A CALCULUS FOR FACTORIAL ARRANGEMENTS[1]

By B. Kurkjian and M. Zelen[2]

Diamond Ordanance Fuze Laboratories; National Bureau of Standards and Mathematics Research Center, U. S. Army, The University of Wisconsin

1. Introduction and summary. This paper introduces a special calculus for the analysis of factorial experiments. The calculus applies to the general case of asymmetric factorial experiments and is not restricted to symmetric factorials as is the current theory which relies on the theory of finite projective geometry. The concise notation and operations of this calculus point up the relationship of treatment combinations to interactions and the effect of patterns of arrangements on the distribution of relevant quantities. One aim is to carry out complex manipulations and operations with relative ease. The calculus enables many large order arithmetic operations, necessary for analyzing factorial designs, to be partly carried out by logical operations. This should be of importance in programming the analysis of factorial designs on high speed computers.

The principal new results of this paper, aside from the new notation and operations, are (i) the further development of a theory of confounding for asymmetrical factorials (Section 4) and (ii) a new approach to the calculation of polynomial regression (Section 5). In particular, the use of the calculus enables one to write the inverse matrix of the normal equations for a polynomial model as a partitioned matrix. As a result it only requires inverting matrices of smaller order.

2. Elements and operations of the calculus. Consider an asymmetric factorial experiment with n factors $A_1, A_2, \cdots, A_n$ such that the number of levels of factor A_i is m_i. A particular selection of levels $i = (i_1, i_2, \cdots, i_n)$ is termed a treatment combination, where i_s denotes the i_sth level of A_s. The total number of treatment combinations is $v = \prod_{i=1}^{n} m_i$.

Let $Y_i (i = 1, 2, \cdots, v)$ denote the observation on the ith treatment combination. Then the effect[3] of the ith treatment combination is defined to be $t_i = E(Y_i) - \sum_{i=1}^{v} E(Y_i)/v$. Due to the factorial structure of the experiment, the model for the treatment effects can be further expressed in terms of the usual main effect and interaction parameters. We shall denote the main effect, two-factor interaction, $\cdots$, n-factor interaction parameters by

$$a_s(i_s), a_{rs}(i_r, i_s), \cdots, a_{12\cdots n}(i_1, i_2, \cdots, i_n).$$

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[2] Present address: Mathematics Research Center, U. S. Army, The University of Wisconsin.

[3] The effects in this paper will only refer to fixed effects.

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These are taken to satisfy the usual linear constraints that the sum over the levels of any factor is zero, e.g.,

$$\sum_{i_s=1}^{m_s} a_s(i_s) = 0, \qquad \sum_{i_r=1}^{m_r} a_{rs}(i_r, i_s) = \sum_{i_s=1}^{m_s} a_{rs}(i_r, i_s) = 0, \cdots .$$

Hence, for the ith treatment combination $i = (i_1, i_2, \cdots, i_n)$, we can write the relation between the treatment effect and interactions as

$$t_i = \sum_{s=1}^{n} a_s(i_s) + \sum_{\substack{s \\ 1 \leqq s < r \leqq n}}^{n} \sum_{r}^{s} a_{rs}(i_r, i_s) + \cdots + a_{12\cdots n}(i_1, i_2, \cdots, i_n). \tag{1}$$

We distinguish between the model (1) and the combinatorial pattern or design which characterizes the scheduling of the measurements. The design describes the number of replicates for each treatment combination, the blocking or grouping of the treatments, the way in which the treatments are assigned to the experimental units, etc. The design dictates the manner in which the treatment effects (t_i) are to be estimated. In turn the estimation of the interaction parameters only depends explicitly on the variance-covariance matrix of the estimated treatment effects. We regard an experiment as consisting of both a model and a design. Our main interest, in this paper, is to investigate relations between the treatment effects and interactions. The statistical properties of the estimated interactions will be completely characterized by the distribution of the estimates of the treatment effects.

2.1 *Direct product, symbolic direct product, and primitive elements.* Throughout this paper, unless otherwise stated, the following notation will be used:

$\mathbf{1}_i : m_i \times 1$ column vector with all elements unity;

$\mathbf{O}_i : m_i \times 1$ column vector with all elements zero;

$\mathbf{I}_i : m_i \times m_i$ unit matrix;

$\mathbf{J}_i = \mathbf{1}_i\mathbf{1}_i' : m_i \times m_i$ matrix with all elements unity.

Let $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{rs})$ be rectangular matrices of dimensions $m \times n$ and $p \times q$ respectively. Then the (right) direct product (DP) or Kronecker product of the two has dimensions $mp \times nq$ and is written (cf., MacDuffee [8])

$$\mathbf{A} \times \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \cdots & a_{2n}\mathbf{B} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix}.$$

In general if $\mathbf{A}_i$ $(i = 1, 2, \cdots, k)$ are matrices with dimensions $m_i \times n_i$, their joint direct product will be written as $\prod_{i=1}^{n} \mathbf{x}\, \mathbf{A}_i$ and has dimensions $\prod_{i=1}^{k} m_i \times \prod_{i=1}^{k} n_i$.

We shall define n primitive elements $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$. The ith primitive element is a vector having m_i components and is given by

$$\mathbf{a}_i' = (a_i(1), a_i(2), \cdots, a_i(m_i)). \tag{2}$$

The primitive elements will be used both as abstract and numerical quantities. When used as numerical quantities they are simply the main effect terms in (1). The formation of new elements from the primitive elements will be carried out using the operation of the symbolic direct product (SDP). The symbol $\otimes$ will be used to denote the SDP. When the SDP operation is used, the primitive elements and the new elements derived from them are abstract elements and *not* numerical quantities. The SDP of (say) $\mathbf{a}_p$ and $\mathbf{a}_q$ is defined by

$$(3)\quad \begin{aligned}(\mathbf{a}_p \otimes \mathbf{a}_q)' &= (a_p(1), a_p(2), \cdots, a_p(m_p)) \otimes (a_q(1), a_q(2), \cdots, a_q(m_q))\\ &= (a_{pq}(1,1), a_{pq}(1,2), \cdots, a_{pq}(1, m_q),\\ &\qquad a_{pq}(2,1), \cdots, a_{pq}(2, m_q), \cdots, a_{pq}(m_p, m_q)).\end{aligned}$$

Similarly the SDP of (say) $\mathbf{a}_p$, $\mathbf{a}_q$, $\mathbf{a}_r$ is

$$(4)\quad \begin{aligned}&(\mathbf{a}_p \otimes \mathbf{a}_q \otimes \mathbf{a}_r)'\\ &\quad = (a_{pqr}(1,1,1), a_{pqr}(1,1,2), \cdots, a_{pqr}(1,1,m_r), a_{pqr}(1,2,1),\\ &\qquad \cdots, a_{pqr}(1,2,m_r), \cdots, a_{pqr}(1, m_q, m_r), a_{pqr}(2, m_q, m_r),\\ &\qquad \cdots, a_{pqr}(m_p, m_q, m_r)).\end{aligned}$$

The SDP is only defined to be an operator on primitive elements or on new elements derived from the SDP in conjunction with primitive elements. Note that the SDP is associative, i.e.,

$$\mathbf{a}_p \otimes \mathbf{a}_q \otimes \mathbf{a}_r = (\mathbf{a}_p \otimes \mathbf{a}_q) \otimes \mathbf{a}_r = \mathbf{a}_p \otimes (\mathbf{a}_q \otimes \mathbf{a}_r).$$

This SDP is the same as that recently used by Shah [9]. It should be noted that Connor used an equivalent operation in [1].

2.2 *Combined operations.* In our operations, simultaneous use will be made of the DP and SDP in the same expression. Since the DP is only defined as an operation on matrices and the SDP is only defined in connection with primitive elements, we shall use the asterisk $*$ to indicate *either* the DP or SDP operation when they both appear in the same expression.

Let $\mathbf{u}' = (u_1, u_2, \cdots, u_{m_p})$ and $\mathbf{w}' = (w_1, w_2, \cdots, w_{m_q})$ be vectors having scalar components. Then the inner product $\mathbf{u}'\mathbf{a}_p$ is

$$(5)\quad \mathbf{u}'\mathbf{a}_p = \sum_{i=1}^{m_p} u_i a_p(i).$$

The expression $\mathbf{u}'\mathbf{a}_p$ when used alone will refer to a numerical quantity. On the other hand, when used in conjunction with the asterisk product, the components $a_p(i)$ will denote abstract quantities. We define the asterisk product

$$(6)\quad \mathbf{u}'\mathbf{a}_p * \mathbf{w}'\mathbf{a}_q = \sum_{i=1}^{m_p} u_i a_p(i) * \sum_{j=1}^{m_q} w_j a_p(j) = \sum_{i=1}^{m_p}\sum_{j=1}^{m_q} u_i w_j a_{pq}(ij).$$

However the right hand side of (6) is exactly $(\mathbf{u}' \times \mathbf{w}')(\mathbf{a}_p \otimes \mathbf{a}_q)$. Hence we can write

$$(7)\quad \mathbf{u}'\mathbf{a}_p * \mathbf{w}'\mathbf{a}_q = (\mathbf{u}' \times \mathbf{w}')(\mathbf{a}_p \otimes \mathbf{a}_q).$$

We shall generalize the relation (7) to matrix operations. Let **B** and **C** be rectangular matrices having dimensions $r \times m_p$ and $s \times m_q$ respectively. Further let the rows of **B** and **C** be denoted by the vectors $(\mathbf{b}_1', \mathbf{b}_2', \cdots, \mathbf{b}_r')$ and $(\mathbf{c}_1', \mathbf{c}_2', \cdots, \mathbf{c}_s')$. Then we define the following operations:

$$(8) \qquad \mathbf{u}'\mathbf{a}_p * \mathbf{C}\mathbf{a}_q = \begin{bmatrix} \mathbf{u}'\mathbf{a}_p * \mathbf{c}_1'\mathbf{a}_q \\ \mathbf{u}'\mathbf{a}_p * \mathbf{c}_2'\mathbf{a}_q \\ \vdots \\ \mathbf{u}'\mathbf{a}_p * \mathbf{c}_s'\mathbf{a}_q \end{bmatrix} = \begin{bmatrix} (\mathbf{u}' \times \mathbf{c}_1')(\mathbf{a}_p \otimes \mathbf{a}_q) \\ (\mathbf{u}' \times \mathbf{c}_2')(\mathbf{a}_p \otimes \mathbf{a}_q) \\ \vdots \\ (\mathbf{u}' \times \mathbf{c}_s')(\mathbf{a}_p \otimes \mathbf{a}_q) \end{bmatrix} = (\mathbf{u}' \times \mathbf{C})(\mathbf{a}_p \otimes \mathbf{a}_q),$$

$$(9) \qquad \mathbf{B}\mathbf{a}_p * \mathbf{C}\mathbf{a}_q = \begin{bmatrix} \mathbf{b}_1'\mathbf{a}_p * \mathbf{C}\mathbf{a}_q \\ \mathbf{b}_2'\mathbf{a}_p * \mathbf{C}\mathbf{a}_q \\ \vdots \\ \mathbf{b}_r'\mathbf{a}_p * \mathbf{C}\mathbf{a}_q \end{bmatrix} = \begin{bmatrix} (\mathbf{b}_1' \times \mathbf{C})(\mathbf{a}_p \otimes \mathbf{a}_q) \\ (\mathbf{b}_2' \times \mathbf{C})(\mathbf{a}_p \otimes \mathbf{a}_q) \\ \vdots \\ (\mathbf{b}_r' \times \mathbf{C})(\mathbf{a}_p \otimes \mathbf{a}_q) \end{bmatrix} = (\mathbf{B} \times \mathbf{C})(\mathbf{a}_p \otimes \mathbf{a}_q).$$

In general if $\mathbf{B}_i$ has dimensions $n_i \times m_i$ for $i = 1, 2, \cdots, n$ we have

$$(10) \qquad (\mathbf{B}_1\mathbf{a}_1 * \mathbf{B}_2\mathbf{a}_2 \cdots * \mathbf{B}_n\mathbf{a}_n) = (\mathbf{B}_1 \times \mathbf{B}_2 \times \cdots \times \mathbf{B}_n)(\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \cdots \otimes \mathbf{a}_n).$$

Consider the vector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ such that α_i takes on only the values zero or one. Define

$$(11) \qquad \mathbf{a}_i^{\alpha_i} = \begin{cases} \mathbf{1}_i & \text{if} \quad \alpha_i = 0, \\ \mathbf{a}_i & \text{if} \quad \alpha_i = 1. \end{cases}$$

Such quantities will arise in expressions like

$$(12) \qquad \mathbf{a}_1^{\alpha_1} * \mathbf{a}_2^{\alpha_2} * \cdots * \mathbf{a}_n^{\alpha_n}.$$

Omitting the null vector, there will be $2^n - 1$ distinct α vectors. Hence there will be $2^n - 1$ distinct terms having the form of (12).

