TABLES TO FACILITATE THE COMPUTATION OF PERCENTAGE POINTS OF THE NON-CENTRAL t-DISTRIBUTION

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1. Introduction. Let z be a random variable distributed normally about zero with unit standard deviation and let w be a random variable distributed independently of z as χ^2/f with f degrees of freedom. If t is defined by $t = (z + \delta)/w^{\frac{1}{2}}$, where δ is some constant, then t is said to have the non-central t-distribution with f degrees of freedom and non-centrality parameter δ.

Among the tables related to the non-central t statistic which have been published the ones which cover the widest range of the parameters f and δ are those of Johnson and Welch [1]. The primary purpose of the Johnson and Welch tables is to facilitate the computation of δ such that, if f, ϵ, and t_ϵ are given, then $\Pr(t > t_\epsilon \mid f, \delta) = \epsilon$.

The Johnson and Welch Table IV gives a quantity λ related to the percentage point t_ϵ by the equations

$$(1) \qquad \lambda = (t_\epsilon - \delta)/(1 + t_\epsilon^2/(2f))^{\frac{1}{2}},$$

$$(2) \qquad t_\epsilon = \frac{\delta + \lambda(1 + \delta^2/(2f) - \lambda^2/(2f))^{\frac{1}{2}}}{1 - \lambda^2/(2f)}.$$

To use the Johnson and Welch Table IV to obtain percentage points, that is, to find t_ϵ when f, δ and ϵ are given, the following iterative procedure is necessary. One begins with an approximation to t_ϵ based on the asymptotic normality of the non-central t statistic; one then transforms the argument t_ϵ, interpolates for λ in the tables, and then solves (1) for a new value of t_ϵ. This process is repeated until the value of t_ϵ stabilizes.

A special table, number V, is given by Johnson and Welch, which gives λ as a function of f, δ and ϵ, for $\epsilon = .05$. In constructing Table V, Johnson and Welch eliminated t_ϵ from the equations (1) and (2) by the iterative procedure. Table V greatly reduces the labor in finding t_ϵ since no iterations are necessary.

The present paper provides a set of Tables, derived from the Johnson and Welch Table IV, which facilitate the computation of the percentage points for a wide range of ϵ. These tables are the set labelled Table A. They are similar in form to the Johnson and Welch Table V. The table for $\epsilon = .05$ is essentially Table V of Johnson and Welch except for occasional differences in the last decimal place. These differences are probably due to the fact that Johnson and Welch used more decimal places to begin with than are now recorded in their Table IV.

To find t_ϵ for $f > 9$, in Table A, one interpolates linearly on $12/f^{\frac{1}{2}}$. To find $t_{1-\epsilon}(\delta)$ one finds $-t_\epsilon(-\delta)$.

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After this paper was accepted for publication tables similar to those of Table A, but covering only the values $\epsilon = 0.01, 0.05$ and 0.10, were published as Table 5.1 of [3].

2. The use of Table A. Table A makes it possible to find the point t_ϵ which is exceeded with probability ϵ in the non-central t-distribution, with f degrees of freedom and non-centrality parameter δ, for seventeen different values of ϵ. The computation may be carried out as follows. First compute

$$r = \delta(2f)^{-\frac{1}{2}} \quad \text{and} \quad \eta = r(1 + r^2)^{-\frac{1}{2}}. \tag{3}$$

The table corresponding to the desired ϵ is entered with η and f as arguments to obtain the tabular value λ. Then the percentage point is obtained by computing

$$t_\epsilon = [\delta + \lambda(1 + r^2 - \lambda^2/(2f))^{\frac{1}{2}}]/[1 - \lambda^2/(2f)]. \tag{4}$$

For values of ϵ greater than $\frac{1}{2}$ the relation

$$t_{1-\epsilon}(\delta) = -t_\epsilon(-\delta) \tag{5}$$

may be used.

