SYMMETRICAL UNEQUAL BLOCK ARRANGEMENTS WITH TWO UNEQUAL BLOCK SIZES[1]

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1. Summary. This paper deals with the analysis and constructions of Symmetrical Unequal Block (SUB) arrangements with two unequal block sizes. The analysis of these designs is obtained on the assumption of equal intrablock error variances for blocks of different sizes. Various methods of constructing these arrangements from known incomplete block designs are considered in this paper. The method of constructing SUB arrangements by the method of finite differences is also considered in this paper.

2. Introduction. The necessity of incomplete blocks in experiments was noticed by experimenters and theoreticians long ago, and, as a result, the Balanced Incomplete Block (BIB) designs of Yates [21] and the Partially Balanced Incomplete Block (PBIB) designs of Bose and Nair [6] were evolved. Amongst all incomplete block designs with constant block size k, the BIB designs were shown to be the most efficient designs by Kiefer [12], Kshirsagar [14] and Mote [15]. But these designs become useless when the natural blocks comprise different number of plots. To meet this contingency, Kishen [13] introduced the SUB arrangements, which share the property of complete balance (in the sense of having a constant λ), but which involve blocks of different sizes.

Let v treatments $1, 2, \cdots, v$ satisfy the following relations:

(a) Any two treatments are either 1st, or 2nd block associates, the relation of block association being symmetrical, i.e., if the treatment θ is the ith block associate of ϕ, then ϕ is the ith block associate of θ.

(b) Each treatment θ has μ_i, ith block associates, the number μ_i being independent of θ.

(c) If any two treatments θ and ϕ are ith block associates, then the number of treatments which are jth block associates of θ and kth block associates of ϕ, is a_{ijk} and is independent of the pair of ith block associates θ and ϕ. Also,

$$a_{ijk} = a_{ikj}.$$

The arrangement of v treatments, satisfying the above conditions, in b blocks where n_1 blocks are of size k_1, and n_2 blocks are of size k_2 is said to be a SUB arrangement with two unequal block sizes, if

(i) each treatment occurs in $n_i k_i / v$ blocks, of size k_i ($i = 1, 2$), and

(ii) every pair of 1st block associates occurs together in u blocks of size k_1 and in $\lambda - u$ blocks of size k_2, while every pair of 2nd block associates occurs together in λ blocks of size k_2.

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From (i), we can easily see that each treatment occurs in

$$(n_1k_1/v) + (n_2k_2/v) = r,$$

say, blocks in all. $v, b, r, k_i, n_i, \mu_i, a_{ijk}, \lambda(i, j, k = 1, 2)$ are known as the parameters of the SUB arrangement with two unequal block sizes. The following relations are easily deducible from the above definition:

$$b = \sum_{i=1}^{2} n_i, \qquad vr = \sum_{i=1}^{2} n_ik_i, \qquad v(v-1)\lambda = \sum_{i=1}^{2} n_ik_i(k_i - 1),$$

$$v - 1 = \sum_{i=1}^{2} \mu_i, \qquad \sum_{k=1}^{2} a_{ijk} = \mu_j - \delta_{ij}, \tag{2.1}$$

$$\mu_ia_{ijk} = \mu_ja_{jki} = \mu_ka_{kij}, \quad i, j, k = 1, 2,$$

where δ_{ij} is the Kronecker delta taking the value 1 or 0 according as $i = j$ or $i \neq j$. It can be easily seen that the SUB arrangement as defined above is equivalent to adjoining two PBIB designs D, D^*, with two associate classes and the same association scheme and unequal block sizes. The restrictions on the λ parameters of the PBIB designs in this case become $\lambda_2 = 0$, and

$$\lambda_1 + \lambda_1^* = \lambda_2^* = \lambda.$$

These restrictions on the parameters were inherent in the definition of Kishen. Further it follows that the existence of two PBIB plans with the above restrictions on λ's and with the same association schemes implies that of a SUB arrangement and conversely. We use this fact in Sections 6, 7 and 8.

Recently, pairwise balanced designs were used in the construction of mutually orthogonal latin squares and orthogonal arrays ([8], [9], and [11]). Let us see the similarities and differences between pairwise balanced designs and SUB arrangements. An arrangement of v objects (called treatments) in b sets (called blocks) will be called a pairwise balanced design of index λ and type $(v; k_1, k_2, \cdots, k_p)$ if each block contains either $k_1, k_2, \cdots, k_p$ treatments which are all distinct ($k_i \leqq v$, $k_i \neq k_j$, for $i \neq j$), and every pair of distinct treatments occurs in λ blocks of the design. The differences between pairwise balanced designs and SUB arrangements are the following:

(1) In a pairwise balanced design, each treatment need not occur the same number of times, while, in a SUB arrangement, every treatment occurs r times.