For example with $n = 3$, let $\boldsymbol{\alpha} = (1, 0, 1)$. Then (12) can be written

$$\mathbf{a}_1^{\alpha_1} * \mathbf{a}_2^{\alpha_2} * \mathbf{a}_3^{\alpha_3} = \mathbf{a}_1 \otimes (\mathbf{1}_2 \times \mathbf{a}_3)$$

and

$$= \mathbf{a}_1 \otimes \begin{pmatrix} \mathbf{a}_3 \\ \mathbf{a}_3 \\ \vdots \\ \mathbf{a}_3 \end{pmatrix} = \begin{bmatrix} (a_1(1) \times \mathbf{1}_2) \otimes \mathbf{a}_3 \\ (a_1(2) \times \mathbf{1}_2) \otimes \mathbf{a}_3 \\ \vdots \\ (a_1(m_1) \times \mathbf{1}_2) \otimes \mathbf{a}_3 \end{bmatrix} = (\mathbf{a}_1 \times \mathbf{1}_2) \otimes \mathbf{a}_3.$$

Finally there is need for the analogue of (10), where $\mathbf{a}_i$ is replaced by $\mathbf{a}_i^{\alpha_i}$. This results in

$$(13) \qquad \mathbf{B}_1\mathbf{a}_1^{\alpha_1} * \mathbf{B}_2\mathbf{a}_2^{\alpha_2} * \cdots * \mathbf{B}_n\mathbf{a}_n^{\alpha_n} = (\mathbf{B}_1 \times \mathbf{B}_2 \times \cdots \times \mathbf{B}_n)(\mathbf{a}_1^{\alpha_1} * \mathbf{a}_2^{\alpha_2} * \cdots * \mathbf{a}_n^{\alpha_n}).$$

2.3 *Linear restraints.* In order to make the operations (10) and (13) internally consistent, it is necessary to assume that the components of the primitive elements and the quantities derived from them using the SDP are linearly dependent. We shall assume that these components always satisfy the restraints:

$$
\begin{aligned}
&\mathbf{1}_s' \mathbf{a}_s = 0, && s = 1, 2, \cdots, n;\\
&\begin{bmatrix} \mathbf{1}_r' \times \mathbf{I}_s \\ \mathbf{I}_r \times \mathbf{1}_s' \end{bmatrix} [\mathbf{a}_r \otimes \mathbf{a}_s] = \begin{bmatrix} \mathbf{O}_s \\ \mathbf{O}_r \end{bmatrix}, && r \neq s = 1, 2, \cdots, n;\\
&\begin{bmatrix} \mathbf{1}_q' \times \mathbf{I}_r \times \mathbf{I}_s \\ \mathbf{I}_q \times \mathbf{1}_r' \times \mathbf{I}_s \\ \mathbf{I}_q \times \mathbf{I}_r \times \mathbf{1}_s' \end{bmatrix} [\mathbf{a}_q \otimes \mathbf{a}_r \otimes \mathbf{a}_s] = \begin{bmatrix} \mathbf{O}_r \times \mathbf{O}_s \\ \mathbf{O}_q \times \mathbf{O}_s \\ \mathbf{O}_q \times \mathbf{O}_r \end{bmatrix}, && q \neq r \neq s = 1, 2, \cdots, n;\\
(14) \qquad &\vdots && \vdots\\
&\begin{bmatrix} \mathbf{1}_1' \times \mathbf{I}_2 \times \cdots \times \mathbf{I}_n \\ \mathbf{I}_1 \times \mathbf{1}_2' \times \cdots \times \mathbf{I}_n \\ \vdots \quad \vdots \quad \vdots \\ \mathbf{I}_1 \times \mathbf{I}_2 \times \cdots \times \mathbf{1}_n' \end{bmatrix} [\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \cdots \otimes \mathbf{a}_n] = \begin{bmatrix} \mathbf{O}_2 \times \mathbf{O}_3 \times \cdots \times \mathbf{O}_n \\ \mathbf{O}_1 \times \mathbf{O}_3 \times \cdots \times \mathbf{O}_n \\ \vdots \quad \vdots \quad \vdots \\ \mathbf{O}_1 \times \mathbf{O}_2 \times \cdots \times \mathbf{O}_{n-1} \end{bmatrix}.
\end{aligned}
$$

As a direct consequence of these restraints, if there is at least one matrix (say) the rth for which $\mathbf{B}_r = \mathbf{1}_r'$ and $\alpha_r = 1$, we can then write

$$(\mathbf{B}_1 \times \mathbf{B}_2 \times \cdots \times \mathbf{1}_r' \times \cdots \times \mathbf{B}_n)(\mathbf{a}_1^{\alpha_1} * \mathbf{a}_2^{\alpha_2} * \cdots * \mathbf{a}_r \otimes \cdots * \mathbf{a}_n^{\alpha_n}) = \mathbf{O}^4$$

by using (10) and (14). This result can be summarized in the following lemma.

LEMMA 1. *Let* $\{\mathbf{B}_i\}$ *be matrices of dimension* $n_i \times m_i$ *for* $i = 1, 2, \cdots, n$. *Further if there exists at least one term, (say) the rth, for which* $\mathbf{B}_r\mathbf{a}_r^{\alpha_r} = \mathbf{1}_r'\mathbf{a}_r$, *then*

$$(\mathbf{B}_1 \times \mathbf{B}_2 \times \cdots \times \mathbf{B}_n)(\mathbf{a}_1^{\alpha_1} * \mathbf{a}_2^{\alpha_2} * \cdots * \mathbf{a}_n^{\alpha_n}) = \mathbf{B}_1\mathbf{a}_1^{\alpha_1} * \mathbf{B}_2\mathbf{a}_2^{\alpha_2} * \cdots * \mathbf{B}_n\mathbf{a}_n^{\alpha_n} = \mathbf{O}.$$

Another useful lemma on operations deals with matrices where each row sums to zero. We shall term such a matrix a contrast matrix and shall denote it by $\mathbf{C}_i$; i.e., $\mathbf{C}_i\mathbf{1}_i = \mathbf{O}$. Then one can readily prove the following:

LEMMA 2. *If there exists at least one term such that* $\mathbf{B}_i\mathbf{a}_i^{\alpha_i} = \mathbf{C}_i\mathbf{1}_i = \mathbf{O}$, *then*

$$(\mathbf{B}_1 \times \mathbf{B}_2 \times \cdots \times \mathbf{B}_n)(\mathbf{a}_1^{\alpha_1} * \mathbf{a}_2^{\alpha_2} * \cdots * \mathbf{a}_n^{\alpha_n}) = \mathbf{B}_1\mathbf{a}_1^{\alpha_1} * \mathbf{B}_2\mathbf{a}_2^{\alpha_2} * \cdots * \mathbf{B}_n\mathbf{a}_n^{\alpha_n} = \mathbf{O}.$$

3. Relation of the calculus to factorial experiments. The previous section outlined notation, operations, and some elementary lemmas. In this section, we discuss the relation of the calculus to factorial experiments. The components of the primitive elements defined by (2) are the main effect terms in the linear model (1); the two-factor interaction parameters are the components of $\mathbf{a}_p \otimes \mathbf{a}_q (p \neq q)$; etc. The n-factor interaction parameters are the components of $\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \cdots \otimes \mathbf{a}_n$. These main effect and interaction parameters are not all linearly independent, but can be taken to satisfy certain linear restraints. These linear restraints are explicitly given by (14).

[4] Throughout this paper, the symbol O will denote a null matrix whose dimensions will be evident from the context.

3.1 *The matrix model.* We shall write the relation of the treatment effects to the interaction parameters (1) in matrix notation. In order to do this it is necessary to order serially the treatment combinations. For this purpose let $\boldsymbol{\theta}_i' = (1, 2, \cdots, m_i)$ denote a vector having m_i elements such that the sth element is s for $s = 1, 2, \cdots, m_i$. The SDP for the $\boldsymbol{\theta}_i$'s is defined as

$$(15) \quad (\boldsymbol{\theta}_p \otimes \boldsymbol{\theta}_q)' = (11, 12, \cdots, 1m_q, 21, \cdots, 2m_q, \cdots, m_p1, \cdots, m_pm_q).$$

The SDP among all the $\boldsymbol{\theta}_i$'s $(i = 1, 2, \cdots, n)$ will be denoted by $\boldsymbol{\Phi}_n$ and is

$$(16) \qquad \boldsymbol{\Phi}_n = \boldsymbol{\theta}_1 \otimes \boldsymbol{\theta}_2 \otimes \cdots \otimes \boldsymbol{\theta}_n .$$

The rows of $\boldsymbol{\Phi}_n$ can be used to serially order the $v = \prod_{i=1}^n m_i$ treatment combinations by designating the ith treatment combination to refer to the ith row of $\boldsymbol{\Phi}_n$. The element in the sth column of the ith row denotes the level of factor A_s in the ith treatment combination.

When working with arrays like (16) it will be necessary to single out particular columns from the full array. This can be conveniently done by using the following lemma.

LEMMA 3. *Let $\boldsymbol{\Phi}_n = \boldsymbol{\theta}_1 \otimes \boldsymbol{\theta}_2 \otimes \cdots \otimes \boldsymbol{\theta}_n$ and select any s columns from $\boldsymbol{\Phi}_n$. Define the variable α_i to be*

$$\alpha_i = \begin{cases} 1 & \text{if } i\text{th column is selected} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\boldsymbol{\theta}_i^{\alpha_i} = \begin{cases} \mathbf{1}_i & \text{if} \quad \alpha_i = 0, \\ \boldsymbol{\theta}_i & \text{if} \quad \alpha_i = 1. \end{cases}$$

Then the array of the selected s columns from $\boldsymbol{\Phi}_n$ can be written

$$(17) \qquad \boldsymbol{\theta}_1^{\alpha_1} * \boldsymbol{\theta}_2^{\alpha_2} * \cdots * \boldsymbol{\theta}_n^{\alpha_n}.$$

In particular, the jth column is

$$(18) \qquad \mathbf{1}_1 \times \mathbf{1}_2 \times \cdots \times \mathbf{1}_{j-1} \times \boldsymbol{\theta}_j \times \mathbf{1}_{j+1} \times \cdots \times \mathbf{1}_n .$$

PROOF. When $n = 2$

$$\boldsymbol{\Phi}_2 = \boldsymbol{\theta}_1 \otimes \boldsymbol{\theta}_2 = [\boldsymbol{\theta}_1 \times \mathbf{1}_2 \mid \mathbf{1}_1 \times \boldsymbol{\theta}_2];$$

for $n = 3$

$$\boldsymbol{\Phi}_3 = \boldsymbol{\theta}_1 \otimes \boldsymbol{\theta}_2 \otimes \boldsymbol{\theta}_3 = \boldsymbol{\Phi}_2 \otimes \boldsymbol{\theta}_3 = [\boldsymbol{\Phi}_2 \times \mathbf{1}_3 \mid \mathbf{1}_1 \times \mathbf{1}_2 \times \boldsymbol{\theta}_3],$$

where $\mathbf{1}_1 \times \mathbf{1}_2 \times \boldsymbol{\theta}_3$ is the third column. In general,

$$\boldsymbol{\Phi}_j = \boldsymbol{\Phi}_{j-1} \otimes \boldsymbol{\theta}_j = [\boldsymbol{\Phi}_{j-1} \times \mathbf{1}_j \mid \mathbf{1}_1 \times \mathbf{1}_2 \times \cdots \times \mathbf{1}_{j-1} \times \boldsymbol{\theta}_j],$$

where $\mathbf{1}_1 \times \mathbf{1}_2 \times \cdots \times \mathbf{1}_{j-1} \times \boldsymbol{\theta}_j$ is the jth column of $\boldsymbol{\Phi}_j$. Note that the first

j columns of $\mathbf{\Phi}_n$ are $\boldsymbol{\theta}_1 \otimes \boldsymbol{\theta}_2 \otimes \cdots \otimes \boldsymbol{\theta}_j \times \prod_{i=j+1}^{n} \times \mathbf{1}_i$. Hence the first j columns of $\mathbf{\Phi}_n$ can be written in the partitioned form

$$\mathbf{\Phi}_j \times \prod_{i=j+1}^{n} \times \mathbf{1}_i = \left[\mathbf{\Phi}_{j-1} \times \prod_{i=j}^{n} \times \mathbf{1}_i \;\vdots\; \prod_{i=1}^{j-1} \times \mathbf{1}_i \times \boldsymbol{\theta}_j \times \prod_{i=j+1}^{n} \times \mathbf{1}_i \right]$$

which proves (18).

