5.23)\qquad h(b, y) \leqq K(b) \leqq H(b, y).$$

Now for $x > 3a$ the continuity of $\theta(b, x)$ can be proved directly from its definition, and hence $\theta_0(b, x)$ is also continuous. Since $\theta_0(b, x)$ achieves the values $h(b, y)$ and $H(b, y)$ in each interval $y - a \leqq x \leqq y$, it must equal $K(b)$ somewhere in $[y - a, y]$ for any y. Thus

$$(5.24)\qquad f(b, x) = \theta(b, x) - (x + \nu)K(b)$$

has a zero in every interval $[y - a, y]$, for $y > 3a$.

Using (5.18) we see that $f(b, x)$ satisfies the relation

$$f(b, x) = 2 \int_1^a dG(u) \int_0^{x-u} f(b, t) f_x(t \mid u)\, dt,$$

and hence for $y > 3a$ and $y - a \leqq x \leqq y$ we have

$$\begin{aligned} f(b, y) &= 2 \int_1^a dG(u) \left[\int_0^{x-u} f(b, t) \frac{f_y(t \mid u)}{f_x(t \mid u)} f_x(t \mid u)\, dt \right. \\ &\quad \left. + \int_{x-u}^{y-u} f(b, t) f_y(t \mid u)\, dt \right] \leqq 2 \int_1^a dG(u) \int_0^{x-u} f(b, t) f_x(t \mid u)\, dt \\ &\quad + 2Ax^{\beta-1} \int_1^a dG(u) \int_0^{x-u} |f(b, t)|\, f_x(t \mid u)\, dt \\ &\quad + 2 \int_1^a dG(u) \int_{x-u}^{y-u} |f(b, t)|\, f_y(t \mid u)\, dt. \end{aligned} \tag{5.25}$$

By the property of the zeros of f we may choose x in $[y - a, y]$ so that the first term on the right side of (5.25) is zero. Thus given any $y > 3a$, there exists an x in $[y - a, y]$ such that

$$\begin{aligned} f(b, y) \leqq 2Ax^{\beta-1} \int_1^a dG(u) \int_0^{x-u} |f(b, t)|\, f_x(t \mid u)\, dt \\ + 2 \int_1^a dG(u) \int_{x-u}^{y-u} |f(b, t)|\, f_y(t \mid u)\, dt. \end{aligned} \tag{5.26}$$

Since it is clear from its definition that $|f(b, x)| \leqq 2x$, we may apply (2.2)–(2.4) to (5.26) and obtain

$$f(b, y) = O(y^{\beta}). \tag{5.27}$$

Resubstituting (5.27) in (5.26) and applying (2.2)–(2.4) again yields $f(b, y) = O(y^{2\beta-1})$, and resubstitution in (5.26) m times, we get $f(b, y) = O(y^{m\beta-m+1})$. Since $\beta < 1$, we may take m so large that for any given n, $m(\beta - 1) + 1 < -n$, and thus that

$$f(b, y) = O(y^{-n}). \tag{5.28}$$

Finally, from the definition of f and θ, (5.28) implies (3.2) and the theorem.

PROOF OF THEOREM 3. When $f_x(v \mid u)$ is the uniform density function on $[0, x - u]$, then (5.4) reads

$$\begin{aligned} \mu_n(b, x) &= \int_1^a \frac{dG_{0x}(u)}{x - u} \int_0^{x-u} \sum_{i=0}^{n} \binom{n}{i} \mu_{n-i}(b, t) \mu_i(b, x - t - u)\, dt \\ &\quad + n \int_b^a \frac{dG_{0x}(u)}{x - u} \int_0^{x-u} \sum_{i=0}^{n-1} \binom{n-1}{i} \mu_{n-1-i}(b, t) \mu_i(b, x - t - u)\, dt. \end{aligned} \tag{5.29}$$