(2) Some pairwise balanced designs do not satisfy the conditions (a), (b) and (c) on the treatments.

It can be further seen that SUB arrangements form a special class of pairwise balanced designs which can be conveniently analysed, while, the analysis of a general pairwise balanced design shall be very cumbersome.

Let $N = (n_{ij})$ be the incidence matrix of the SUB arrangements, where $n_{ij} = 1$ or 0 according as the ith treatment occurs in the jth block or not. Then we have

$$b \geqq v, \tag{2.2}$$

since rank $(NN') = v =$ rank $(N) \leqq b$.

Kishen [13] has classified SUB arrangements with two block sizes according as the values of n_1k_1/v, and λ, and provided approximate analyses in particular cases, assuming the same intrablock error variance for blocks of all sizes. It seems more appropriate to assume equal intrablock variance only for the blocks of the same size, but this leads to a complicated analysis and is a problem for further research. The assumption of equal intrablock variance is of very doubtful general validity, but may be reasonable in some cases, e.g., if the experiment is an agronomic one and the block sizes do not vary much. The analysis of SUB arrangements is obtained in this paper with the help of association matrices.

3. Association matrices. Let us further assume that each treatment is its own 0th block associate and of no other treatment. Then, we have

$$\mu_0 = 1, \qquad a_{0ij} = \delta_{ij}\mu_i, \qquad i, j = 0, 1, 2,$$
$$a_{i0k} = \delta_{ik}, \qquad i, k = 1, 2, \tag{3.1}$$
$$a_{i00} = 0, \qquad i = 1, 2,$$

We now define the matrices B_0, B_1, and B_2 as the association matrices of SUB arrangements with two unequal block sizes as $B_i = (b_{\theta\phi i})$, where $b_{\theta\phi i} = 1$ or 0 according as θ and ϕ are ith block associates or otherwise. We can easily see that

$$B_0 + B_1 + B_2 = E_{vv},$$
$$B_jB_k = \sum_{i=0}^{2} a_{ijk}B_i, \qquad j, k = 0, 1, 2, \tag{3.2}$$

where E_{mn} stands for the $m \times n$ matrix with positive unit elements everywhere.

Now, let $A_k = (a_{ijk})$, $k = 0, 1, 2$ be a 3×3 matrix, where the first subscript stands for the column, the second for the row and the third for the matrix. Following an argument similar to Bose and Mesner [5], it can be shown that A's provide a regular representation in 3×3 matrices of the algebra given by the B's, which are $v \times v$ matrices.

4. Analysis. We easily see that

$$u = n_1k_1(k_1 - 1)/v\mu_1 . \tag{4.1}$$

Let us assume the model to be

$$y_{ij} = m + t_i + b_j + \epsilon_{ij}, \tag{4.2}$$

where y_{ij} is the yield of the plot in the jth block to which the ith treatment is applied, m is the general effect, t_i is the effect of the ith treatment, b_j is the effect of the jth block and the ϵ_{ij}'s are independent normal variates with mean zero and variance σ^2. Let T_i be the total yield of all plots having the ith treatment, B_j be the total yield of the jth block, and $\hat{t}_i$ be a solution for t_i in the normal equations. Further, denote the column vectors

$$\{T_1, T_2, \cdots, T_v\}, \qquad \{B_1, B_2, \cdots, B_b\}, \qquad \{t_1, t_2, \cdots, t_v\},$$

and $\{\hat{t}_1, \hat{t}_2, \cdots, \hat{t}_v\}$ by T, B, t, and $\hat{t}$ respectively. It is well known that the reduced normal equations are

$$Q = C\hat{t}, \tag{4.3}$$

where

$$Q = T - N[k_1^{-1}I_{n_1} \dotplus k_2^{-1}I_{n_2}]B, \tag{4.4}$$

and

$$C = rI_v - N[k_1^{-1}I_{n_1} \dotplus k_2^{-1}I_{n_2}]N'. \tag{4.5}$$

In equations (4.4) and (4.5), the symbol $\dotplus$ stands for the Kronecker sum (direct sum) of matrices. If N_1 and N_2 are the incidence matrices for the first n_1 blocks and the last n_2 blocks respectively, we have

$$\begin{aligned} C &= rI_v - k_1^{-1}N_1N_1' - k_2^{-1}N_2N_2' \\ &= (r - v^{-1}b)B_0 - k_1^{-1}k_2^{-1}\{uk_2 + (\lambda - u)k_1\}B_1 - k_2^{-1}\lambda B_2 . \end{aligned} \tag{4.6}$$

Setting

$$r - v^{-1}b = \alpha_0, \qquad -k_1^{-1}k_2^{-1}\{uk_2 + (\lambda - u)k_1\} = \alpha_1, \qquad -k_2^{-1}\lambda = \alpha_2, \tag{4.7}$$