Example. To find the point having 80% of the probability above it on the non-central t-distribution with $f = 20$ degrees of freedom and non-centrality parameter $\delta = 5$, we calculate the upper 20% point on the non-central t-distribution with $f = 20$ and $\delta = -5$. We find

$$r = .79057, \quad \eta = -.62017, \quad \text{and} \quad 12/f^{\frac{1}{2}} = 2.6833.$$

From Table A, $\epsilon = .20$, we find a grid of four λ values corresponding to the interpolation arguments in η and $12/f^{\frac{1}{2}}$.

η \ $12/f^{\frac{1}{2}}$	3	2
−.7	.4592	.4802
−.6	.4717	.4882

We interpolate linearly on $12/f^{\frac{1}{2}}$ with argument 2.6833 to obtain the pair of λ values corresponding to the interpolation arguments in η.

η	λ
−.7	.4659
−.6	.4769

We now interpolate linearly on η with argument $-.62017$ to obtain $\eta = .4747$. (If interpolation is performed first on η and then on $12/f^{\frac{1}{2}}$ the result, for this example, will be the same to four decimal places). This value of η is now inserted in (4) to obtain $t_{.20} = 4.421$ for $f = 20$ and $\delta = -5$. The desired result is $t_{.80} = -4.421$ for $f = 20$ and $\delta = 5$.

3. On the accuracy of the results obtainable from Table A. There are two sources of error in the percentage points yielded by Table A. One of these is the result of linear interpolation in η and in $12/f^{\frac{1}{2}}$; the other is the effect of small f because of the asymptotic nature of the expansion used to obtain the original values of λ. For very small f these two errors are indistinguishable. However, for $f \geqq 16$ only the error due to interpolation affects the last decimal place, when t_ϵ is rounded to the number of decimal places given for λ. To assess the magnitude of the error due to interpolation many values of $t_\epsilon/f^{\frac{1}{2}}$, for $f \geqq 16$, were computed from Table A, for the values of δ covered in the tables of Resnikoff and Lieberman [2]. These were then compared with corresponding entries in the latter tables. It was found that the effect of linear interpolation in both η and $12/f^{\frac{1}{2}}$ on $t_\epsilon/f^{\frac{1}{2}}$ was slight, resulting in a maximum error of two units in the last decimal place. Actually this occurs only rarely. Most of the comparisons showed no discrepancies in the last decimal place.

The effect of the combination of small f, small ϵ and large positive δ (or large ϵ and large negative δ) is more serious. The values of λ in [1] were obtained by Johnson and Welch by the use of an Edgeworth type series so that the equation (2) yields an asymptotic approximation. The magnitude of the errors, as a function of f, δ, and ϵ, for small f, can be judged from the following short table, Table B. The entries in Table B are $t_\epsilon/f^{\frac{1}{2}}$. As noted in the preceding paragraph, the entries labelled "Table A" may have, though rarely, an error of one or two

TABLE B

f	ϵ = .005		.01		.05		.10	
	Table A	Correct Value	Table A	Correct Value	Table A	Correct Value	Table A	Correct Value
				$\delta = 3.0902\sqrt{f+1}$				
4	15.73	15.60	13.11	13.02	8.393	8.386	6.835	6.833
5	12.24	12.15	10.48	10.45	7.245	7.243	6.087	6.086
6	10.27	10.25	9.02	9.02	6.550	6.547	5.618	5.618
7	9.08	9.05	8.11	8.09	6.081	6.079	5.297	5.297
8	8.21	8.25	7.44	7.44	5.743	5.742	5.060	5.061
9	7.651	7.650	6.965	6.960	5.485	5.484	4.879	4.879
				$\delta = .6745\sqrt{f+1}$				
4	4.65	4.67	3.86	3.86	2.405	2.403	1.898	1.898
5	3.66	3.65	3.11	3.12	2.076	2.076	1.687	1.687
6	3.09	3.09	2.69	2.69	1.871	1.871	1.550	1.550
7	2.73	2.73	2.41	2.41	1.729	1.729	1.454	1.454
8	2.48	2.48	2.21	2.21	1.625	1.625	1.381	1.381
9	2.294	2.293	2.060	2.060	1.544	1.544	1.324	1.325