The proof for (17) is by induction. Let the s selected columns be the $j_1 < j_2 < \cdots < j_s$ columns and assume (17) is true for the $(s - 1)$ columns $j_1, j_2, \cdots, j_{s-1}$. (For notational convenience we shall let $j_{s-1} = y$ and $j_s = z$.) Then the s columns may be written

$$\left[\boldsymbol{\theta}_1^{\alpha_1} * \boldsymbol{\theta}_2^{\alpha_2} * \cdots * \boldsymbol{\theta}_y \times \prod_{i=y+1}^{n} \times \mathbf{1}_i \;\vdots\; \prod_{i=1}^{z-1} \times \mathbf{1}_i \times \boldsymbol{\theta}_z \times \prod_{i=z+1}^{n} \times \mathbf{1}_i \right]$$

$$= \boldsymbol{\theta}_1^{\alpha_1} * \boldsymbol{\theta}_2^{\alpha_2} * \cdots * \boldsymbol{\theta}_y \times \prod_{i=y+1}^{z-1} \times \mathbf{1}_i \otimes \boldsymbol{\theta}_z \times \prod_{i=z+1}^{n} \times \mathbf{1}_i$$

$$= \boldsymbol{\theta}_1^{\alpha_1} * \boldsymbol{\theta}_2^{\alpha_2} * \cdots * \boldsymbol{\theta}_n^{\alpha_n}.$$

We shall use Lemma 3 for writing the model (1) in matrix notation. Let $\mathbf{t}$ be the $v \times 1$ vector of treatment effects as ordered by the array $\mathbf{\Phi}_n$; i.e., the ith treatment combination $i = (i_1, i_2, \cdots, i_n)$ corresponds to the ith row of $\mathbf{\Phi}_n$. Then the level associated with the main effect parameter of A_s is the sth element of i; e.g., $a_s(i_s)$. The levels associated with the two factor interaction parameter for A_r and A_s are the rth and sth elements of i, e.g., $a_{rs}(i_r, i_s)$, etc. Thus by choosing the appropriate columns of $\mathbf{\Phi}_n$ we can achieve a correct identification of factor levels with the interaction parameters. That is, for the interaction involving the s factors $A_{j_1}, A_{j_2}, \cdots, A_{j_s}$ it is only necessary to select the corresponding s columns from $\mathbf{\Phi}_n$. Replacing the $\boldsymbol{\theta}_i$ by $\mathbf{a}_i$ in (17) results in the interaction parameters associated with these factors being written in the correct serial order.

In order to illustrate ideas, consider a three factor experiment with factors A_1, A_2, A_3 and respective numbers of levels m_1, m_2, m_3. Using Lemma 3, the first column of $\mathbf{\Phi}_3$ is $\boldsymbol{\theta}_1 \times \mathbf{1}_2 \times \mathbf{1}_3$; the second column is $\mathbf{1}_1 \times \boldsymbol{\theta}_2 \times \mathbf{1}_3$, etc. The first two columns are $\boldsymbol{\theta}_1 \otimes (\boldsymbol{\theta}_2 \times \mathbf{1}_3)$, the first and third columns are $\boldsymbol{\theta}_1 \otimes (\mathbf{1}_2 \times \boldsymbol{\theta}_3)$, etc. Replacing the $\boldsymbol{\theta}_i$ in these expressions by $\mathbf{a}_i$ gives the vector of main effect and interaction parameters in the desired order. We thus have the model

$$\begin{aligned} \mathbf{t} = {} & (\mathbf{a}_1 \times \mathbf{1}_2 \times \mathbf{1}_3) + (\mathbf{1}_1 \times \mathbf{a}_2 \times \mathbf{1}_3) + (\mathbf{1}_1 \times \mathbf{1}_2 \times \mathbf{a}_3) \\ & + (\mathbf{a}_1 \otimes \mathbf{a}_2 \times \mathbf{1}_3) + (\mathbf{a}_1 \otimes \mathbf{1}_2 \times \mathbf{a}_3) + (\mathbf{1}_1 \times \mathbf{a}_2 \otimes \mathbf{a}_3) + (\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3). \end{aligned} \tag{19}$$

Note that each interaction vector is made up of $n = 3$ terms and can be written as $(\mathbf{a}_1^{\alpha_1} * \mathbf{a}_2^{\alpha_2} * \mathbf{a}_3^{\alpha_3})$. Therefore (19) can be written in the alternative form

$$\mathbf{t} = \sum_{k=1}^{3} \left\{ \sum_{\alpha_1+\alpha_2+\alpha_3=k} (\mathbf{a}_1^{\alpha_1} * \mathbf{a}_2^{\alpha_2} * \mathbf{a}_3^{\alpha_3}) \right\}, \tag{20}$$

where $\sum_{\alpha_1+\alpha_2+\alpha_3=k}$ denotes the summation over all terms for which $\alpha_1 + \alpha_2 + \alpha_3 = k$. The model for the general case can be summarized in the following theorem.

THEOREM 1. *Let* $\mathbf{t}$ *denote the* $v \times 1$ *vector of treatment effects which are ordered serially by* $\mathbf{\Phi}_n$. *Let the variable* α_i *take on the values zero or one. Then the model relating the treatment effects to the main effect and interaction parameters corresponding to* (1) *can be written*

$$\mathbf{t} = \sum_{k=1}^{n} \left\{ \sum_{\alpha_1+\alpha_2+\cdots+\alpha_n=k} (\mathbf{a}_1^{\alpha_1} * \mathbf{a}_2^{\alpha_2} * \cdots * \mathbf{a}_n^{\alpha_n}) \right\}, \tag{21}$$

where the summation $\sum_{\alpha_1+\alpha_2+\cdots+\alpha_n=k}$ *goes through all combinations of* α_i *for which the sum is exactly equal to* k.

The proof of (21) is immediate by direct application of Lemma 3.

3.2 *Operations on the matrix model.* Since the model for treatment effects can be written as a sum of terms of the type $(\mathbf{a}_1^{\alpha_1} * \mathbf{a}_2^{\alpha_2} * \cdots * \mathbf{a}_n^{\alpha_n})$, it is possible to now directly carry Lemmas 1 and 2 over to matrix multiplication on the model. The two theorems presented below are the analogues of Lemmas 1 and 2.

THEOREM 2. *Let* $\mathbf{B}_1, \mathbf{B}_2, \cdots, \mathbf{B}_n$ *be matrices such that* $\mathbf{B}_i$ *has dimensions* $n_i \times m_i$. *Then*

$$(\mathbf{B}_1 \times \mathbf{B}_2 \times \cdots \times \mathbf{B}_n)\mathbf{t} = \sum_{k=1}^{n} \sum_{\alpha_1+\alpha_2+\cdots\alpha_n=k} (\mathbf{B}_1\mathbf{a}_1^{\alpha_1} * \mathbf{B}_2\mathbf{a}_2^{\alpha_2} * \cdots * \mathbf{B}_n\mathbf{a}_n^{\alpha_n}). \tag{22}$$

THEOREM 3. *Let the matrices* $\mathbf{B}_i$ *be of three types; viz.,* $\mathbf{B}_i = \mathbf{I}_i$, $\mathbf{B}_i = \mathbf{1}_i'$, *and* $\mathbf{B}_i = \mathbf{C}_i$ *(a contrast matrix, i.e.,* $\mathbf{C}_i\mathbf{1}_i = \mathbf{O}$*). Then the result of the matrix operation* $\mathbf{B}_i\mathbf{a}_i^{\alpha_i}$ *is summarized in the following multiplication table where* $\mathbf{O}$ *denotes an* $n_i \times 1$ *null vector.*

(23) $\mathbf{B}_i\,\mathbf{a}_i^{\alpha_i} =$

	$\alpha_i = 0$	$\alpha_i = 1$
$\mathbf{B}_i = \mathbf{I}_i$	$\mathbf{1}_i$	$\mathbf{a}_i$
$\mathbf{B}_i = \mathbf{1}_i'$	m_i	$\mathbf{O}$
$\mathbf{B}_i = \mathbf{C}_i$	$\mathbf{O}$	$\mathbf{C}_i\mathbf{a}_i$

THEOREM 4. *Let* $\mathbf{B}_{i_j}$ *for* $j = 1, 2, \cdots, s$ *be contrast matrices, i.e.* $\mathbf{B}_{i_j} = \mathbf{C}_{i_j}$. *Further let the remaining* $\mathbf{B}_i$ *matrices be row vectors with unity elements, i.e.,* $\mathbf{B}_i = \mathbf{1}_i'$. *Then*

$$\begin{aligned}(\mathbf{B}_1 \times \mathbf{B}_2 \times \cdots \times \mathbf{B}_n)\mathbf{t} &= \left(v \Big/ \prod_{j=1}^{s} m_{i_j}\right)\{\mathbf{C}_{i_1}\,\mathbf{a}_{i_1} * \mathbf{C}_{i_2}\,\mathbf{a}_{i_2} * \cdots * \mathbf{C}_{i_s}\,\mathbf{a}_{i_s}\} \\ &= \left(v \Big/ \prod_{j=1}^{s} m_{i_j}\right)\{\mathbf{C}_{i_1} \times \mathbf{C}_{i_2} \times \cdots \times \mathbf{C}_{i_s}\}\{\mathbf{a}_{i_1} \otimes \mathbf{a}_{i_2} \otimes \cdots \otimes \mathbf{a}_{i_s}\}.\end{aligned} \tag{24}$$

PROOF. Using Theorem 3

$$\mathbf{B}_1\,\mathbf{a}_1^{\alpha_1} * \mathbf{B}_2\,\mathbf{a}_2^{\alpha_2} * \cdots * \mathbf{B}_n\,\mathbf{a}_n^{\alpha_n} = \begin{cases} \left(v \Big/ \prod_{j=1}^{s} m_{i_j}\right)[\mathbf{C}_{i_1}\,\mathbf{a}_{i_1} * \mathbf{C}_{i_2}\,\mathbf{a}_{i_2} * \cdots * \mathbf{C}_{i_s}\,\mathbf{a}_{i_s}] \\ \quad \text{if } \alpha_{i_1} = \alpha_{i_2} = \cdots = \alpha_{i_s} = 1 \\ \mathbf{O} \text{ otherwise } \left(\mathbf{O} \text{ has dimension } \prod_{i=1}^{n} n_i \times 1\right). \end{cases}$$

Since there exists only one vector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ for which $\alpha_{i_j} = 1$ for $j = 1, 2, \cdots, s$ with the remaining $\alpha_i = 0$ we have the desired result.