(4.6) reduces to

$$C = \alpha_0B_0 + \alpha_1B_1 + \alpha_2B_2 . \tag{4.8}$$

A little calculation shows that

$$\alpha_0 + \alpha_1\mu_1 + \alpha_2\mu_2 = 0. \tag{4.9}$$

From Corollary 3.2.1 of Shah [17], it is known that the solution of the normal equations (4.3) is

$$\hat{t} = DQ, \tag{4.10}$$

where D is a linear combination of B_0, B_1, and B_2, such that

$$CD = DC = (vI_v - E_{vv})/v. \tag{4.11}$$

Letting

$$D = \beta_0B_0 + \beta_1B_1 + \beta_2B_2 , \tag{4.12}$$

with a little algebra it can be shown that the β_i's are solutions of the equations

$$\begin{aligned} \sum_{i=0}^{2}\sum_{j=0}^{2} a_{lij}\alpha_i\beta_j &= (v - 1)/v, \qquad \text{if } l = 0; \\ &= -1/v, \qquad \text{otherwise.} \end{aligned} \tag{4.13}$$

From the relation (4.9), we easily see that the above three equations are not independent. Further, since the least square estimate of an estimable function is unique whatever be the solution of the normal equations, we can take any two of the equations (4.13) and solve them with an additional convenient restraint like $\sum_{j=0}^{2}\beta_j = 0$, or for some j, $\beta_j = 0$. A solution for the equations (4.13) can easily be obtained as

$$\begin{aligned}\beta_0 &= -(\mu_1\alpha_1^2 - \Delta\alpha_0)^{-1}\{\Delta + (\mu_1\alpha_1 - \Delta)/v\},\\ \beta_1 &= (\mu_1\alpha_1^2 - \Delta\alpha_0)^{-1}\{\alpha_1 + (\alpha_0 - \alpha_1)/v\},\\ \beta_2 &= 0.\end{aligned} \tag{4.14}$$

where $\Delta = \alpha_0 + a_{111}\,\alpha_1 + a_{112}\,\alpha_2$.

Substituting these values in (4.12), we can find out D, and the matrix D, when substituted in (4.10) determines the solution of the normal equations. The covariance matrix of $\hat{t}$ is

$$\begin{aligned}\sigma^2DCD' = \sigma^2[\{\beta_0 - v^{-1}(\beta_0 + \mu_1\beta_1)\}B_0 \\ + \{\beta_1 - v^{-1}(\beta_0 + \mu_1\beta_1)\}B_1 - v^{-1}(\beta_0 + \mu_1\beta_1)B_2].\end{aligned} \tag{4.15}$$

Hence

$$\begin{aligned}\text{Var}(\hat{t}_\theta - \hat{t}_\phi) &= 2(\beta_0 - \beta_1)\sigma^2, && \text{if } \theta \text{ and } \phi \text{ are first block associates;}\\ &= 2\beta_0\sigma^2, && \text{otherwise.}\end{aligned} \tag{4.16}$$

The average variance is given by

$$\begin{aligned}2\{v\ \text{trace}(DCD') - E_{1v}DCD'E_{v1}\}\sigma^2/v(v-1) \\ = 2\{(v-1)\beta_0 - \mu_1\beta_1\}\sigma^2/(v-1).\end{aligned} \tag{4.17}$$

Hence the efficiency of SUB arrangements is

$$(v-1)/[r\{(v-1)\beta_0 - \mu_1\beta_1\}]. \tag{4.18}$$

5. Characteristic roots of the C matrix of a SUB arrangement. In his concluding remarks, Rao [16] mentions that some balanced designs exist with unequal block sizes also. In this section we prove that no SUB arrangement with two unequal block sizes is balanced. To prove this result we require the characteristic roots of the C matrix. The distinct characteristic roots of the C matrix can be shown, in a manner similar to that of Connor and Clatworthy [11a], to be 0, ϕ_1, and ϕ_2, where

$$\phi_i = \alpha_0 - \tfrac{1}{2}[(\alpha_1 - \alpha_2)(-p + (-1)^i w^{\frac{1}{2}}) + (\alpha_1 + \alpha_2)], \quad i = 1, 2. \tag{5.1}$$

where $p = a_{212} - a_{112}$, $q = a_{112} + a_{212}$ *and* $w = p^2 + 2q + 1$. We know that the design is balanced if the non zero characteristic roots of C are all equal (cf., Rao [16]). We can easily see that ϕ_1 and ϕ_2 are equal if and only if

$$(\alpha_1 - \alpha_2)^2 w = 0. \tag{5.2}$$

Since a's are positive or zero, w is always non zero and (5.2) is impossible. Hence we have