units in the last decimal place as a consequence of the linear interpolation on η, in addition to that due to the asymptotic nature of the expansion from which the λ values were derived. From Table B it can be seen that, except for $f < 9$, large positive δ, and $\epsilon = .005$ and $\epsilon = .01$, (or large negative δ and $\epsilon = .995$ and $\epsilon = .99$) Table A yields excellent results. The entries labelled "correct" were obtained from [2].

4.

TABLE A

Computational formulas needed to apply the tables

$$r = \delta/\sqrt{2f} \qquad \eta = r/\sqrt{1 + r^2}$$

$$t_\epsilon = \frac{\delta + \lambda\sqrt{1 + r^2 - \lambda^2/(2f)}}{1 - \lambda^2/(2f)} \qquad t_{1-\epsilon}(\delta) = -t_\epsilon(-\delta)$$

REFERENCES

[1] JOHNSON, N. L. and WELCH, B. L. (1940). Applications of the non-central t-distrubution. *Biometrika*. **31** 362–89.

[2] RESNIKOFF, G. J. and LIEBERMAN, G. J. (1957). *Tables of the Non-Central t-Distribution*. Stanford Univ. Press.

[3] OWEN, D. B. (1962). *Handbook of Statistical Tables*. Addison-Wesley, Reading, Mass.

TABLE A

Values of λ as a function of δ

$\epsilon = .005$

$12/\sqrt{f}$						4	3	2	1	0
η \ f	4	5	6	7	8	9	16	36	144	∞
-1.0	2.62	2.62	2.63	2.63	2.63	2.626	2.622	2.613	2.598	2.576
-.9	2.52	2.54	2.54	2.55	2.56	2.560	2.574	2.583	2.584	2.576
-.8	2.52	2.52	2.53	2.53	2.53	2.538	2.553	2.566	2.575	2.576
-.7	2.57	2.55	2.55	2.54	2.54	2.542	2.548	2.559	2.570	2.576
-.6	2.60	2.59	2.58	2.57	2.56	2.560	2.554	2.558	2.568	2.576
-.5	2.61	2.61	2.60	2.59	2.59	2.579	2.565	2.562	2.567	2.576
-.4	2.59	2.61	2.61	2.60	2.60	2.594	2.578	2.568	2.569	2.576
-.3	2.54	2.58	2.60	2.60	2.60	2.601	2.587	2.575	2.571	2.576
-.2	2.50	2.55	2.58	2.60	2.60	2.600	2.594	2.581	2.574	2.576
-.1	2.45	2.52	2.56	2.58	2.59	2.592	2.597	2.586	2.578	2.576
.0	2.41	2.49	2.53	2.55	2.57	2.580	2.595	2.590	2.580	2.576
.1	2.37	2.45	2.50	2.53	2.55	2.563	2.589	2.591	2.582	2.576
.2	2.34	2.42	2.47	2.50	2.53	2.544	2.581	2.589	2.583	2.576
.3	2.31	2.39	2.45	2.48	2.50	2.523	2.570	2.585	2.584	2.576
.4	2.28	2.37	2.42	2.46	2.48	2.503	2.557	2.579	2.583	2.576
.5	2.26	2.35	2.40	2.43	2.46	2.482	2.542	2.572	2.581	2.576
.6	2.24	2.33	2.38	2.41	2.44	2.461	2.525	2.562	2.576	2.576
.7	2.23	2.31	2.36	2.39	2.42	2.440	2.507	2.551	2.571	2.576
.8	2.22	2.29	2.34	2.38	2.40	2.420	2.489	2.537	2.564	2.576
.9	2.20	2.28	2.32	2.35	2.38	2.399	2.470	2.522	2.556	2.576
1.0	2.19	2.26	2.30	2.34	2.36	2.380	2.450	2.504	2.545	2.576