Define the square matrix $\mathbf{M}_i$ of order m_i to be

$$\mathbf{M}_i = m_i\mathbf{I}_i - \mathbf{J}_i\,. \tag{25}$$

Since $\mathbf{M}_i\mathbf{1}_i = \mathbf{O}_i$, $\mathbf{M}_i$ is a contrast matrix. Now

$$\mathbf{M}_i\mathbf{a}_i = m_i\mathbf{a}_i - \mathbf{J}_i\mathbf{a}_i = m_i\mathbf{a}_i$$

by virtue of $\mathbf{1}_i'\mathbf{a}_i = 0$. Hence if $\mathbf{C}_{i_j} = \mathbf{M}_{i_j}$, the right-hand side of (24) reduces to $v\{\mathbf{a}_{i_1} \otimes \mathbf{a}_{i_2} \otimes \cdots \otimes \mathbf{a}_{i_s}\}$. We summarize this result in a corollary.

COROLLARY. *Consider the contrast matrices* $\mathbf{M}_i = m_i\mathbf{I}_i - \mathbf{J}_i$ *and define*

$$\mathbf{M}_i^{x_i} = \begin{cases} \mathbf{1}_i' & \text{if } x_i = 0, \\ \mathbf{M}_i & \text{if } x_i = 1. \end{cases} \tag{26}$$

Let $x_{i_1} = x_{i_2} = \cdots = x_{i_s} = 1$ *with the remaining* $x_i = 0$. *Then*

$$\mathbf{a}_{i_1} \otimes \mathbf{a}_{i_2} \otimes \cdots \otimes \mathbf{a}_{i_s} = (1/v)(\mathbf{M}_1^{x_1} \times \mathbf{M}_2^{x_2} \times \cdots \times \mathbf{M}_n^{x_n})\mathbf{t}. \tag{27}$$

Good [3] has derived a closely related expression for the interaction in terms of the direct product of certain matrices. As an example of the use of this corollary consider the case where $n = 3$. Then we have

$$\mathbf{a}_1 = (1/v)(\mathbf{M}_1 \times \mathbf{1}_2' \times \mathbf{1}_3')\mathbf{t}, \qquad \mathbf{a}_2 = (1/v)(\mathbf{1}_1' \times \mathbf{M}_2 \times \mathbf{1}_3')\mathbf{t}$$

$$\mathbf{a}_3 = (1/v)(\mathbf{1}_1' \times \mathbf{1}_2' \times \mathbf{M}_3)\mathbf{t}, \qquad \mathbf{a}_1 \otimes \mathbf{a}_2 = (1/v)(\mathbf{M}_1 \times \mathbf{M}_2 \times \mathbf{1}_3')\mathbf{t}$$

$$\mathbf{a}_1 \otimes \mathbf{a}_3 = (1/v)(\mathbf{M}_1 \times \mathbf{1}_2' \times \mathbf{M}_3)\mathbf{t}, \qquad \mathbf{a}_2 \otimes \mathbf{a}_3 = (1/v)(\mathbf{1}_1' \times \mathbf{M}_2 \times \mathbf{M}_3)\mathbf{t}$$

$$\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3 = (1/v)(\mathbf{M}_1 \times \mathbf{M}_2 \times \mathbf{M}_3)\mathbf{t}.$$

4. The estimation of interaction effects and designs for asymmetric confounding. In this section we consider some properties of the estimators for the various interaction parameters. It is shown that, when an incomplete block design has a variance-covariance matrix of a certain form, it will be possible to use the design for confounding in an asymmetrical factorial experiment. Let $\hat{\mathbf{t}}$ be the minimum variance unbiased estimate (among the class of estimators linear in the observations) of $\mathbf{t}$ and denote by $\mathbf{V}(\hat{\mathbf{t}})$ the variance-covariance matrix of $\hat{\mathbf{t}}$. Then the minimum variance linear unbiased estimate of an interaction parameter is obtained by substituting $\hat{\mathbf{t}}$ for $\mathbf{t}$ in (27). We summarize this in the following theorem, along with the expression for the variance-covariance matrix.

THEOREM 5. *The minimum variance linear unbiased estimate of a p-factor interaction* $(1 \leqq p \leqq n)$ *is*

$$\hat{\mathbf{a}}_{i_1} \otimes \hat{\mathbf{a}}_{i_2} \otimes \cdots \otimes \hat{\mathbf{a}}_{i_p} = (1/v)(\mathbf{M}_1^{x_1} \times \mathbf{M}_2^{x_2} \times \cdots \times \mathbf{M}_n^{x_n})\hat{\mathbf{t}}. \tag{28}$$

where $x_{i_1} = x_{i_2} = \cdots = x_{i_p} = 1$ *and the remaining* $x_i = 0$. *The variance-covariance matrix of a p-factor interaction estimate and the covariance matrix between two different vectors of interaction estimates are given respectively by*

$$\begin{aligned} &\operatorname{Var}(\hat{\mathbf{a}}_{i_1} \otimes \hat{\mathbf{a}}_{i_2} \otimes \cdots \otimes \hat{\mathbf{a}}_{i_p}) \\ &\quad = (1/v^2)\{(\mathbf{M}_1^{x_1} \times \mathbf{M}_2^{x_2} \times \cdots \times \mathbf{M}_n^{x_n})\mathbf{V}(\hat{\mathbf{t}})(\mathbf{M}_1^{x_1} \times \mathbf{M}_2^{x_2} \times \cdots \times \mathbf{M}_n^{x_n})'\}, \end{aligned} \tag{29}$$

and

$$\begin{aligned} &\operatorname{Cov}(\hat{\mathbf{a}}_{i_1} \otimes \cdots \otimes \hat{\mathbf{a}}_{i_p}, \hat{\mathbf{a}}_{j_1} \otimes \cdots \otimes \hat{\mathbf{a}}_{j_q}) \\ &\quad = (1/v^2)\{(\mathbf{M}_1^{x_1} \times \mathbf{M}_2^{x_1} \times \cdots \times \mathbf{M}_n^{x_n})\mathbf{V}(\hat{\mathbf{t}})(\mathbf{M}_1^{x_1^*} \times \mathbf{M}_2^{x_2^*} \times \cdots \times \mathbf{M}_n^{x_n^*})\}, \end{aligned} \tag{30}$$

where $x_{i_1} = x_{i_2} = \cdots = x_{i_p} = 1$, $x_{j_1}^* = x_{j_2}^* = \cdots = x_{j_q}^* = 1$ *and the remaining* x_i, x_j^* *are zero.*

Let the experiment design pattern to which the factorial arrangement is superimposed be a block design with b blocks. Denote by Y_{ij} the measurement of the ith treatment combination in the jth block. (This only has meaning if the ith treatment appears in the jth block.) Consider the model

$$E(Y_{ij}) = \mu + t_i + b_j, \qquad i = 1, 2, \cdots, v; j = 1, 2, \cdots, b,$$

where μ is a constant, t_i is the effect of the ith treatment combination, and b_j is the effect of the jth block. Further assume the measurements are uncorrelated with common variance σ^2.

When the design is a completely randomized design, or a randomized block design, the expressions for the variances of the interaction estimates reduced to simple forms. Let $\bar{y}_i$ denote the average for the ith treatment and let $\mathbf{Y}' = (\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_v)$. Then the treatment effects vector is estimated by $\hat{\mathbf{t}} = (\mathbf{I} - \mathbf{J}/\mathbf{v})\mathbf{Y}$, where $\mathbf{I}$ and $\mathbf{J}$ have dimension $v \times v$. Since $\mathbf{V}(\mathbf{Y}) = (\sigma^2/r)\mathbf{I}$ (r is number of replicates), $\mathbf{V}(\hat{\mathbf{t}}) = (\sigma^2/r)(\mathbf{I} - \mathbf{J}/v)$. Note that

$$\mathbf{I} = \mathbf{I}_1 \times \mathbf{I}_2 \times \cdots \times \mathbf{I}_n \quad \text{and} \quad \mathbf{J} = \mathbf{J}_1 \times \mathbf{J}_2 \times \cdots \times \mathbf{J}_n$$

as $v = \prod_{i=1}^n m_i$. Therefore (29), and (30) simplify to

$$\operatorname{Var}(\hat{\mathbf{a}}_{i_1} \otimes \hat{\mathbf{a}}_{i_2} \otimes \cdots \otimes \hat{\mathbf{a}}_{i_p}) = (1/rv)\{\mathbf{M}_{i_1} \times \mathbf{M}_{i_2} \times \cdots \times \mathbf{M}_{i_p}\}\sigma^2 \tag{29a}$$

$$\operatorname{Cov}(\hat{\mathbf{a}}_{i_1} \otimes \cdots \otimes \hat{\mathbf{a}}_{i_p}, \quad \hat{\mathbf{a}}_{j_1} \otimes \cdots \otimes \hat{\mathbf{a}}_{j_q}) = \mathbf{O}_p \times \mathbf{O}_q'. \tag{30a}$$

Also by virtue of (29a) and (30a), the sum of squares

$$vr\left(\prod_{i=1}^n m_i^{x_i}\right)^{-1} \{(\hat{\mathbf{a}}_{i_1} \otimes \hat{\mathbf{a}}_{i_2} \otimes \cdots \otimes \hat{\mathbf{a}}_{i_p})'(\hat{\mathbf{a}}_{i_1} \otimes \hat{\mathbf{a}}_{i_2} \otimes \cdots \otimes \hat{\mathbf{a}}_{i_p})\}$$

will have a $\sigma^2\chi^2$ distribution under the null hypothesis of no interaction effect.

As another example, suppose the block design is a balanced incomplete block design with efficiency factor E. The the variance-covariance matrix of $\hat{\mathbf{t}}$ is

$$\mathbf{V}(\hat{\mathbf{t}}) = \sigma^2(Er)^{-1}\left(\mathbf{I} - \frac{1}{v}\mathbf{J}\right).$$

Thus the variance-covariance matrix for a p-factor interaction and the sum of squares are the same as for a randomized block design except that r is replaced by (Er).

The sums of squares, variances, and covariances when confounding with a balanced incomplete block design are the same as for randomized block designs with the exception of a multiplying factor. The problem arises—under what conditions will confounding with an arbitrary block design also result in *simple* sums of squares, variances, and covariances? Theorem 6 provides a partial solution. It is well known that if the variance-covariance matrix of a generalized p-factor interaction term (say) $\mathbf{V}(\hat{\mathbf{a}}_p)$ satisfies the condition $\mathbf{V}(\hat{\mathbf{a}}_p)^2 = \lambda\mathbf{V}(\hat{\mathbf{a}}_p)$ (λ a scalar), then $\hat{\mathbf{a}}_p'\hat{\mathbf{a}}_p/\lambda$ follows a $\sigma^2\chi^2$ distribution when the null hypothesis is true and the observations are independent normal variates with common variance. If this condition holds for all estimates of interaction terms (possibly with different λ's), then it is equivalent to having all covariances between different interactions identically zero, cf., Lancaster [6]. Further these conditions depend only on the $\mathbf{V}(\hat{\mathbf{t}})$ associated with the particular incomplete block design. When $\mathbf{V}(\hat{\mathbf{t}})$ takes the form given in Theorem 6 all these conditions are met.