Let

$$\begin{aligned} \bar{\mu}_n(b, x) &= \int_0^x \frac{dG(u)}{x - u} \int_0^{x-u} \sum_{i=0}^{n} \binom{n}{i} \mu_{n-i}(b, t) \mu_i(b, x - t - u)\, dt \\ &\quad + n \int_0^x \frac{dG(b, u)}{x - u} \int_0^{x-u} \sum_{i=0}^{n-1} \binom{n-1}{i} \mu_{n-1-i}(b, t) \mu(b, x - t - u)\, dt, \end{aligned} \tag{5.30}$$

where it is understood that $\mu_0(b, x) \equiv 1$, and where

$$G(b, u) = \begin{cases} G(u) & \text{when } u \geqq b \\ 0 & \text{otherwise.} \end{cases}$$

Let $m_n(b, s)$ and $\bar{m}_n(b, s)$ be the Laplace transforms of $\mu_n(b, x)$ and $\bar{\mu}_n(b, x)$ respectively, and $g(b, s)$ be the Laplace-Sieltjes transform of $G(b, u)$. Finally, let

$$M_n(b, s) = \int_0^{3a} e^{-xs}\mu_n(b, x)\,dx; \qquad \bar{M}_n(b, s) = \int_0^{3a} e^{-xs}\bar{\mu}_n(b, x)\,dx.$$

Since $\mu_n(b, x + 3a) = \bar{\mu}_n(b, x + 3a)$ for all $x \geqq 0$, we have $m_n(b, s) - M_n(b, s) = \bar{m}_n(b, s) - \bar{M}_n(b, s)$. Furthermore

$$\text{(5.31)} \qquad \begin{aligned} \bar{m}_n(b, s) = g(s) \int_s^\infty \sum_{i=0}^{n} \binom{n}{i} m_{n-i}(b, t) m_i(b, t)\,dt \\ + ng(b, s) \int_s^\infty \sum_{i=0}^{n-1} \binom{n-1}{i} m_{n-1-i}(b, t) m_i(b, t)\,dt, \end{aligned}$$

and hence, setting

$$\text{(5.32)} \qquad A_n(b, s) = \frac{ng(b, s)}{g(s)} \int_s^\infty \sum_{i=0}^{n-1} m_{n-1-i}(b, t) m_i(b, t)\,dt,$$

$$\text{(5.33)} \qquad B_n(b, s) = \sum_{i=1}^{n-1} \binom{n}{i} \int_s^\infty m_{n-i}(b, t) m_i(b, t)\,dt,$$

and

$$\text{(5.34)} \qquad r_n(b, s) = -M_n' + \bar{M}_n' + \frac{g'}{g} M_n - \frac{g'}{g} \bar{M}_n - gA_n' - gB_n',$$

where primes denote derivatives with respect to s, we may write

$$\text{(5.35)} \qquad m_n'(b, s) + \left\{2\frac{g(s)}{s} - \frac{g'(s)}{g(s)}\right\} m(b, s) + r_n(b, s) = 0.$$

The solution of (5.35) is

$$\text{(5.36)} \qquad m_n(b, s) = w(s) \left\{\frac{m_n(b, s_0)}{w(s_0)} + \int_s^{s_0} \frac{r_n(b, t)}{w(t)}\,dt\right\},$$

where w is as defined in (2.12) and (2.13).

We shall show that as $s \to 0$

$$\text{(5.37)} \qquad m_n(b, s) \sim [C(b)]^n [n!/s^{n+1}]$$

and the theorem will then follow directly from the Karameta and Hardy-Littlewood theorems quoted in Section 5. The proof of (5.37) is by induction. It has been proved for $n = 1$. Assume it true for $n = 1, \cdots, N - 1$. Fix s_0. By direct calculation and use of the appropriate definitions we can show that for $n \leqq N - 1$, M'_n, $\bar{M}'_n$, $(g'/g)M_n$, $(g'/g)\bar{M}_n$ and gA'_n are all $0(1/s^{n+1})$ as $s \to 0$. Furthermore

$$-B'_N(b, s) = \sum_{i=1}^{N-1} \binom{N-1}{i} m_{N-i}(b, s) m_i(b, s) \sim \frac{(N-1)N!}{s^{N+2}} [C(b)]^N,$$

and, as was shown before, $w(s) \sim k/s^2$, where k is given by (2.14).