$\epsilon = .01$

$12/\sqrt{f}$						4	3	2	1	0
η \ f	4	5	6	7	8	9	16	36	144	∞
-1.0	2.32	2.33	2.33	2.34	2.34	2.340	2.343	2.342	2.337	2.326
-.9	2.26	2.27	2.28	2.28	2.30	2.295	2.311	2.322	2.327	2.326
-.8	2.26	2.26	2.27	2.28	2.28	2.282	2.296	2.310	2.321	2.326
-.7	2.30	2.29	2.29	2.29	2.29	2.288	2.295	2.306	2.318	2.326
-.6	2.34	2.33	2.31	2.31	2.30	2.304	2.302	2.307	2.317	2.326
-.5	2.36	2.35	2.34	2.33	2.33	2.323	2.313	2.311	2.317	2.326
-.4	2.36	2.36	2.36	2.35	2.35	2.339	2.324	2.317	2.319	2.326
-.3	2.35	2.36	2.36	2.36	2.35	2.350	2.335	2.324	2.322	2.326
-.2	2.32	2.35	2.36	2.36	2.36	2.355	2.343	2.330	2.324	2.326
-.1	2.29	2.33	2.34	2.35	2.35	2.354	2.348	2.336	2.327	2.326
.0	2.26	2.30	2.33	2.34	2.34	2.348	2.350	2.340	2.330	2.326
.1	2.23	2.28	2.31	2.32	2.33	2.339	2.349	2.342	2.333	2.326
.2	2.20	2.25	2.29	2.31	2.32	2.328	2.344	2.343	2.334	2.326
.3	2.17	2.23	2.27	2.29	2.31	2.314	2.337	2.342	2.335	2.326
.4	2.15	2.21	2.25	2.27	2.29	2.298	2.329	2.339	2.335	2.326
.5	2.13	2.19	2.23	2.25	2.27	2.282	2.318	2.334	2.334	2.326
.6	2.11	2.17	2.21	2.23	2.25	2.266	2.306	2.327	2.331	2.326
.7	2.10	2.15	2.19	2.22	2.24	2.249	2.293	2.318	2.328	2.326
.8	2.09	2.14	2.18	2.20	2.22	2.233	2.279	2.308	2.323	2.326
.9	2.08	2.13	2.16	2.19	2.20	2.216	2.264	2.296	2.316	2.326
1.0	2.06	2.11	2.15	2.17	2.19	2.199	2.247	2.283	2.309	2.326

$\epsilon = .025$

$12/\sqrt{f}$						4	3	2	1	0
η \ f	4	5	6	7	8	9	16	36	144	∞
-1.0	1.892	1.904	1.912	1.917	1.922	1.925	1.939	1.950	1.956	1.960
-.9	1.869	1.881	1.890	1.897	1.903	1.907	1.924	1.939	1.951	1.960
-.8	1.882	1.888	1.892	1.897	1.901	1.905	1.920	1.935	1.948	1.960
-.7	1.913	1.910	1.911	1.911	1.912	1.914	1.923	1.935	1.948	1.960
-.6	1.949	1.939	1.934	1.931	1.930	1.930	1.931	1.938	1.948	1.960
-.5	1.980	1.967	1.958	1.953	1.949	1.946	1.941	1.942	1.950	1.960
-.4	1.999	1.987	1.978	1.972	1.967	1.963	1.952	1.948	1.952	1.960
-.3	2.006	1.999	1.992	1.986	1.981	1.977	1.963	1.955	1.954	1.960
-.2	2.003	2.004	1.999	1.994	1.991	1.987	1.972	1.962	1.957	1.960
-.1	1.994	2.001	2.001	1.999	1.996	1.993	1.979	1.968	1.960	1.960
.0	1.981	1.994	1.998	1.999	1.998	1.995	1.985	1.972	1.963	1.960
.1	1.965	1.984	1.991	1.995	1.996	1.995	1.988	1.976	1.966	1.960
.2	1.950	1.972	1.983	1.988	1.990	1.992	1.989	1.979	1.968	1.960
.3	1.933	1.959	1.972	1.980	1.984	1.987	1.988	1.981	1.970	1.960
.4	1.918	1.946	1.961	1.970	1.976	1.979	1.985	1.980	1.971	1.960
.5	1.903	1.933	1.949	1.960	1.967	1.971	1.981	1.979	1.971	1.960
.6	1.890	1.920	1.937	1.948	1.956	1.962	1.975	1.977	1.970	1.960
.7	1.877	1.908	1.925	1.937	1.946	1.952	1.968	1.972	1.969	1.960
.8	1.866	1.896	1.913	1.926	1.935	1.941	1.959	1.967	1.967	1.960
.9	1.856	1.884	1.902	1.915	1.924	1.930	1.950	1.961	1.964	1.960
1.0	1.844	1.872	1.890	1.903	1.912	1.919	1.940	1.953	1.959	1.960