THEOREM 6. *Let the x_i be defined as in Theorem 5. Define the $m_i \times m_i$ matrix $\mathbf{D}_i^{\delta_i}$ by*

$$\mathbf{D}_i^{\delta_i} = \begin{cases} \mathbf{I}_i & \text{if } \delta_i = 1, \\ \mathbf{J}_i & \text{if } \delta_i = 0. \end{cases}$$

Let constants $c(\delta_1, \delta_2, \cdots, \delta_n)$ be defined as a function of the δ's. Then if $\mathbf{V}(\hat{\mathbf{t}})$ can be written in the form

$$\mathbf{V}(\hat{\mathbf{t}}) = \sigma^2 \sum_{s=0}^{n} \left\{ \sum_{\delta_1+\cdots+\delta_n=s} c(\delta_1, \delta_2, \cdots, \delta_n)[\mathbf{D}_1^{\delta_1} \times \mathbf{D}_2^{\delta_2} \times \cdots \times \mathbf{D}_n^{\delta_n}] \right\}, \tag{31}$$

with $c(1, 1, \cdots, 1) \neq 0$, we will have

$$\text{(i) Var}\,(\hat{\mathbf{a}}_{i_1} \otimes \hat{\mathbf{a}}_{i_2} \otimes \cdots \otimes \hat{\mathbf{a}}_{i_p}) = \frac{\sigma^2}{rvE(c)}[\mathbf{M}_{i_1} \times \mathbf{M}_{i_2} \times \cdots \times \mathbf{M}_{i_p}], \tag{32}$$

where $E(c)$ is a function of the c's in $\mathbf{V}(\hat{\mathbf{t}})$ and depends on the particular factors present in the interaction.

(ii) *All covariances* (30) *between interaction terms having at least one factor not in common are identically zero;*

(iii) *the quadratic forms*

$$\left(E(c)vr \Big/ \prod_{i=1}^{n} m_i^{x_i}\right)(\hat{\mathbf{a}}_{i_1} \otimes \hat{\mathbf{a}}_{i_2} \otimes \cdots \otimes \hat{\mathbf{a}}_{i_p})'(\hat{\mathbf{a}}_{i_1} \otimes \hat{\mathbf{a}}_{i_2} \otimes \cdots \otimes \hat{\mathbf{a}}_{i_p}) \tag{33}$$

will be distributed as $\chi^2\sigma^2$ *variate with* $\prod_{i=1}^{n}(m_i - 1)^{x_i}$ *degrees of freedom under the hypothesis of no interaction effects and assuming the observations are independent, following a normal distribution with common variance* σ^2.

Block designs which can be written in the form (31) are randomized block designs, balanced incomplete block designs, group divisibles designs, designs obtained by taking the direct product of incidence matrices such as those investigated by Shah [10] and Vartak [11]. This latter class of designs includes many higher order associate class partially balanced designs. Extensive applications of these results are given in Kurkjian [5].

With respect to the randomized block designs, $\mathbf{V}(\hat{\mathbf{t}}) = (\sigma^2/r)(\mathbf{I} - (1/v)\mathbf{J})$. Hence $c(1, 1, \cdots, 1) = r^{-1}$, $c(0, 0, \cdots, 0) = -(rv)^{-1}$ and the remaining $c(\delta_1, \delta_2, \cdots, \delta_n)$ are zero. A similar result holds for balanced incomplete block designs except that (Er) replaces r.

When a group divisible design is used for a factorial design, the number of treatments can be written as the product of two integers (say) $v = gh$. Assign the n factors into two groups of n_1 and n_2 factors respectively ($n = n_1 + n_2$), such that the product of the number of levels in the first group is g, i.e. $g = \prod_{i=1}^{n_1} m_i$. Similarly the product of the number of levels in the second group is h. Then the variance-covariance matrix of the design can be written

$$\mathbf{V}(\hat{\mathbf{t}}) = [r(k-1)]^{-1}\{(k - c_1)(\mathbf{I}_g \times \mathbf{I}_h) + (c_1 - c_2)(\mathbf{J}_g \times \mathbf{I}_h) + c_3(\mathbf{J}_g \times \mathbf{J}_h)\}\sigma^2,$$

where $\mathbf{I}_g$, $\mathbf{I}_h$, $\mathbf{J}_g$, $\mathbf{J}_h$ refer to square matrices of dimension g or h and the constants c_1, c_2, and c_3 depend on the particular design. (The constants c_1 and c_2 are functions of the design parameters and are included with most tabulations of group divisible designs. The value of the constant c_3 is not necessary for the analysis.) Therefore $\mathbf{V}(\hat{\mathbf{t}})$ can be put in the form of (31) with

$$c(1, 1, \cdots, 1) = (k - c_1)/[r(k-1)], \; c(0, 0, \cdots, 0) = c_3/[r(k-1)]$$

$$c(\underbrace{0, 0, \cdots, 0}_{n_1\text{times}}, \underbrace{1, 1, \cdots, 1}_{n_2\text{times}}) = (c_1 - c_2)/[r(k-1)].$$

The efficiency factors $E(c)$ can be found by applying (29).

Theorem 6 is the key theorem for confounding in asymmetric experiments. Any incomplete block design having a variance-covariance matrix for the treatment estimates which can be put in the form given by (31) may be used for the confounding of a factorial experiment. The resulting analysis is straightforward and relatively easy.

5. The polynomial model. When all the factors in an experiment are quantitative, the model (1) can be written as a polynomial in n independent variables. In this section we shall adapt the operations and notation of Section 3 to the polynomial model and show how the usual calculations for computing a regression equation may be eased.

5.1. *Scaled model.* Since all factors are quantitative, the ith treatment combination may be designated by $i = (x_{1i}, x_{2i}, \cdots, x_{ni})$ where x_{si} denotes the quantitative level of factor A_s. Define the $m_s \times q_s$ matrix $\mathbf{X}_s (s = 1, 2, \cdots, n)$ by

$$\mathbf{X}_s = \begin{bmatrix} x_{s1} & x_{s1}^2 & \cdots & x_{s1}^{q_s} \\ x_{s2} & x_{s2}^2 & \cdots & x_{s2}^{q_s} \\ \vdots & & & \vdots \\ x_{sm_s} & x_{sm_s}^2 & \cdots & x_{sm_s}^{q_s} \end{bmatrix},$$

where q_s need only satisfy the condition $q_s < m_s$. In order to make a direct correspondence with Section 3, it is necessary to consider a transformation of X_s. For this purpose define the $m_s \times q_s$ matrix $\mathbf{\Xi}_s$ by

$$\mathbf{\Xi}_s = \mathbf{X}_s - \mathbf{J}_s\mathbf{X}_s/m_s = \mathbf{M}_s\mathbf{X}_s/m_s. \tag{35}$$

Note that $\mathbf{1}'\mathbf{\Xi}_s = \mathbf{O}$ and that each element in the rth column of $\mathbf{\Xi}_s$ is of degree r in the variables $x_{s1}, x_{s2}, \cdots, x_{sm_s}$. We shall refer to the development in terms of $\mathbf{\Xi}_s$ as the scaled model.

Also define new primitive elements $\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n$ by

$$\mathbf{b}_s' = (b_s^{(1)}, b_s^{(2)}, \cdots, b_s^{(q_s)}), \tag{36}$$

where new elements are formed by using the SDP. The $\mathbf{b}_s$ will be used in the same manner as the $\mathbf{a}_s$. However note that the number of components in $\mathbf{b}_s$ is different from $\mathbf{a}_s$. The components of $\mathbf{b}_s$ will be used to denote the coefficients of the linear, quadratic, $\cdots$, up to the q_s degree of the $\mathbf{\Xi}_s$ variables. The elements of $\mathbf{b}_p \otimes \mathbf{b}_q$ will denote the linear by linear coefficients of $\mathbf{\Xi}_p$ and $\mathbf{\Xi}_q$, etc.

The relation between the $\mathbf{a}_s$ and the $\mathbf{b}_s$ primitive elements is

$$\mathbf{a}_s = \mathbf{\Xi}_s\mathbf{b}_s. \tag{37}$$

Using the analogy with (11)

$$(\mathbf{\Xi}_i\mathbf{b}_i)^{\alpha_i} = \begin{cases} \mathbf{1}_i & \text{if} \quad \alpha_i = 0, \\ \mathbf{\Xi}_i\mathbf{b}_i & \text{if} \quad \alpha_i = 1. \end{cases}$$

By convention we take $(\mathbf{\Xi}_i\mathbf{b}_i)^{\alpha_i} = \mathbf{\Xi}_i^{\alpha_i}\mathbf{b}_i^{\alpha_i}$. In the case of quantitative variables it will be convenient to define $\beta_1, \beta_2, \cdots, \beta_n$ to be variables which take on only the values zero or one and use these in place of the α_i. The only additional conventions are

$$\begin{cases} (\mathbf{\Xi}_i)^{\beta_i} = \begin{cases} \mathbf{1}_i & \text{if } \beta_i = 0 \\ \mathbf{\Xi}_i & \text{if } \beta_i = 1 \end{cases} \\ \mathbf{b}_i^{\beta_i} = \begin{cases} 1 & \text{if } \beta_i = 0 \\ \mathbf{b}_i & \text{if } \beta_i = 1. \end{cases} \end{cases} \tag{38}$$

Then we have

$$
\begin{aligned}
(39)\qquad \mathbf{a}_1^{\alpha_1} * \mathbf{a}_2^{\alpha_2} * \cdots * \mathbf{a}_n^{\alpha_n} &= (\boldsymbol{\Xi}_1\mathbf{b}_1)^{\beta_1} * (\boldsymbol{\Xi}_2\mathbf{b}_2)^{\beta_2} * \cdots * (\boldsymbol{\Xi}_n\mathbf{b}_n)^{\beta_n} \\
&= \{\boldsymbol{\Xi}_1^{\beta_1} \times \boldsymbol{\Xi}_2^{\beta_2} \times \cdots \times \boldsymbol{\Xi}_n^{\beta_n}\}\{\mathbf{b}_1^{\beta_1} \otimes \mathbf{b}_2^{\beta_2} \otimes \cdots \otimes \mathbf{b}_n^{\beta_n}\},
\end{aligned}
$$

where $\beta_i = \alpha_i$. The second bracket in the right-hand side of (39) will contain terms such as (say) $(\mathbf{b}_1)^1 \otimes (\mathbf{b}_2)^0 = \mathbf{b}_1 \otimes 1$. This quantity is to be taken as simply $\mathbf{b}_1$; i.e.

$$
(40)\qquad \mathbf{b}_s \otimes 1 = \mathbf{b}_s .
$$

Using this convention is equivalent to suppressing all $\mathbf{b}$ elements which are raised to the zero power.

In order to satisfy the linear restraints (14) it is necessary that $\boldsymbol{\Xi}_i$ satisfy $\mathbf{1}_i'\boldsymbol{\Xi}_i = \mathbf{O}$. Then we have $\mathbf{1}_i'\mathbf{a}_i = \mathbf{1}_i'\boldsymbol{\Xi}_i\mathbf{b}_i = 0$ and the remaining restraints given by (14) are satisfied by virtue of (39). The requirement $\mathbf{1}'\boldsymbol{\Xi}_i = \mathbf{O}$ involves no loss in generality as it is simply a change of scale.