An elementary analysis therefore yields

$$\begin{aligned}\int_s^{s_0} \frac{r_N(b, t)}{w(t)}\, dt &\sim \int_s^{s_0} \frac{(N-1)N!}{kt^N} [C(b)]^N\, dt \qquad \text{as} \quad s \to 0, \\ &= \frac{N!\,[C(b)]^N}{ks^{N-1}} + K(b, s_0),\end{aligned} \tag{5.38}$$

where $K(b, s_0)$ is a constant depending on b and s_0. Finally (5.37), and hence the theorem, follow from (5.36), (5.38), and the already established behavior of $w(s)$ near zero.

Using the same techniques it can be shown that the non-central moments exhibit similar behavior. For example, if $\sigma^2(b, x)$ denotes the variance of $N(b, x)$, then it can be shown (see [3]) that $\sigma^2(b, x) \sim D(b)x$ as $x \to \infty$.

6. Unsolved problems. An obvious shortcoming of the results discussed above is that the function $K(b)$ of Theorem 1 has been explicitly solved for only when $f_x(v \mid u)$ was the uniform density on $[0, x - u]$. In this case we showed that $K(b) = C(b)$ as specified in (2.16). It would be desirable to extend such a result to the case of a general f_x. The author has made only spotty progress in this direction.

Consider for example the case when F is any distribution function on [0, 1], and suppose that $F_x(v \mid u)$ has the special form $F(v/(x - u))$. Thus $F_x(v \mid u)$ is the function F with its scale stretched from [0, 1] to $[0, x - u]$. Its associated density function will be of the form

$$f_x(v \mid u) = \frac{1}{x-u} f\left(\frac{v}{x-u}\right). \tag{6.1}$$

The analogue to (5.5) then becomes

$$\mu(b, x) = \int_b^a dG_{0x}(u) + 2 \int_1^a dG_{0x}(u) \int_0^{x-u} \mu(b, v) f\left(\frac{v}{x-u}\right) \frac{dv}{x-u}. \tag{6.2}$$

After defining $M(b, x)$ as before, and constructing an appropriate analogue of $A(b, x)$, say $D(b, x)$, we will get

$$M(b, x) = D(b, x) + 2 \int_1^x dG(u) \int_0^{x-u} M(b, v) f\left(\frac{v}{x-u}\right) \frac{dv}{x-u}. \tag{6.3}$$

Taking the Laplace transform of both sides of (6.3), and letting $d(b, s)$ be the transform of $D(b, x)$, we get

$$m(b, s) = d(b, s) + 2g(s) \int_s^\infty \frac{m(b, z)}{z} f\left(\frac{s}{z}\right) dz. \tag{6.4}$$

For special cases of f we may, of course, solve (6.4). If for instance, for some $\alpha > -1$, $f(x) = (\alpha + 1)x^\alpha$ for $0 \leqq x \leqq 1$ and $f(x) = 0$ elsewhere, then (6.4) becomes

$$\frac{m(b, s) - d(b, s)}{s^\alpha g(s)} = 2(\alpha + 1) \int_s^\infty \frac{m(b, z)}{z^{\alpha+1}} dz, \tag{6.5}$$

and we may proceed with our analysis as in the case of the lemma. In general, however, it is not clear how to solve (6.4) sufficiently explicitly to yield $K(b)$. Thus, even for the special form of $f_x(v \mid u)$ employed here, it is not clear how to obtain $K(b)$.

A much deeper class of unsolved problems involves extensions of the present theory to higher dimensions. The simplest generalization would be a two-dimensional analog of the case $U_x \equiv 1$ and $f_x(v \mid u)$ uniform on $[0, x - u]$. This would involve the placing of unit squares at random in a square of side $[0, x]$. Even for this case there are no known analogues of the theorems of this paper. The difficulty lies in the fact that after the first unit square has been placed in the larger square, the remaining set is not a union of separated squares or rectangles. The fact that such a property did hold in the one-dimensional problem, was the basis for attacking that case.

There is no doubt that such multi-dimensional problems are extremely difficult. Their solution, beside being of intrinsic interest in its own right, would be of considerable interest in a number of problems of physics and chemistry which involve the filling of a volume by molecules or crystals. One such problem is that of the idealized gas, which is treated in [2], and whose relation to the problems of this paper is briefly outlined in [3].

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