$\epsilon = .05$

$12/\sqrt{f}$					4	3	2	1	0
η \ f	4	5	6	7	8	9	16	36	144	∞
-1.0	1.528	1.543	1.554	1.563	1.569	1.5744	1.5952	1.6141	1.6307	1.6449
-.9	1.531	1.544	1.553	1.561	1.567	1.5722	1.5925	1.6116	1.6292	1.6449
-.8	1.551	1.559	1.565	1.571	1.575	1.5792	1.5954	1.6122	1.6290	1.6449
-.7	1.582	1.582	1.585	1.587	1.589	1.5911	1.6018	1.6150	1.6299	1.6449
-.6	1.615	1.609	1.608	1.606	1.607	1.6065	1.6109	1.6196	1.6313	1.6449
-.5	1.646	1.636	1.629	1.627	1.624	1.6230	1.6213	1.6251	1.6334	1.6449
-.4	1.670	1.658	1.650	1.645	1.642	1.6391	1.6322	1.6313	1.6359	1.6449
-.3	1.688	1.676	1.667	1.662	1.657	1.6534	1.6428	1.6379	1.6387	1.6449
-.2	1.698	1.688	1.680	1.674	1.670	1.6657	1.6524	1.6443	1.6417	1.6449
-.1	1.702	1.695	1.690	1.683	1.679	1.6755	1.6610	1.6503	1.6448	1.6449
.0	1.703	1.699	1.694	1.690	1.686	1.6828	1.6682	1.6558	1.6477	1.6449
.1	1.699	1.699	1.696	1.693	1.690	1.6874	1.6738	1.6606	1.6505	1.6449
.2	1.693	1.696	1.696	1.695	1.693	1.6898	1.6778	1.6646	1.6529	1.6449
.3	1.687	1.692	1.694	1.693	1.692	1.6903	1.6804	1.6676	1.6550	1.6449
.4	1.679	1.687	1.690	1.690	1.691	1.6896	1.6816	1.6698	1.6567	1.6449
.5	1.672	1.681	1.685	1.687	1.687	1.6871	1.6817	1.6709	1.6580	1.6449
.6	1.664	1.674	1.680	1.682	1.683	1.6838	1.6804	1.6714	1.6587	1.6449
.7	1.657	1.668	1.674	1.677	1.679	1.6796	1.6782	1.6707	1.6589	1.6449
.8	1.650	1.661	1.668	1.672	1.674	1.6747	1.6750	1.6691	1.6586	1.6449
.9	1.643	1.655	1.662	1.666	1.669	1.6695	1.6711	1.6667	1.6576	1.6449
1.0	1.636	1.648	1.655	1.660	1.662	1.6638	1.6665	1.6634	1.6559	1.6449