Using (37), (38), (39), and (40) the model may now be written in the polynomial form

$$
(41)\qquad \mathbf{t} = \sum_{k=1}^{n} \left\{ \sum_{\beta_1+\beta_2+\cdots+\beta_n=k} [\boldsymbol{\Xi}_1^{\beta_1} \times \boldsymbol{\Xi}_2^{\beta_2} \times \cdots \times \boldsymbol{\Xi}_n^{\beta_n}][\mathbf{b}_1^{\beta_1} \otimes \mathbf{b}_2^{\beta_2} \otimes \cdots \otimes \mathbf{b}_n^{\beta_n}] \right\}.
$$

Note that the parameters of the generalized pth factor interaction can also be written as

$$
\begin{aligned}
(42)\qquad & \mathbf{a}_{i_1} \otimes \mathbf{a}_{i_2} \otimes \cdots \otimes \mathbf{a}_{i_p} \\
&\qquad = (\boldsymbol{\Xi}_{i_1} \times \boldsymbol{\Xi}_{i_2} \times \cdots \times \boldsymbol{\Xi}_{i_p})(\mathbf{b}_{i_1} \otimes \mathbf{b}_{i_2} \otimes \cdots \otimes \mathbf{b}_{i_p}),
\end{aligned}
$$

and substituting (27) in (42)

$$
\begin{aligned}
(43)\qquad & (\boldsymbol{\Xi}_{i_1} \times \boldsymbol{\Xi}_{i_2} \times \cdots \times \boldsymbol{\Xi}_{i_p})(\mathbf{b}_{i_1} \otimes \mathbf{b}_{i_2} \otimes \cdots \otimes \mathbf{b}_{i_p}) \\
&\qquad = (1/v)(\mathbf{M}_1^{x_1} \times \mathbf{M}_2^{x_2} \times \cdots \times \mathbf{M}_n^{x_n})\mathbf{t}
\end{aligned}
$$

where $x_{i_1} = x_{i_2} = \cdots = x_{i_p} = 1$ and the remaining $x_i = 0$. Pre-multiplying (43) by $(\boldsymbol{\Xi}_{i_1} \times \boldsymbol{\Xi}_{i_2} \times \cdots \times \boldsymbol{\Xi}_{i_p})'$ and solving for $\mathbf{b}_{i_1} \otimes \mathbf{b}_{i_2} \otimes \cdots \otimes \mathbf{b}_{i_p}$ results in

$$
\begin{aligned}
(44)\qquad \mathbf{b}_{i_1} \otimes \mathbf{b}_{i_2} \otimes \cdots \otimes \mathbf{b}_{i_p} &= (1/v)[\mathbf{W}_{i_1} \otimes \mathbf{W}_{i_2} \otimes \cdots \otimes \mathbf{W}_{i_p}] \\
&\quad \cdot [\boldsymbol{\Xi}_{i_1}' \times \boldsymbol{\Xi}_{i_2}' \times \cdots \times \boldsymbol{\Xi}_{i_p}'][\mathbf{M}_1^{x_1} \times \mathbf{M}_2^{x_2} \times \cdots \times \mathbf{M}_n^{x_n}]\mathbf{t}
\end{aligned}
$$

where $\mathbf{W}_s = (\boldsymbol{\Xi}_s'\boldsymbol{\Xi}_s)^{-1}$.

Since $\mathbf{M}_s^2 = m_s\mathbf{M}_s$, (44) can be re-written in terms of the original variables as

$$
(45)\qquad \mathbf{b}_{i_1} \otimes \mathbf{b}_{i_2} \otimes \cdots \otimes \mathbf{b}_{i_p} = \left(\prod_{i=1}^{n} \frac{m_i^{x_i}}{v}\right)[\mathbf{G}_1^{x_1} \times \mathbf{G}_2^{x_2} \times \cdots \times \mathbf{G}_n^{x_n}]\mathbf{t}
$$

where

$$
\mathbf{G}_i^{x_i} = \begin{cases} \mathbf{1}_i' & \text{if } x_i = 0 \\ (\mathbf{X}_i'\mathbf{M}_i\mathbf{X}_i)^{-1}\mathbf{X}_i'\mathbf{M}_i & \text{if } x_i = 1. \end{cases}
$$

The significance of (44) or (45) is that it can be used to estimate the coefficients in a large order polynomial model without inverting a large order matrix. One need only invert matrices of the type $(\boldsymbol{\Xi}_s'\boldsymbol{\Xi}_s) = (\mathbf{X}_s'\mathbf{M}_s\mathbf{X}_s/m_s)$ which are of dimension q_s. This result is a generalization of work by Cornish [2] and is related to recent work by Lieblein [7].

The above development was predicated on transforming the original independent variables $\mathbf{X}_s$ into the scaled variables $\boldsymbol{\Xi}_s$. This was done so that the columns of $\boldsymbol{\Xi}_s$ sum to zero; i.e., $\mathbf{1}_s'\boldsymbol{\Xi}_s = \mathbf{O}$. On the other hand if one further imposed the requirement that the columns of $\boldsymbol{\Xi}_s$ were mutually orthogonal, then the development would correspond to curve fitting using orthogonal polynomials.

5.2. *The unscaled model.* Often one requires that both the model and the coefficients be written in terms of the original unscaled variables. Substituting $\boldsymbol{\Xi}_s = \mathbf{M}_s\mathbf{X}_s/m_s$ in the polynomial model (41) and re-arranging terms enables one to write

$$\mathbf{t} = (\mathbf{1}_1 \times \mathbf{1}_2 \times \cdots \times \mathbf{1}_n)c_0 + \sum_{k=1}^{n}\left\{\sum_{\beta_1+\beta_2+\cdots+\beta_n=k}[\mathbf{X}_1^{\beta_1} \times \mathbf{X}_2^{\beta_2} \times \cdots \times \mathbf{X}_n^{\beta_n}][\mathbf{c}_1^{\beta_1} \otimes \mathbf{c}_2^{\beta_2} \otimes \cdots \otimes \mathbf{c}_n^{\beta_n}]\right\} \tag{46}$$

where the $\mathbf{c}$ coefficients now replace the $\mathbf{b}$ coefficients. In order to write the $\mathbf{c}$ coefficients as a function of the $\mathbf{b}$ coefficients define

$$\mathbf{B}_s(x_s) = \begin{cases}\mathbf{I}_{q_s}(q_s \times q_s \text{ identity matrix}) & \text{if } x_s = 1\\ \dfrac{\mathbf{1}_s'\mathbf{X}_s^{\beta_s}}{m_s} & \text{if } x_s = 0.\end{cases} \tag{47}$$

After some algebra the generalized pth order coefficient can be written in terms of the $\mathbf{b}$ coefficients as

$$c_0 = v^{-1}\sum_{k=1}^{n}(-1)^k \cdot\left\{\sum_{\beta_1+\beta_2+\cdots+\beta_n=k}[\mathbf{1}_1'\mathbf{X}_1^{\beta_1} \times \mathbf{1}_2'\mathbf{X}_2^{\beta_2} \times \cdots \times \mathbf{1}_n'\mathbf{X}_n^{\beta_n}][\mathbf{b}_1^{\beta_1} \otimes \mathbf{b}_2^{\beta_2} \otimes \cdots \otimes \mathbf{b}_n^{\beta_n}]\right\}. \tag{48}$$

$$\mathbf{c}_{i_1} \otimes \mathbf{c}_{i_2} \otimes \cdots \otimes \mathbf{c}_{i_p} = \sum_{k=0}^{n-p}(-1)^k \cdot\left\{\sum_{\sum_{i=1}^{n}(\beta_i-\beta_i^{x_i})=k-(n-p)}[\mathbf{B}_1(x_1) \times \mathbf{B}_2(x_2) \times \cdots \times \mathbf{B}_n(x_n)] \cdot[\mathbf{b}_1^{1+\beta_1-\beta_1^{x_1}} \otimes \mathbf{b}_2^{1+\beta_2-\beta_2^{x_2}} \otimes \cdots \otimes \mathbf{b}_n^{1+\beta_n-\beta_n^{x_n}}]\right\} \tag{49}$$

where in (49) $x_{i_1} = x_{i_2} = \cdots = x_{i_p} = 1$ and the remaining $x_i = 0$.

5.3. *Estimates and variance-covariance matrices.* The estimates for the $\mathbf{b}$ or $\mathbf{c}$ coefficients are obtained by replacing the vector $\mathbf{t}$ by its minimum variance unbiased estimate. Also since the **b** coefficients can be written as a function of the **a** parameters, it is an easy matter to find the variance-covariance matrix of the estimates for the **b** regression coefficients. Similarly, since the **c** coefficients can be expressed in terms of the **b** coefficients, the variance-covariance matrix of the estimates for the **c** coefficients can be readily obtained. The results for the **b** coefficients are summarized below in Theorem 7.

THEOREM 7. *Let* $\hat{\mathbf{t}}$ *be the minimum variance linear unbiased estimate of* $\mathbf{t}$ *with variance-covariance matrix* $\mathbf{V}(\hat{\mathbf{t}})$. *Let* $x_{i_1} = x_{i_2} = \cdots = x_{i_p} = 1$ *and the remaining* $x_i = 0$. *Then*

$$\begin{aligned}(50)\quad &\operatorname{Var}(\hat{\mathbf{b}}_{i_1} \otimes \hat{\mathbf{b}}_{i_2} \otimes \cdots \otimes \hat{\mathbf{b}}_{i_p}) \\ &= \left[\prod_{s=1}^{p} \times \mathbf{W}_{i_s}\boldsymbol{\Xi}'_{i_s}\right][\operatorname{Var}(\hat{\mathbf{a}}_{i_1} \otimes \hat{\mathbf{a}}_{i_2} \otimes \cdots \otimes \hat{\mathbf{a}}_{i_p})]\left[\prod_{s=1}^{p} \times \mathbf{W}_{i_s}\boldsymbol{\Xi}'_{i_s}\right]' \\ &= \frac{1}{v^2}\left[\prod_{s=1}^{p} \times \mathbf{W}_{i_s}\boldsymbol{\Xi}'_{i_s}\right]\left[\left(\prod_{j=1}^{n} \times \mathbf{M}_j^{x_j}\right)\mathbf{V}(\hat{\mathbf{t}})\left(\prod_{j=1}^{n} \times \mathbf{M}_j^{x_j}\right)'\right] \\ &\qquad \cdot\left[\prod_{s=1}^{p} \times \mathbf{W}_{i_s}\boldsymbol{\Xi}'_{i_s}\right]';\end{aligned}$$

$$\begin{aligned}(51)\quad &\operatorname{Cov}(\hat{\mathbf{b}}_{i_1} \otimes \hat{\mathbf{b}}_{i_2} \otimes \cdots \otimes \hat{\mathbf{b}}_{i_p}, \hat{\mathbf{b}}_{j_1} \otimes \hat{\mathbf{b}}_{j_2} \otimes \cdots \otimes \hat{\mathbf{b}}_{j_q}) \\ &= \left[\prod_{s=1}^{p} \times \mathbf{W}_{i_s}\boldsymbol{\Xi}'_{i_s}\right][\operatorname{Cov}(\hat{\mathbf{a}}_{i_1} \otimes \cdots \otimes \hat{\mathbf{a}}_{i_p}, \hat{\mathbf{a}}_{j_1} \otimes \cdots \otimes \hat{\mathbf{a}}_{j_q})] \\ &\qquad \cdot\left[\prod_{s=1}^{q} \times \mathbf{W}_{j_s}\boldsymbol{\Xi}'_{j_s}\right]' \\ &= \frac{1}{v^2}\left[\prod_{s=1}^{p} \times \mathbf{W}_{i_s}\boldsymbol{\Xi}'_{i_s}\right]\left[\left(\prod_{i=1}^{n} \times \mathbf{M}_i^{x_i}\right)\mathbf{V}(\hat{\mathbf{t}})\left(\prod_{j=1}^{n} \times \mathbf{M}_j^{x_j^*}\right)'\right] \\ &\qquad \cdot\left[\prod_{s=1}^{q} \times \mathbf{W}_{j_s}\boldsymbol{\Xi}'_{j_s}\right]',\end{aligned}$$

where $x_{j_1}^* = x_{j_2}^* = \cdots = x_{j_q}^* = 1$ *and the remaining* $x_j^* = 0$; *if*

$$\mathbf{V}(\hat{\mathbf{t}}) = \sigma^2/r(\mathbf{I} - (1/v)\mathbf{J}),$$

then (50) *and* (51) *reduce to*

$$(52)\quad \operatorname{Var}(\hat{\mathbf{b}}_{i_1} \otimes \hat{\mathbf{b}}_{i_2} \otimes \cdots \otimes \hat{\mathbf{b}}_{i_p}) = \sigma^2\left(\prod_{i=1}^{n} m_i^{x_i} \Big/ rv\right)\left\{\prod_{s=1}^{p} \times \mathbf{W}_{i_s}\right\},$$

and

$$(53)\quad \operatorname{Cov}(\hat{\mathbf{b}}_{i_1} \otimes \hat{\mathbf{b}}_{i_2} \otimes \cdots \otimes \hat{\mathbf{b}}_{i_p}, \quad \hat{\mathbf{b}}_{j_1} \otimes \hat{\mathbf{b}}_{j_2} \otimes \cdots \otimes \mathbf{b}_{j_q}) = \mathbf{O}$$

where $\mathbf{O}$ *is a* $(\prod_{r=1}^{p} m_{i_s}) \times (\prod_{s=1}^{q} m_{i_s})$ *null matrix.*

The corresponding results for the **c** coefficients are more complicated. However for the case of no block effects, which is the usual regression model, the results can be expressed simply. These are summarized in Theorem 8.