THEOREM 5.1. *No* SUB *arrangement with two unequal block sizes is balanced.*

6. Construction of SUB arrangements from Group Divisible designs. A Group Divisible (GD) design is defined by Bose and Connor [4] as an incomplete

TABLE 6.1

Parameters of SUB *arrangements Obtainable by the method of Theorem* 6.1

Serial No.	Parameters of SUB Arrangements											*
	v	b	r	k_1	k_2	n_1	n_2	μ_1	μ_2	a_{111}	λ	
1	6	7	3	2	3	3	4	1	4	0	1	SR 1
2	6	11	7	3	4	2	9	2	3	1	4	SR 4
3	6	15	9	2	4	3	12	1	4	0	5	R 4
4	8	12	4	2	3	4	8	1	6	0	1	R 5
5	8	26	10	4	3	2	24	3	4	2	3	R 7
6	9	12	7	3	6	3	9	2	6	1	4	SR 14
7	12	13	4	3	4	4	9	2	9	1	1	SR 20
8	12	13	7	3	8	4	9	2	9	1	4	SR 26
9	12	15	7	4	6	3	12	3	8	2	3	SR 25
10	12	19	5	4	3	3	16	3	8	2	1	SR 21
11	12	26	6	2	3	6	20	1	10	0	1	R 16
12	14	21	5	2	4	7	14	1	12	0	1	R 24
13	14	35	7	2	3	7	28	1	12	0	1	R 25
14	15	20	5	3	4	5	15	2	12	1	1	R 27
15	15	28	6	5	3	3	25	4	10	3	1	SR 36
16	15	33	9	5	4	3	30	4	10	3	2	R 31
17	16	36	7	4	3	4	32	3	12	2	1	R 35
18	18	39	7	6	3	3	36	5	12	4	1	SR 45
19	18	57	9	2	3	9	48	1	16	0	1	R 40
20	20	21	5	4	5	5	16	3	16	2	1	SR 51
21	20	29	6	5	4	4	25	4	15	3	1	SR 52
22	20	70	10	2	3	10	60	1	18	0	1	R 42
23	21	52	8	7	3	3	49	6	14	5	1	SR 59
24	24	30	6	4	5	6	24	3	20	2	1	R 45
25	24	50	8	3	4	8	42	2	21	1	1	R 46
26	24	67	9	8	3	3	64	7	16	6	1	SR 61
27	24	76	10	6	3	4	72	5	18	4	1	R 47
28	26	65	9	2	4	13	52	1	24	0	1	R 53
29	27	63	9	3	4	9	54	2	24	1	1	R 54
30	27	84	10	9	3	3	81	8	18	7	1	SR 66
31	28	53	8	7	4	4	49	6	21	5	1	SR 68
32	30	31	6	5	6	6	25	4	25	3	1	SR 70
33	32	68	9	8	4	4	64	7	24	6	1	SR 74
34	35	54	8	7	5	5	49	6	28	5	1	SR 76
35	36	85	10	9	4	4	81	8	27	7	1	SR 78
36	40	69	9	8	5	5	64	7	32	6	1	SR 79
37	40	82	10	4	5	10	72	3	36	2	1	R 61
38	42	55	8	7	6	6	49	6	35	5	1	SR 80
39	45	86	10	9	5	5	81	8	36	7	1	SR 81
40	48	56	8	6	7	8	48	5	42	4	1	R 63
41	48	70	9	8	6	6	64	7	40	6	1	SR 82
42	54	87	10	9	6	6	81	8	45	7	1	SR 84
43	56	57	8	7	8	8	49	6	49	5	1	SR 85
44	56	71	9	8	7	7	64	7	48	6	1	SR 86
45	63	72	9	7	8	9	63	6	56	5	1	R 66
46	63	88	10	9	7	7	81	8	54	7	1	SR 87
47	72	73	9	8	9	9	64	7	64	6	1	SR 89
48	72	89	10	9	8	8	81	8	63	7	1	SR 90
49	80	90	10	8	9	10	80	7	72	6	1	R 68

* Reference to the GD design from the Tables, that can be used to construct the SUB arrangement.

block arrangement with v^* treatments, each replicated r^* times, in b^* blocks, of size k^*, where the treatments can be divided into m groups, of n treatments each, so that the treatments belonging to the same group occur together in λ_1^* blocks, while the treatments belonging to different groups occur together in λ_2^* blocks. We now prove