$\epsilon = .10$

$12/\sqrt{f}$					4	3	2	1	0
η \ f	4	5	6	7	8	9	16	36	144	∞
-1.0	1.528	1.543	1.554	1.563	1.569	1.5744	1.5952	1.6141	1.6307	1.6449
-1.0	1.116	1.136	1.150	1.161	1.169	1.1765	1.2049	1.2319	1.2575	1.2816
-.9	1.139	1.155	1.166	1.175	1.182	1.1880	1.2121	1.2359	1.2591	1.2816
-.8	1.169	1.179	1.187	1.193	1.198	1.2024	1.2209	1.2409	1.2612	1.2816
-.7	1.200	1.204	1.208	1.212	1.215	1.2179	1.2311	1.2466	1.2636	1.2816
-.6	1.231	1.230	1.231	1.231	1.233	1.2341	1.2418	1.2526	1.2661	1.2816
-.5	1.259	1.254	1.252	1.251	1.250	1.2503	1.2527	1.2591	1.2688	1.2816
-.4	1.285	1.277	1.272	1.269	1.267	1.2657	1.2634	1.2656	1.2718	1.2816
-.3	1.306	1.297	1.290	1.286	1.283	1.2801	1.2738	1.2720	1.2748	1.2816
-.2	1.323	1.312	1.305	1.299	1.296	1.2931	1.2836	1.2784	1.2778	1.2816
-.1	1.337	1.326	1.318	1.312	1.308	1.3047	1.2926	1.2846	1.2809	1.2816
.0	1.348	1.337	1.329	1.323	1.319	1.3149	1.3009	1.2903	1.2838	1.2816
.1	1.355	1.346	1.338	1.332	1.328	1.3236	1.3083	1.2957	1.2866	1.2816
.2	1.360	1.352	1.346	1.339	1.335	1.3309	1.3149	1.3007	1.2893	1.2816
.3	1.364	1.357	1.351	1.345	1.341	1.3368	1.3207	1.3052	1.2919	1.2816
.4	1.366	1.360	1.354	1.350	1.345	1.3413	1.3256	1.3093	1.2943	1.2816
.5	1.367	1.362	1.357	1.353	1.349	1.3455	1.3294	1.3128	1.2964	1.2816
.6	1.368	1.364	1.360	1.355	1.352	1.3484	1.3329	1.3158	1.2983	1.2816
.7	1.369	1.365	1.361	1.358	1.354	1.3507	1.3357	1.3184	1.3001	1.2816
.8	1.369	1.366	1.362	1.359	1.356	1.3526	1.3380	1.3206	1.3015	1.2816
.9	1.370	1.367	1.363	1.360	1.357	1.3540	1.3397	1.3224	1.3028	1.2816
1.0	1.370	1.368	1.364	1.361	1.358	1.3554	1.3413	1.3238	1.3038	1.2816

$\epsilon = .20$

$12/\sqrt{f}$					4	3	2	1	0
η \ f	4	5	6	7	8	9	16	36	144	∞
-1.0	.632	.656	.673	.686	.697	.7055	.7408	.7753	.8089	.8416
-.9	.668	.686	.700	.711	.719	.7261	.7552	.7842	.8130	.8416
-.8	.702	.716	.726	.735	.741	.7463	.7692	.7928	.8169	.8416
-.7	.735	.744	.750	.757	.761	.7649	.7824	.8009	.8207	.8416
-.6	.765	.770	.773	.777	.780	.7824	.7944	.8083	.8241	.8416
-.5	.793	.793	.794	.796	.797	.7987	.8059	.8156	.8275	.8416
-.4	.816	.814	.814	.814	.813	.8138	.8168	.8225	.8308	.8416
-.3	.839	.834	.831	.829	.828	.8278	.8270	.8291	.8340	.8416
-.2	.858	.852	.848	.845	.843	.8409	.8367	.8354	.8371	.8416
-.1	.877	.868	.862	.859	.856	.8532	.8459	.8415	.8401	.8416
.0	.892	.883	.876	.871	.868	.8648	.8547	.8475	.8431	.8416
.1	.908	.897	.890	.884	.880	.8759	.8632	.8532	.8460	.8416
.2	.921	.910	.902	.895	.890	.8864	.8715	.8589	.8489	.8416
.3	.935	.922	.913	.906	.901	.8964	.8795	.8645	.8519	.8416
.4	.946	.933	.923	.916	.911	.9062	.8873	.8701	.8548	.8416
.5	.958	.945	.934	.926	.920	.9152	.8948	.8756	.8577	.8416
.6	.969	.955	.945	.936	.930	.9247	.9024	.8809	.8605	.8416
.7	.980	.966	.955	.946	.940	.9341	.9101	.8865	.8636	.8416
.8	.990	.976	.965	.956	.949	.9433	.9179	.8923	.8668	.8416
.9	1.001	.986	.975	.966	.959	.9525	.9258	.8983	.8700	.8416
1.0	1.012	.997	.986	.977	.969	.9623	.9341	.9044	.8734	.8416