THEOREM 8. *Let* $E(Y_{ij}) = \mu + t_i$ *for* $i = 1, 2, \cdots, v$; $j = 1, 2, \cdots, r$, *and* Var $\mathbf{Y} = \sigma^2 \mathbf{I}/r$, *where the* t_i *are given in matrix form by* (46). *Further let*

$$z_i = 1 + m_i^{-1}(\mathbf{1}_i' \mathbf{X}_i \mathbf{W}_i \mathbf{X}_i' \mathbf{1}_i), \qquad x_{i_1} = x_{i_2} = \cdots = x_{i_p} = 1,$$
$$x_{j_1}^* = x_{j_2}^* = \cdots = x_{j_q}^* = 1,$$

and take the remaining x_i *and* x_j^* *to be equal to zero. Then the variance-covariance matrices of the* $\hat{\mathbf{c}}$ *coefficients are*

$$\text{Var}\, \hat{\mathbf{c}}_0 = \frac{\sigma^2}{rv}\left[\prod_{i=1}^{n} z_i - 1\right]; \tag{54}$$

$$\text{Var}\,(\hat{\mathbf{c}}_{i_1} \otimes \hat{\mathbf{c}}_{i_2} \otimes \cdots \otimes \hat{\mathbf{c}}_{i_p}) = \frac{\sigma^2}{rv}\left[\prod_{i=1}^{n} m_i^{x_i} z_i^{1-x_i}\right]\left[\prod_{s=1}^{p} \times \mathbf{W}_{i_s}\right]; \tag{55}$$

$$\begin{aligned} &\text{Cov}\,(\hat{\mathbf{c}}_{i_1} \otimes \hat{\mathbf{c}}_{i_2} \otimes \cdots \otimes \hat{\mathbf{c}}_{i_p}, \hat{\mathbf{c}}_{j_1} \otimes \hat{\mathbf{c}}_{j_2} \otimes \cdots \otimes \hat{\mathbf{c}}_{j_q}) \\ &\quad = (-1)^{p+q} \frac{\sigma^2}{rv}\left[\prod_{i=1}^{n} m_i^{x_i x_i^*} z_i^{(1-x_i)(1-x_i^*)}\right]\left[\prod_{i=1}^{n} \times \mathbf{W}_i(x_i, x_i^*)\right], \\ &\text{Cov}\,(\hat{c}_0, \hat{\mathbf{c}}_{i_1} \otimes \hat{\mathbf{c}}_{i_2} \otimes \cdots \otimes \hat{\mathbf{c}}_{i_p}) = \frac{(-1)^p \sigma^2}{rv}\left[\prod_{i=1}^{n} z_i^{1-x_i}\right]\left[\prod_{i=1}^{n} \times \mathbf{W}(x_i, 0)\right]; \end{aligned} \tag{56}$$

where

$$\mathbf{W}_i(0,0) = 1, \qquad \mathbf{W}_i(0,1) = \mathbf{1}_i' \mathbf{X}_i \mathbf{W}_i$$
$$\mathbf{W}_i(1,0) = W_i X_i' 1_i, \qquad \mathbf{W}_i(1,1) = \mathbf{W}_i.$$

Due to the fact that $\sum_{i=1}^{v} t_i = 0$, the estimate for μ is the grand average, i.e., $\hat{\mu} = \sum_{i=1}^{v} \sum_{j=1}^{r} Y_{ij}/vr$. Furthermore $\hat{\mu}$ is not correlated with the $\hat{\mathbf{t}}_i$ and hence not with any linear function of them. In particular $\hat{\mu}$ is not correlated with the estimates of the **b** or **c** coefficients. Since both μ and c_0 are constant terms, we could combine them to write $c_{00} = \mu + c_0$ and denote the estimate by $\hat{c}_{00} = \hat{\mu} + \hat{c}_0$. Consequently Var $\hat{c}_{00}$ = Var $\hat{\mu}$ + Var $\hat{c}_0$ and the covariances between $\hat{c}_{00}$ and the $\hat{c}$ coefficients remain exactly the same. We can summarize this in a corollary.

COROLLARY. *Let* $c_{00} = \mu + c_0$, *and* $\hat{\mu} = (1/vr) \sum_{i=1} \sum_{j=1} Y_{ij}$ *with* $\hat{c}_0$ *as the estimate for* c_0. *Then* $\hat{c}_{00} = \hat{\mu} + \hat{c}_0$ *and*

$$\begin{aligned} &\text{Var}\, c_{00} = \frac{\sigma^2}{rv} \prod_{i=1}^{n} z_i; \\ &\text{Cov}\,(\hat{\mathbf{c}}_{i_1} \otimes \hat{\mathbf{c}}_{i_2} \otimes \cdots \otimes \hat{\mathbf{c}}_{i_p}, \hat{c}_{00}) = \text{Cov}\,(\hat{\mathbf{c}}_{i_1} \otimes \hat{\mathbf{c}}_{i_2} \otimes \cdots \otimes \hat{c}_{i_p}, \hat{c}_0). \end{aligned} \tag{57}$$

5.4. *An example:* To illustrate the preceding ideas consider an example with $n = 3$. Then (41) becomes

$$\begin{aligned} \mathbf{t} = (\boldsymbol{\Xi}_1 \times \mathbf{1}_2 \times \mathbf{1}_3)\mathbf{b}_1 + (\mathbf{1}_1 \times \boldsymbol{\Xi}_2 \times \mathbf{1}_3)\mathbf{b}_2 + (\mathbf{1}_1 \times \mathbf{1}_2 \times \boldsymbol{\Xi}_3)\mathbf{b}_3 \\ + (\boldsymbol{\Xi}_1 \times \boldsymbol{\Xi}_2 \times \mathbf{1}_3)(\mathbf{b}_1 \otimes \mathbf{b}_2) + (\boldsymbol{\Xi}_1 \times \mathbf{1}_2 \times \boldsymbol{\Xi}_3)(\mathbf{b}_1 \otimes \mathbf{b}_3) \\ + (\mathbf{1}_1 \times \boldsymbol{\Xi}_2 \times \boldsymbol{\Xi}_3)(\mathbf{b}_2 \otimes \mathbf{b}_3) + (\boldsymbol{\Xi}_1 \times \boldsymbol{\Xi}_2 \times \boldsymbol{\Xi}_3)(\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3). \end{aligned}$$

The solutions for the **b** coefficients in terms of **t** result in (using (44))

$$\mathbf{b}_1 = v^{-1}\{\mathbf{W}_1\mathbf{\Xi}_1'[\mathbf{M}_1 \times \mathbf{1}_2' \times \mathbf{1}_3']\mathbf{t}\}$$

$$\mathbf{b}_2 = v^{-1}\{\mathbf{W}_2\mathbf{\Xi}_2'[\mathbf{1}_1' \times \mathbf{M}_2 \times \mathbf{1}_3']\mathbf{t}\}$$

$$\mathbf{b}_3 = v^{-1}\{\mathbf{W}_3\mathbf{\Xi}_3'[\mathbf{1}_1' \times \mathbf{1}_2' \times \mathbf{M}_3]\mathbf{t}\}$$

$$\mathbf{b}_1 \otimes \mathbf{b}_2 = v^{-1}\{[\mathbf{W}_1 \times \mathbf{W}_2][\mathbf{\Xi}_1' \times \mathbf{\Xi}_2'][\mathbf{M}_1 \times \mathbf{M}_2 \times \mathbf{1}_3']\mathbf{t}\}$$

$$\mathbf{b}_1 \otimes \mathbf{b}_3 = v^{-1}\{[\mathbf{W}_1 \times \mathbf{W}_3][\mathbf{\Xi}_1' \times \mathbf{\Xi}_3'][\mathbf{M}_1 \times \mathbf{1}_2' \times \mathbf{M}_3]\mathbf{t}\}$$

$$\mathbf{b}_2 \otimes \mathbf{b}_3 = v^{-1}\{[\mathbf{W}_2 \times \mathbf{W}_3][\mathbf{\Xi}_2' \times \mathbf{\Xi}_3'][\mathbf{1}_1' \times \mathbf{M}_2 \times \mathbf{M}_3]\mathbf{t}\}$$

$$\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3 = v^{-1}\{[\mathbf{W}_1 \times \mathbf{W}_2 \times \mathbf{W}_3][\mathbf{\Xi}_1' \times \mathbf{\Xi}_2' \times \mathbf{\Xi}_3'][\mathbf{M}_1 \times \mathbf{M}_2 \times \mathbf{M}_3]\mathbf{t}\}.$$

In terms of the unscaled model, the **c** coefficients can be written using (47), (48), and (49) as

$$c_0 = -\left[\left(\frac{\mathbf{1}_1'\mathbf{X}_1}{m_1}\right)\mathbf{b}_1 + \left(\frac{\mathbf{1}_2'\mathbf{X}_2}{m_2}\right)\mathbf{b}_2 + \left(\frac{\mathbf{1}_3'\mathbf{X}_3}{m_3}\right)\mathbf{b}_3\right] + \left[\left(\frac{\mathbf{1}_1'\mathbf{X}_1}{m_1} \times \frac{\mathbf{1}_2'\mathbf{X}_2}{m_2}\right)(\mathbf{b}_1 \otimes \mathbf{b}_2) + \left(\frac{\mathbf{1}_1'\mathbf{X}_1}{m_1} \times \frac{\mathbf{1}_3'\mathbf{X}_3}{m_3}\right)(\mathbf{b}_1 \otimes \mathbf{b}_3) + \left(\frac{\mathbf{1}_2'\mathbf{X}_2}{m_2} \times \frac{\mathbf{1}_3'\mathbf{X}_3}{m_3}\right)(\mathbf{b}_2 \otimes \mathbf{b}_3)\right] - \left[\left(\frac{\mathbf{1}_1'\mathbf{X}_1}{m_1} \times \frac{\mathbf{1}_2'\mathbf{X}_2}{m_3} \times \frac{\mathbf{1}_3'\mathbf{X}_3}{m_3}\right)(\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3)\right],$$

$$\mathbf{c}_1 = \mathbf{b}_1 - \left[\left(\mathbf{I}_{q_1} \times \frac{\mathbf{1}_2'\mathbf{X}_2}{m_2}\right)(\mathbf{b}_1 \otimes \mathbf{b}_2) + \left(\mathbf{I}_{q_1} \times \frac{\mathbf{1}_3'\mathbf{X}_3}{m_3}\right)(\mathbf{b}_1 \otimes \mathbf{b}_3)\right] + \left(\mathbf{I}_{q_1} \times \frac{\mathbf{1}_2'\mathbf{X}_2}{m_2} \times \frac{\mathbf{1}_3'\mathbf{X}_3}{m_3}\right)(\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3),$$

$$c_2 = \mathbf{b}_2 - \left[\left(\frac{\mathbf{1}_1'\mathbf{X}_1}{m_1} \times \mathbf{I}_{q_2}\right)(\mathbf{b}_1 \otimes \mathbf{b}_2) + \left(\mathbf{I}_{q_2} \times \frac{\mathbf{1}_3'\mathbf{X}_3}{m_3}\right)(\mathbf{b}_2 \otimes \mathbf{b}_3)\right] + \left(\frac{\mathbf{1}_1'\mathbf{X}_1}{m_1} \times \mathbf{I}_{q_2} \times \frac{\mathbf{1}_3'\mathbf{X}_3}{m_3}\right)(\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3),$$