THEOREM 6.1. *If a* GD *design exists with parameters*

$$(6.1)\qquad v^* = mn, \quad b^*, \quad r^*, \quad k^* \neq n, \quad \lambda_1^* = \lambda - 1, \quad \lambda_2^* = \lambda,$$

then, a SUB *arrangement with parameters*

$$(6.2)\qquad \begin{gathered} v = mn, \quad b = b^* + m, \quad r = r^* + 1, \quad k_1 = n, \quad k_2 = k^*, \\ n_1 = m, \quad n_2 = b^*, \quad \mu_1 = n - 1, \\ \mu_2 = n(m-1), \quad a_{111} = n - 2, \quad \lambda, \end{gathered}$$

can be constructed by adding m blocks to the plan of the GD *design where the i*th *added block contains the treatments of the i*th *group* ($i = 1, 2, \cdots, m$). *Conversely, if a* SUB *arrangement exists with parameters* (6.2), *then, by removing the blocks of size* k_1, *we get a* GD *design with parameters* (6.1) *where the i*th *group contains treatments of the i*th *block of size* k_1 ($i = 1, 2, \cdots, m$).

PROOF. The first part of the theorem is obvious and requires no proof. To prove the second part, it is sufficient to show that the blocks of size k_1 form a PBIB design with group divisible association scheme, which can easily be seen to be so in this case from the definition of SUB arrangements.

The first part of the above theorem is the same as Lemma 5 of Bose and Shrikhande [8], but is included here for completeness.

7. Construction of SUB arrangements from PBIB designs with two associate classes having the triangular association scheme. A PBIB design is said to have a triangular association scheme (cf., Bose and Shimamoto [7]) if the number of treatments $v^* = n(n-1)/2$ and the association scheme is an array of n rows and n columns with the following properties:

(i) The positions in the principal diagonal (running from upper left hand corner to lower right hand corner) are left blank.

(ii) The $n(n-1)/2$ positions above the principal diagonal are filled by the numbers $1, 2, \cdots, n(n-1)/2$ corresponding to the treatments.

(iii) The $n(n-1)/2$ positions below the principal diagonal are filled so that the arrangement is symmetrical about the principal diagonal.

(iv) For any treatment i, the first associates are exactly those treatments which lie in the same row and column as i.

We now prove

THEOREM 7.1. *If a* PBIB *design with two associate classes having a triangular association scheme exists with parameters*

$$(7.1)\qquad v^* = n(n-1)/2, \quad b^*, \quad r^*, \quad k^* \neq n - 1, \quad \lambda_1^* = \lambda - 1, \quad \lambda_2^* = \lambda,$$

then a SUB *arrangement with parameters*

$$v = n(n-1)/2, \qquad b = b^* + n, \qquad r = r^* + 2,$$
$$(7.2) \qquad k_1 = n - 1, \qquad k_2 = k^*, \qquad n_1 = n, \qquad n_2 = b^*,$$
$$\mu_1 = 2n - 4, \qquad \mu_2 = (n-2)(n-3)/2, \qquad a_{111} = n - 2, \qquad \lambda,$$

can be constructed by adjoining n new blocks to the plan of the PBIB *design, where the* ith *added block contains treatments of the* ith *row of the association scheme* ($i = 1, 2, \cdots, n$). *Conversely, if a* SUB *arrangement exists with parameters* (7.2), *by removing the blocks of size* k_1, *we obtain a* PBIB *design with parameters* (7.1) *having a triangular association scheme.*

PROOF. The first part of the theorem is simple and requires no proof. To prove the second part, from the comment in Section 2, it is sufficient to show that the n_1 blocks of size k_1 form a PBIB design with triangular association scheme. It can be easily seen that each treatment occurs twice in blocks of size k_1 and the two blocks in which a particular treatment θ occurs may be written as

$$(7.3) \qquad \begin{matrix} (\theta, \theta_1, \theta_2, \cdots, \theta_{n-2}) \\ (\theta, \theta_{n-1}, \theta_n, \cdots, \theta_{2n-4}), \end{matrix}$$

where $\theta_1, \theta_2, \cdots, \theta_{2n-4}$ are the first block associates of θ. It can further be seen that the first block associates of θ can be divided into two sets $(\theta_1, \theta_2, \cdots, \theta_{n-2})$ and $(\theta_{n-1}, \theta_n, \cdots, \theta_{2n-4})$ such that the treatment pairs (θ_i, θ_j) and $(\theta_{i'}, \theta_{i'})$ are first block associates for $i \neq j = 1, 2, \cdots, n-2$ and $i' \neq j' = n-1, n, \cdots, 2n-4$. Now using an argument similar to that of Shrikhande [18], we prove that the blocks of size k_1 form a PBIB design with triangular association scheme (see Table 7.1).