ε = .30

12/√f					4	3	2	1	0
η \ f	**4**	**5**	**6**	**7**	**8**	**9**	**16**	**36**	**144**	**∞**

η \ f	4	5	6	7	8	9	16	36	144	∞
12/√f						4	3	2	1	0
-1.0	.295	.320	.339	.353	.365	.3739	.4124	.4503	.4876	.5244
-.9	.332	.353	.368	.380	.389	.3973	.4293	.4611	.4928	.5244
-.8	.368	.384	.396	.406	.413	.4192	.4449	.4710	.4975	.5244
-.7	.401	.413	.422	.428	.434	.4391	.4592	.4802	.5019	.5244
-.6	.429	.437	.443	.449	.453	.4565	.4717	.4882	.5057	.5244
-.5	.456	.460	.465	.467	.471	.4729	.4837	.4959	.5095	.5244
-.4	.480	.481	.483	.485	.487	.4879	.4946	.5030	.5129	.5244
-.3	.501	.501	.501	.501	.501	.5018	.5048	.5097	.5162	.5244
-.2	.521	.518	.516	.515	.515	.5148	.5145	.5160	.5193	.5244
-.1	.539	.535	.531	.529	.528	.5272	.5237	.5221	.5223	.5244
.0	.558	.551	.546	.543	.541	.5391	.5326	.5281	.5253	.5244
.1	.575	.566	.561	.556	.553	.5508	.5414	.5340	.5282	.5244
.2	.591	.582	.575	.570	.565	.5624	.5502	.5398	.5312	.5244
.3	.608	.597	.589	.582	.578	.5741	.5591	.5459	.5343	.5244
.4	.625	.612	.603	.596	.590	.5859	.5682	.5521	.5375	.5244
.5	.642	.627	.617	.610	.603	.5979	.5777	.5585	.5407	.5244
.6	.659	.644	.632	.624	.616	.6103	.5870	.5649	.5441	.5244
.7	.677	.660	.648	.637	.630	.6234	.5974	.5721	.5479	.5244
.8	.695	.677	.663	.652	.644	.6371	.6082	.5797	.5518	.5244
.9	.713	.694	.680	.668	.659	.6515	.6198	.5879	.5560	.5244
1.0	.733	.712	.697	.685	.675	.6668	.6320	.5966	.5607	.5244

