SOME MODIFIED KOLMOGOROV-SMIRNOV TESTS OF APPROXIMATE HYPOTHESES AND THEIR PROPERTIES[1]

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1. Summary and introduction. The paradox of almost certain rejection of the null hypothesis in the Chi Square test-of-fit, when many observations are used, has been pointed out by Cochran [4], and largely removed by Lehmann and Hodges [7]. The same paradox arises in most tests of goodness-of-fit. In this paper the Kolmogorov-Smirnov tests are modified to remove this difficulty and some properties of this modification are investigated. In particular, a rigorous method for choosing sample size (Theorem 3.2 and corollaries) is presented.

Given independent random variables $X_1, \cdots, X_n$ with common distribution function F, suppose that we desire to determine whether or not F is in some class $\mathcal{H}_o$. If we are only interested in whether F is close, in some sense, to some distribution function in $\mathcal{H}_o$, we can let $\mathcal{H}_o^* \supset \mathcal{H}_o$, where $\mathcal{H}_o^*$ is the class of distribution functions "close" to those in $\mathcal{H}_o$, and test the more reasonable hypothesis that $F \varepsilon \mathcal{H}_o^*$.

In what follows we consider tests based on the uniform metric d_1, given by $d_1(F, H) = \sup_x |F(x) - H(x)|$, where $\mathcal{H}_o$ consists of a single distribution function F_o.

2. The proposed tests and asymptotic probability of type I error. Let F_o be a distribution function, and let H_1 and H_2 be monotone functions satisfying $H_1 \leqq F_o \leqq H_2$. We define the null hypothesis $\mathcal{H}_o^*$ by

$$\mathcal{H}_o^* = \{G{:}H_1(x) \leqq G(x) \leqq H_2(x), \text{ all } x, G \text{ a distribution function}\}. \tag{2.1}$$

For each vector $(x_1, \cdots, x_n)$ of possible observed values, the empirical distribution function F_n is defined by

$$F_n(x) = \text{proportion of } \quad x_1, \cdots, x_n \quad \text{not exceeding} \quad x. \tag{2.2}$$

With any distribution function G, we associate the distribution function G^*, given by

$$G^*(x) = \begin{cases} H_1(x) & \text{for} \quad G(x) < H_1(x), \\ G(x) & \text{for} \quad H_1(x) \leqq G(x) \leqq H_2(x), \\ H_2(x) & \text{for} \quad G(x) > H_2(x). \end{cases} \tag{2.3}$$

Received December 10, 1959; revised November 13, 1961.

[1] Research supported in part by ONR and in part under the auspices of the United States Atomic Energy Commission. This work was begun while the author was at Purdue University.

It is easily seen that

$$d_1(G, G^*) = \inf_{H \varepsilon \mathcal{H}_o^*} d_1(G, H), \tag{2.4}$$

where d_1 is the uniform metric. For $X_1, \cdots, X_n$ independent random variables with common distribution function F, let

$$T_1(z) = \inf_{F \text{ a d.f.}} \lim_{n\to\infty} P_F\{n^{\frac{1}{2}} d_1(F_n, F) < z\}. \tag{2.5}$$

It can be shown that $T_1(z) = \lim_{n\to\infty} P_F\{n^{\frac{1}{2}} d_1(F_n, F) < z\}$ for any continuous F; see e.g., Anderson and Darling [1]. Finally, let

$$T_1(h_{1,\alpha}) = 1 - \alpha, 0 < \alpha < 1. \tag{2.6}$$

(See [1] for formula for T_1.)

The proposed test, for sample size n, has as rejection region, rej $\mathcal{H}_o^*$, the set for which

$$n^{\frac{1}{2}} d_1(F_n, F_n^*) \geqq h_{1,\alpha}. \tag{2.7}$$

A simple consequence of (2.4) is the following:

THEOREM 2.1. *When* $F \varepsilon \mathcal{H}_o^*$, $\lim\sup_{n\to\infty} P_F\{\text{rej } \mathcal{H}_o^*\} \leqq \alpha$. On the basis of numerical computations, see Birnbaum [2], it has been conjectured that $P_F\{\text{rej } \mathcal{H}_o^*\} \leqq \alpha$ when $F \varepsilon \mathcal{H}_o^*$, no matter what the value of n is. From Birnbaum's work, [3], it can be shown that

$$\lim_{n\to\infty} \sup_{F \varepsilon \mathcal{H}_o^*} P_F\{\text{rej } \mathcal{H}_o^*\} \geqq \alpha/2. \tag{2.8}$$

This result demonstrates that, with regard to the probability of type I error, the proposed tests are not overly conservative.

3. The choice of sample size. Let $[\![r]\!]$ denote the greatest integer contained in r. The following theorem is well known; see e.g., [8].

THEOREM 3.1. *If the sample size n for the test given by* (2.6) *satisfies*

$$\inf_{p \varepsilon (0,1-l]} \sum_{\nu=0}^{[\![n(p+l) - n^{\frac{1}{2}} h_{1,\alpha}]\!]} \binom{n}{\nu} p^{\nu}(1-p)^{n-\nu} \geqq 1 - \beta, \tag{3.1}$$

then

$$P_F\{\text{rej } \mathcal{H}_o^*\} \geqq 1 - \beta \textit{ when } \inf_{H \varepsilon \mathcal{H}_o^*} d_1(F H) \geqq l > 0. \tag{3.2}$$

It can easily be seen from the Chebychev inequality that there is an n satisfying (3.1) for each $l \varepsilon (0, 1)$. However, use of Chebychev's inequality to obtain such an n would give a much larger value than is needed.

Thus, we are led to seek a workable technique for determining n satisfying (3.1) but much smaller than that n obtainable from Chebychev's inequality.