8. Construction of SUB arrangements from PBIB deisgns with two associate classes having an L_2 association scheme. A PBIB design with two associate classes is said to have an L_2 association scheme, if the number of treatments is

TABLE 7.1

Parameters of SUB *arrangements obtainable by the method of Theorem* 7.1

Serial No.	Parameters of SUB Arrangements											*
	v	b	r	k_1	k_2	n_1	n_2	μ_1	μ_2	a_{111}	λ	
1	10	11	5	4	5	5	6	6	3	3	2	T 9
2	10	15	8	4	6	5	10	6	3	3	4	T 18
3	10	25	8	4	3	5	20	6	3	3	2	T 14
4	15	16	6	5	6	6	10	8	6	4	2	T 22
5	15	16	8	5	9	6	10	8	6	4	4	T 27
6	15	21	5	5	3	6	15	8	6	4	1	T 28

* Reference to the PBIB design with triangular association scheme from the Tables, that can be used to construct the SUB arrangement.

s^2 where s is a positive integer and the treatments can be arranged in an $s \times s$ square, such that any two treatments in the same row and the same column are the first associates. We now prove

THEOREM 8.1. *If a* PBIB *design with two associate classes having an* L_2 *association scheme exists with parameters*

$$v^* = s^2, \quad b^*, \quad r^*, \quad k^* \neq s, \quad \lambda_1^* = \lambda - 1, \quad \lambda_2^* = \lambda, \tag{8.1}$$

then, a SUB *arrangement with parameters*

$$\begin{aligned} &v = s^2, \quad b = b^* + 2s, \quad r = r^* + 2, \quad k_1 = s, \quad k_2 = k^*, \\ &n_1 = 2s, \quad n_2 = b^*, \quad \mu_1 = 2s - 2, \\ &\mu_2 = (s-1)^2, \quad a_{111} = s - 2, \quad \lambda, \end{aligned} \tag{8.2}$$

can be constructed by adding 2s *more blocks to the plan of the* PBIB *design where the* ith *added block contains the treatments in the* ith *row of the association scheme* $(i = 1, 2, \cdots, s)$ *and the* $(s + j)$th *added block* $(j = 1, 2, \cdots, s)$ *contains the treatments in the* jth *column* $(j = 1, 2, \cdots, s)$ *of the association scheme. Conversely, if a* SUB *arrangement exists with parameters* (8.2) *and* $s \neq 4$, *then by removing the blocks of size* k_1, *we obtain a* PBIB *design with parameters* (8.1) *having an* L_2 *association scheme.*

PROOF. Proof for the first part is obvious. To prove the second part, it is sufficient to show that the blocks of size k_1 form a PBIB design with L_2 association scheme and with parameters

$$\begin{aligned} &v' = s^2, \quad b' = 2s, \quad r' = 2, \quad k' = k_1, \quad \lambda_1' = 1, \\ &\lambda_2' = 0, \quad n_1' = 2s - 2, \quad p_{11}^1 = s - 2. \end{aligned} \tag{8.3}$$

From the definition it follows that the blocks of size k_1 form a PBIB design with parameters (8.3) and it can easily be proved to have an L_2 association scheme, when $s \neq 4$ by making an appeal to Theorem 1 of Shrikhande [19].

9. Construction of SUB arrangements from affine resolvable BIB designs. A BIB design with parameters v', b', r', k', λ' is said to be affine resolvable if the b' blocks of the design are grouped into r' sets of b'/r' blocks each, such that each treatment occurs once among the blocks of a given set and blocks of different sets will have k'^2/v' treatments in common. Bose [3] proved that the parameters v', b', r', k', λ' of an affine resolvable BIB design can be expressed in terms of only two integral parameters $n > 0$ and $t \geqq 0$ as follows:

$$\begin{aligned} &v' = n^2\{(n-1)t + 1\}, \quad b' = n(n^2t + n + 1), \\ &r' = n^2t + n + 1, \quad k' = n\{(n-1)t + 1\}, \\ &\lambda' = nt + 1. \end{aligned} \tag{9.1}$$

We now prove

THEOREM 9.1. *If an affine resolvable* BIB *design exists with parameters* (9.1)

TABLE 9.1

Parameters of SUB *arrangements derivable from affine resolvable* BIB *designs by removing a block and the treatments contained in it*

Serial No.	Parameters of Affine Resolvable BIB Designs					Parameters of the Corresponding SUB Arrangements										
	v'	b'	r'	k'	λ'	v	b	r	k_1	k_2	n_1	n_2	μ_1	μ_2	a_{111}	λ
1	9	12	4	3	1	6	11	4	3	2	2	9	2	3	1	1
2	16	20	5	4	1	12	19	5	4	3	3	16	3	8	2	1
3	25	30	6	5	1	20	29	6	5	4	4	25	4	15	3	1
4	49	56	8	7	1	42	55	8	7	6	6	49	6	35	5	1
5	64	72	9	8	1	56	71	9	8	7	7	64	7	48	6	1
6	81	90	10	9	1	72	89	10	9	8	8	81	8	63	7	1