ε = .40

η \ f	4	5	6	7	8	9	16	36	144	∞
12/√f						4	3	2	1	0
-1.0	.016	.041	.060	.075	.087	.0964	.1362	.1755	.2146	.2533
-.9	.051	.073	.089	.102	.112	.1200	.1536	.1870	.2203	.2533
-.8	.085	.102	.116	.126	.135	.1413	.1693	.1973	.2252	.2533
-.7	.116	.130	.141	.149	.156	.1612	.1839	.2067	.2299	.2533
-.6	.144	.155	.163	.169	.174	.1784	.1966	.2151	.2339	.2533
-.5	.169	.177	.183	.187	.191	.1946	.2085	.2229	.2378	.2533
-.4	.192	.197	.201	.204	.207	.2095	.2195	.2300	.2413	.2533
-.3	.213	.216	.218	.220	.222	.2232	.2296	.2368	.2446	.2533
-.2	.233	.233	.234	.235	.235	.2361	.2392	.2431	.2478	.2533
-.1	.251	.250	.249	.249	.248	.2484	.2483	.2491	.2508	.2533
.0	.269	.266	.264	.262	.261	.2604	.2573	.2551	.2537	.2533
.1	.287	.282	.279	.276	.274	.2724	.2662	.2610	.2567	.2533
.2	.305	.298	.293	.290	.287	.2844	.2753	.2671	.2598	.2533
.3	.323	.315	.309	.304	.300	.2969	.2846	.2733	.2629	.2533
.4	.343	.333	.324	.319	.314	.3098	.2944	.2798	.2662	.2533
.5	.364	.350	.341	.334	.328	.3234	.3046	.2867	.2696	.2533
.6	.384	.369	.358	.350	.343	.3373	.3152	.2939	.2733	.2533
.7	.407	.390	.377	.367	.359	.3530	.3272	.3020	.2773	.2533
.8	.430	.411	.397	.386	.377	.3696	.3398	.3105	.2818	.2533
.9	.456	.434	.418	.406	.396	.3877	.3537	.3200	.2865	.2533
1.0	.482	.458	.441	.427	.416	.4067	.3686	.3303	.2919	.2533

ε = .50

η \ f	4	5	6	7	8	9	16	36	144	∞
12/√f						4	3	2	1	0
-1.0	-.237	-.212	.194	-.179	-.167	-.1578	-.1182	-.0787	-.0393	.0000
-.9	-.206	-.183	.167	-.154	-.144	-.1359	-.1015	-.0674	-.0336	.0000
-.8	-.175	-.157	.142	-.132	-.123	-.1158	-.0864	-.0574	-.0286	.0000
-.7	-.148	-.132	.120	-.111	-.104	-.0974	-.0725	-.0481	-.0240	.0000
-.6	-.122	-.108	.098	-.092	-.085	-.0802	-.0598	-.0397	-.0198	.0000
-.5	-.099	-.088	.080	-.074	-.069	-.0648	-.0483	-.0320	-.0159	.0000
-.4	-.077	-.068	.063	-.057	-.054	-.0504	-.0376	-.0250	-.0124	.0000
-.3	-.056	-.050	.046	-.042	-.039	-.0369	-.0275	-.0182	-.0091	.0000
-.2	-.036	-.033	.030	-.028	-.026	-.0242	-.0181	-.0120	-.0060	.0000
-.1	-.018	-.016	.015	-.013	-.012	-.0119	-.0089	-.0059	-.0030	.0000
.0	.000	.000	.000	.000	.000	.0000	.0000	.0000	.0000	.0000
.1	.018	.016	.015	.013	.012	.0119	.0089	.0059	.0030	.0000
.2	.036	.033	.030	.028	.026	.0242	.0181	.0120	.0060	.0000
.3	.056	.050	.046	.042	.039	.0369	.0275	.0182	.0091	.0000
.4	.077	.068	.063	.057	.054	.0504	.0376	.0250	.0124	.0000
.5	.099	.088	.080	.074	.069	.0648	.0483	.0320	.0159	.0000
.6	.122	.108	.098	.092	.085	.0802	.0598	.0397	.0198	.0000
.7	.148	.132	.120	.111	.104	.0974	.0725	.0481	.0240	.0000
.8	.175	.157	.142	.132	.123	.1158	.0864	.0574	.0286	.0000
.9	.206	.183	.167	.154	.144	.1359	.1015	.0674	.0336	.0000
1.0	.237	.212	.194	.179	.167	.1578	.1182	.0787	.0393	.0000