$$\mathbf{c}_3 = \mathbf{b}_3 - \left[\left(\frac{\mathbf{1}_1'\mathbf{X}_1}{m_1} \times \mathbf{I}_{q_3}\right)(\mathbf{b}_1 \otimes \mathbf{b}_3) + \left(\frac{\mathbf{1}_2'\mathbf{X}_2}{m_2} \times \mathbf{I}_{q_3}\right)(\mathbf{b}_2 \otimes \mathbf{b}_3)\right] + \left(\frac{\mathbf{1}_1'\mathbf{X}_1}{m_1} \times \frac{\mathbf{1}_2'\mathbf{X}_2}{m_2} \times \mathbf{I}_{q_3}\right)(\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3),$$

$$\mathbf{c}_1 \otimes \mathbf{c}_2 = \mathbf{b}_1 \otimes \mathbf{b}_2 - \left(\mathbf{I}_{q_1} \times \mathbf{I}_{q_2} \times \frac{\mathbf{1}_3'\mathbf{X}_3}{m_3}\right)(\mathbf{b}_1 \otimes_1 \mathbf{b}_2 \otimes \mathbf{b}_3),$$

$$\mathbf{c}_1 \otimes \mathbf{c}_3 = \mathbf{b}_1 \otimes \mathbf{b}_3 - \left(\mathbf{I}_{q_1} \times \frac{\mathbf{1}_2'\mathbf{X}_2}{m_2} \times \mathbf{I}_{q_3}\right)(\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3),$$

$$\mathbf{c}_2 \otimes \mathbf{c}_3 = \mathbf{b}_2 \otimes \mathbf{b}_3 - \left(\frac{\mathbf{1}_1'\mathbf{X}_1}{m_1} \times \mathbf{I}_{q_2} \times \mathbf{I}_{q_3}\right)(\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3),$$

$$\mathbf{c}_1 \otimes \mathbf{c}_2 \otimes \mathbf{c}_3 = \mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3.$$

If there are no block effects, the variance-covariance matrix for the **c** coefficients can be written using Theorem 8. A few representative variance-covariance matrices are

$$\operatorname{Var} \hat{c}_0 = (\sigma^2/rv)[z_1z_2z_3 - 1], \qquad \operatorname{Var} \hat{c}_{00} = (\sigma^2/rv)[z_1z_2z_3]$$

$$\operatorname{Var} \hat{\mathbf{c}}_1 = (m_1\sigma^2/rv)(z_2z_3)\mathbf{W}_1, \qquad \operatorname{Var} (\hat{\mathbf{c}}_1 \otimes \hat{\mathbf{c}}_2) = (m_1m_2\sigma^2/rv)(z_3)(\mathbf{W}_1 \times \mathbf{W}_2)$$

$$\operatorname{Var} (\hat{\mathbf{c}}_1 \otimes \hat{\mathbf{c}}_2 \otimes \hat{\mathbf{c}}_3) = (m_1m_2m_3\sigma^2/rv)(\mathbf{W}_1 \times \mathbf{W}_2 \times \mathbf{W}_3)$$

$$\operatorname{Cov} (\hat{\mathbf{c}}_1, \hat{c}_{00}) = -(\sigma^2/rv)(z_2z_3)(\mathbf{W}_1\mathbf{X}_1'\mathbf{1}_1)$$

$$\operatorname{Cov} (\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2) = (\sigma^2/rv)(z_3)(\mathbf{W}_1\mathbf{X}_1'\mathbf{1}_1 \times \mathbf{1}_2'\mathbf{X}_2\mathbf{W}_2)$$

$$\operatorname{Cov} (\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2 \otimes \hat{\mathbf{c}}_3) = -(\sigma^2/rv)(\mathbf{W}_1\mathbf{X}_1'\mathbf{1}_1 \times \mathbf{1}_2'\mathbf{X}_2\mathbf{W}_2 \times \mathbf{1}_3'\mathbf{X}_3\mathbf{W}_3)$$

$$\operatorname{Cov} (\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_1 \otimes \hat{\mathbf{c}}_2 \otimes \hat{\mathbf{c}}_3) = (m_1\sigma^2/rv)(\mathbf{W}_1 \times \mathbf{1}_2'\mathbf{X}_2\mathbf{W}_2 \times \mathbf{1}_3'\mathbf{X}_3\mathbf{W}_3)$$

$$\operatorname{Cov} (\hat{\mathbf{c}}_1 \otimes \hat{\mathbf{c}}_2, \hat{\mathbf{c}}_1 \otimes \hat{\mathbf{c}}_2 \otimes \hat{\mathbf{c}}_3) = -(m_1m_2\sigma^2/rv)(\mathbf{W}_1 \times \mathbf{W}_2 \times \mathbf{1}_3'\mathbf{X}_3\mathbf{W}_3).$$

5.5. *Inverse matrix of normal equations.* A by-product of Theorem 8 is that it enables one to immediately write the inverse matrix (in partitioned form) of the normal equations for the case of polynomial regression. Consider the case of no block effects, Var $\mathbf{Y} = \sigma^2\mathbf{I}$, and take $n = 2$. Using (46), $E(\mathbf{Y})$ can be written

$$E(\mathbf{Y}) = (\mathbf{1}_1 \times \mathbf{1}_2)c_{00} + (\mathbf{X}_1 \times \mathbf{1}_2)\mathbf{c}_1 + (\mathbf{1}_1 \times \mathbf{X}_2)\mathbf{c}_2 + (\mathbf{X}_1 \times \mathbf{X}_2)(\mathbf{c}_1 \otimes \mathbf{c}_2)$$

where $c_{00} = \mu + c_0$ is a scalar, $\mathbf{c}_i$ are $q_i \times 1$ vectors, and $\mathbf{c}_1 \otimes \mathbf{c}_2$ is a $q_1q_2 \times 1$ vector. Hence the matrix of normal equations for solving for the estimates of the c's has dimension $(1 + q_1)(1 + q_2)$. It can be written in the partitioned form:

$$(58) \qquad \begin{bmatrix} m_1 m_2 & m_2 \mathbf{1}_1' \mathbf{X}_1 & m_1 \mathbf{1}_2' \mathbf{X}_2 & \mathbf{1}_1' \mathbf{X}_1 \times \mathbf{1}_2' \mathbf{X}_2 \\ m_2 \mathbf{X}_1' \mathbf{1}_1 & m_2 \mathbf{X}_1' \mathbf{X}_1 & \mathbf{X}_1' \mathbf{1}_1 \times \mathbf{1}_2' \mathbf{X}_2 & \mathbf{X}_1' \mathbf{X}_1 \times \mathbf{1}_2' \mathbf{X}_2 \\ m_1 \mathbf{X}_2' \mathbf{1}_2 & \mathbf{1}_1' \mathbf{X}_1 \times \mathbf{X}_2' \mathbf{1}_2 & m_1 \mathbf{X}_2' \mathbf{X}_2 & \mathbf{1}_1' \mathbf{X}_1 \times \mathbf{X}_2' \mathbf{X}_2 \\ \mathbf{X}_1' \mathbf{1}_1 \times \mathbf{X}_2' \mathbf{1}_2 & \mathbf{X}_1' \mathbf{X}_1 \times \mathbf{X}_2' \mathbf{1}_2 & \mathbf{X}_1' \mathbf{1}_1 \times \mathbf{X}_2' \mathbf{X}_2 & \mathbf{X}_1' \mathbf{X}_1 \times \mathbf{X}_2' \mathbf{X}_2 \end{bmatrix}.$$

Since the elements of the inverse matrix are the corresponding variances and covariances of the $\hat{\mathbf{c}}$ coefficients (except for the multiplier σ^2), Theorem 8 can be used to write the inverse matrix in partitioned form. The inverse to (58) takes the form

(59)

$$\frac{1}{m_1 m_2} \begin{bmatrix} z_1 z_2 & -z_2(\mathbf{1}_1' \mathbf{X}_1 \mathbf{W}_1) & -z_1(\mathbf{1}_2' \mathbf{X}_2 \mathbf{W}_2) & \mathbf{1}_1' \mathbf{X}_1 \mathbf{W}_1 \times \mathbf{1}_2' \mathbf{X}_2 \mathbf{W}_2 \\ -z_2(\mathbf{W}_1 \mathbf{X}_1' \mathbf{1}_1) & m_1 z_2 \mathbf{W}_1 & \mathbf{W}_1 \mathbf{X}_1' \mathbf{1}_1 \times \mathbf{1}_2' \mathbf{X}_2 \mathbf{W}_2 & -m_1(\mathbf{W}_1 \times \mathbf{1}_2' \mathbf{X}_2 \mathbf{W}_2) \\ -z_1(\mathbf{W}_2 \mathbf{X}_2' \mathbf{1}_2) & \mathbf{1}_1' \mathbf{X}_1 \mathbf{W}_1' \times \mathbf{W}_2 \mathbf{X}_2' \mathbf{1}_2 & m_2 z_1 \mathbf{W}_2 & -m_2(\mathbf{1}_1' \mathbf{X}_1 \mathbf{W}_1 \times \mathbf{W}_2) \\ \mathbf{W}_1 \mathbf{X}_1' \mathbf{1}_1 \times \mathbf{W}_2 \mathbf{X}_2' \mathbf{1}_2 & -m_1(\mathbf{W}_1 \times \mathbf{W}_2 \mathbf{X}_2' \mathbf{1}_2) & -m_2(\mathbf{W}_1 \mathbf{X}_1' \mathbf{1}_1 \times \mathbf{W}_2) & m_1 m_2(\mathbf{W}_1 \times \mathbf{W}_2) \end{bmatrix}.$$

Note that the inverse matrix only involves determining $\mathbf{W}_i = m_i(\mathbf{X}_i'\mathbf{M}_i\mathbf{X}_i)^{-1}$, $i = 1, 2$. Thus the matrix of normal equations which is of dimension $(1 + q_1) \cdot (1 + q_2)$ can be inverted by inverting two smaller matrices of dimension q_1

and q_2, and doing the indicated direct product multiplication. Note that if $q_1 = q_2 = 1$, the model is

$$E(Y_i) = c_{00} + c_1(1)x_{1i} + c_2(1)x_{2i} + c_{12}(11)x_{1i}\,x_{2i}$$

and the $\mathbf{W}_i$ which need to be inverted are

$$\mathbf{W}_i = (\mathbf{X}_i'\mathbf{M}_i\mathbf{X}_i)^{-1}m_i = \left[\sum_{j=1}^{m_i} (x_{ij} - \bar{x}_i)^2\right]^{-1}, \qquad \bar{x}_i = \left(\sum_{j=1}^{m_i} x_{ij}\right)\bigg/ m_i .$$

Hence the $\mathbf{W}_i$ are scalars and the inverse matrix (59) can be written explicitly without any real matrix inversion. If (59) is not used, then a 4×4 matrix inversion is necessary.

The matrix of normal equations (58) and its corresponding inverse (59) refer to the case of $n = 2$ independent variables (factors). Theorem 8, however, applies to the general case of n factors. Hence it is an easy matter to write the corresponding inverse matrix of normal equations for an arbitrary number of factors. In general for n factors, the normal equations will be of dimension $\prod_{i=1}^{n} (1 + q_i)$. Using Theorem 8, the inverse matrix can be written in partitioned form by inverting n matrices having dimensions $q_1, q_2, \cdots, q_n$ respectively. Therefore one can solve for the $\mathbf{c}$ regression coefficient in the original model, without first solving for the $\mathbf{b}$ regression coefficients of the scaled model. Furthermore, when all $q_i = 1$, the inverse matrix for the normal equations can be readily written without calculating any inverse matrices.

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