and $n > 2$, then by cutting out a block and the treatments occurring in that block, we obtain a SUB *arrangement with parameters*

$$v = n(n-1)\{(n-1)t+1\}, \qquad b = n(n^2t+n+1) - 1,$$
$$r = n^2t+n+1, \qquad k_1 = n\{(n-1)t+1\},$$
$$k_2 = (n-1)\{(n-1)t+1\}, \qquad n_1 = n-1, \qquad n_2 = n^2(nt+1), \tag{9.2}$$
$$\mu_1 = n\{(n-1)t+1\} - 1, \qquad \mu_2 = n(n-2)\{(n-1)t+1\},$$
$$a_{111} = n\{(n-1)t+1\} - 2, \qquad \lambda = nt+1.$$

PROOF. Without loss of generality, let us assume that the first block in the first replicate is cut out along with the treatments contained in it. This block has zero treatments in common with any other block in the first replicate while it has $k'^2/v' = (n-1)t+1$ treatments in common with any other block of the ith replicate ($i = 2, 3, \cdots, n^2t+n+1$). Thus, by the removal of this block and the treatments contained in it we get an unequal block arrangement with the parameters v, b, r, k_1, k_2, n_1, n_2 and λ as in (9.2). Every treatment occurs with $k_1 - 1$ treatments in blocks of size k_1 once and it occurs with these treatments $\lambda - 1$ times in blocks of size k_2, while it occurs λ times with the other $v - k_1$ treatments in blocks of size k_2. Thus, we get $k_1 - 1$ to be the number of first block associates and $v - k_1$ to be the number of second block associates, i.e., we obtain μ_1 and μ_2. Finally, let us consider two first block associates. The number of treatments common to the first block associates of the first treatment, and the first block associates of the second treatment is the number of treatments contained in the block in which they occur, excluding the two treatments under consideration. Thus, $a_{111} = k_1 - 2$, completing the proof of the theorem.

10. Construction of SUB arrangements from GD designs and BIB designs. We prove

TABLE 10.1

Parameters of SUB arrangements obtainable by the method of Theorem 10.1

Serial No.	Parameters of SUB Arrangements											*
	v	b	r	k_1	k_2	n_1	n_2	μ_1	μ_2	a_{111}	λ	
1	6	15	8	2	4	6	9	2	3	1	4	SR 4, a
2	8	24	9	2	4	12	12	3	4	2	3	SR 9, b
3	9	18	5	2	3	9	9	2	6	1	1	SR 12, a
4	9	18	8	2	6	9	9	2	6	1	4	SR 14, a
5	9	30	9	2	3	9	21	2	6	1	2	R 11, a
6	12	21	5	2	4	12	9	2	9	1	1	SR 20, a
7	12	21	8	2	8	12	9	2	9	1	4	SR 26, a
8	12	30	9	2	6	18	12	3	8	2	3	SR 25, b
9	12	34	7	2	3	18	16	3	8	2	1	SR 21, b
10	15	30	6	2	4	15	15	2	12	1	1	R 27, a
11	15	45	8	2	3	15	30	2	12	1	1	R 28, a
12	15	55	9	2	3	30	25	4	10	3	1	SR 36, c
13	16	40	7	2	4	24	16	3	12	2	1	SR 40, b
14	16	56	9	2	3	24	32	3	12	2	1	R 35, b
15	20	46	7	2	5	30	16	3	16	2	1	SR 51, b
16	20	65	9	2	4	40	25	4	15	3	1	SR 52, c
17	24	60	8	2	5	36	24	3	20	2	1	R 45, b
18	24	66	9	2	4	24	42	2	21	1	1	R 46, a
19	25	75	9	2	5	50	25	4	20	3	1	SR 64, c
20	27	81	10	2	4	27	54	2	24	1	1	R 54, a
21	28	77	10	3	4	28	49	6	21	5	1	SR 68, d
22	30	85	9	2	6	60	25	4	25	3	1	SR 70, c
23	35	84	10	3	5	35	49	6	28	5	1	SR 76, d
24	42	91	10	3	6	42	49	6	35	5	1	SR 80, d
25	49	94	10	3	7	49	49	6	42	5	1	SR 83, d

* Reference to GD design from the Tables, and a, b, c, d stand for BIB designs with parameters

$$v' = 3 = b', \quad r' = 2 = k', \quad \lambda' = 1,$$
$$v' = 4, \quad b' = 6, \quad r'=3, \quad k' = 2, \quad \lambda' = 1,$$
$$v' = 5, \quad b' = 10, \quad r' = 4, \quad k' = 2, \quad \lambda' = 1,$$
$$v' = 7 = b', \quad r' = 3 = k', \quad \lambda' = 1,$$

respectively.

THEOREM 10.1. *Let N_2^* be the incidence matrix of a* GD *design with parameters*

$$(10.1) \qquad v^* = mn, \quad b^*, \quad r^*, \quad k^*, \quad \lambda_1^*, \quad \lambda_2^* > \lambda_1^*,$$

and let N_1^* be the incidence matrix of a BIB *design with parameters*

$$(10.2) \qquad v' = n, \quad b', \quad r', \quad k' \neq k^*, \quad \lambda' = \lambda_2^* - \lambda_1^*.$$

Then

$$(10.3) \qquad N = [I_m X N_1^* \mid N_2^*],$$

where "X" denotes the Kronecker product of matrices, is the incidence matrix of a SUB *arrangement with parameters*

$$v = mn, \quad b = b^* + mb', \quad r = r^* + r', \quad k_1 = k',$$
$$(10.4) \qquad k_2 = k^*, \quad n_1 = mb', \quad n_2 = b^*, \quad \mu_1 = n - 1,$$
$$\mu_2 = n(m-1), \quad a_{111} = n - 2, \quad \lambda = \lambda_2^*.$$

PROOF. The parameters $v, b, r, k_1, k_2, n_1, n_2$, and λ need no explanation. From the method of construction we see that $n - 1$ treatments occur together $\lambda_2^* - \lambda_1^*$ times with a particular treatment θ in blocks of size k_1 and they occur together λ_1^* times with θ in blocks of size k_2, while the other $n(m - 1)$ treatments occur together λ_2^* times with θ in blocks of size k_2. Hence μ_1 and μ_2. From the association scheme of *GD* design, we obtain $a_{111} = n - 2$, completing the proof of the theorem.

11. Construction of SUB arrangements by the method of finite differences. The method of differences had been extensively used by Bose [1], [2], and Sprott [20] to obtain BIB designs. We shall give here the application of this method in the construction of SUB arrangements with two unequal block sizes and $\lambda = 1$.

Consider a finite module M with exactly v elements. Let a treatment correspond to each element of this module. Suppose, it is possible to find t_1 blocks each containing k_1 elements and t_2 blocks each containing k_2 elements satisfying the following conditions:

(1) Among the $\{t_1k_1(k_1 - 1) + t_2k_2(k_2 - 1)\}$ differences formed, each non zero element of the module occurs once,

(2) Let $\theta_1^1, \theta_2^1, \cdots, \theta_{t_1k_1(k_1-1)}^1$ be the differences formed from blocks of size k_1, and among the $t_1k_1\ (k_1 - 1)\{t_1k_1(k_1 - 1) - 1\}$ differences $\theta_i^1 - \theta_j^1\{i \neq j = 1, 2, \cdots, t_1k_1(k_1 - 1)\}$ every number of the set $\theta_1^1, \theta_2^1, \cdots, \theta_{t_1k_1(k_1-1)}^1$ is repeated a_{111} times.

Let

$$(11.1) \qquad \begin{array}{ll} (x_{i1}, x_{i2}, \cdots, x_{ik_1}), & i = 1, 2, \cdots, t_1, \\ (x_{j1}, x_{j2}, \cdots, x_{jk_2}), & j = 1, 2, \cdots, t_2, \end{array}$$

be the $t_1 + t_2$ blocks satisfying the above conditions. Then

THEOREM 11.1. *The $v(t_1 + t_2)$ blocks obtained by adding ϕ, an element of M, to every member of a block of* (11.1) *and reducing it to* (mod v), *is a* SUB *arrangement with parameters*

$$v, \quad b = v(t_1 + t_2), \quad r = t_1k_1 + t_2k_2, \quad k_1, \quad k_2, \quad n_1 = vt_1,$$
$$n_2 = vt_2, \quad \mu_1 = t_1k_1(k_1 - 1), \quad \mu_2 = t_2k_2(k_2 - 1), \quad a_{111}, \quad \lambda = 1.$$

The proof can be given on similar lines to Section 6 of Bose and Nair [6].

12. Bounds on r for SUB arrangements with two unequal block sizes and $b = v$. When $b = v$, we have from relations (2.1) that

$$(12.1) \qquad n_1(r - k_1) = n_2(k_2 - r).$$

Since n_1 and n_2 are positive integers, $r - k_1 \gtreqless 0$, according as $k_2 - r \gtreqless 0$. If $k_2 - r = 0$, then $r - k_1 = 0$ and the SUB arrangement becomes an ordinary symmetrical BIB design. Otherwise, we get $k_1 < r < k_2$ or $k_2 < r < k_1$. Thus, we have

THEOREM 12.1. *In a* SUB *arrangement with two unequal block sizes* k_1 *and* k_2, *and* $b = v$, r *lies in the open interval* (k_1, k_2).

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