One technique which is quick and not too inefficient is based on a theorem of Okamoto [9], stating that if X is a random variable with the binomial distribution, based on sample size n, probability p, then

$$P\{(X/n) - p \geqq |c|\} \leqq \exp[-2nc^2], \quad P\{(X/n) - p \leqq -|c|\} \leqq \exp[-2nc^2]. \tag{3.3}$$

From this we see that if

$$n = [\![\{[(-\log_e \beta)/2]^{\frac{1}{2}} + h_{1,\alpha}\}^2/l^2 + 1]\!], \tag{3.4}$$

then (3.1) will be satisfied.

A simple technique for choosing n which often achieves a considerable reduction in sample size over that given by (3.4), and which satisfies (3.1) will now be given. After an outline of the main theorem, a table will be given comparing these results to those of (3.4), and also to those given by the usual intuitive procedure which is obtained under the invalid assumption that the normal approximation to the distribution function of

$$(n^{\frac{1}{2}}[F_n(x) - F(x)])/(F(x)[1 - F(x)])^{\frac{1}{2}}$$

is exact. The procedure to be given results from a rigorous modification of this intuitive procedure.

Now let Φ be the standard normal distribution function and define ϕ_λ by

$$\Phi(\phi_\lambda) = 1 - \lambda, \qquad \lambda \varepsilon (0, 1). \tag{3.5}$$

The sample size

$$n_e = [\![(h_{1,\alpha} + \phi_\beta/2)^2/l^2 + 1]\!] \tag{3.6}$$

specified by the usual intuitive procedure not using a continuity correction will be seen to be smaller than that demanded by the rigorous procedure to be outlined. Let

$$p^* = \frac{\left(\phi_\beta + \frac{\phi_\beta}{4n_e} - \frac{1}{2\sqrt{n_e}}\right) + \left[\left(\phi_\beta + \frac{\phi_\beta}{4n_e} - \frac{1}{2\sqrt{n_e}}\right)^2 - \frac{\phi_\beta}{n_e}\left(2\phi_\beta + \frac{2\phi_\beta l}{1 - 2l} - \frac{1}{2\sqrt{n_e}}\right)\right]^{\frac{1}{2}}}{2[2\phi_\beta + 2\phi_\beta\, l/(1 - 2l) - \frac{1}{2}n_e^{-\frac{1}{2}}]} \tag{3.7}$$

and

$$p^{**} = \phi_\beta/(2\phi_\beta - 2n_e^{-\frac{1}{2}}). \tag{3.8}$$

Let

(3.9) p' be whichever of the two numbers p^*, p^{**} is further from $\frac{1}{2}$. Let

$$p(n) = \frac{\left(\phi_\beta + \frac{\phi_\beta}{4n_e} - \frac{1}{2\sqrt{n_e}}\right) + \left[\left(\phi_\beta + \frac{\phi_\beta}{4n_e} - \frac{1}{2\sqrt{n_e}}\right)^2 - \frac{\phi_\beta}{n_e}\left(2\phi_\beta + \frac{2\phi_\beta\left[l - \frac{h_{1,\alpha}}{\sqrt{n}}\right]}{1 - 2\left[l - \frac{h_{1,\alpha}}{\sqrt{n}}\right]} - \frac{1}{2\sqrt{n_e}}\right)\right]^{\frac{1}{2}}}{2[2\phi_\beta + 2\phi_\beta(l - h_{1,\alpha} n^{-\frac{1}{2}})/(1 - 2\{l - h_{1,\alpha} n^{-\frac{1}{2}}\}) - (\frac{1}{2})n_e^{-\frac{1}{2}}]}, \tag{3.10}$$

$$p'(n) = \tfrac{1}{2} + [\sup \tfrac{1}{2} - p(n), p^{**} - \tfrac{1}{2}] \operatorname{sgn} [p(n) + p^{**} - 1], \tag{3.11}$$

i.e., $p'(n)$ is whichever of $p(n)$, p^{**} is further from $\frac{1}{2}$, and

$$\begin{aligned} &\eta(n, p) \\ &= |1 - 2p|/13.4(2\pi np[1 - p])^{\frac{1}{2}} + (.073 + .09|1 - 2p|)/np(1 - p). \end{aligned} \tag{3.12}$$

THEOREM 3.2. *Given* $\alpha \leqq .1$, $\beta \leqq .1$, $l \leqq .2$, *let* n_1 *be the smallest integer* n *for which*

$$2(n^{\frac{1}{2}}l - h_{1,\alpha}) - n^{-\frac{1}{2}} \geqq \phi_{\beta - \eta(n,p')} . \tag{3.13}$$

Recursively define n_κ, $\kappa > 1$ *to be the smallest integer* n *for which*

$$2(n^{\frac{1}{2}}l - h_{1,\alpha}) - n^{-\frac{1}{2}} \geqq \phi_{\beta - \eta[n, p'(n_{\kappa-1})]} . \tag{3.14}$$

Then if for some κ, $n \geqq$ max $(109, n_\kappa)$ *and*

$$1/([\![nl - n^{\frac{1}{2}}h_{1,\alpha}]\!]!) \leqq \beta, \tag{3.15}$$

then n *satisfies* (3.2).

Further, if we let

$$\begin{aligned} \chi_\kappa &= [n_\kappa^{\frac{1}{2}}l - h_{1,\alpha} - 1/(2n_\kappa^{\frac{1}{2}})]/\{p'(n_{\kappa-1})[1 - p'(n_{\kappa-1})]\}^{\frac{1}{2}}, \quad \kappa = 2, 3, \cdots, \\ \chi_1 &= [n_1^{\frac{1}{2}}l - h_{1,\alpha} - 1/(2n_1^{\frac{1}{2}})]/[p'(1 - p')]^{\frac{1}{2}}, \end{aligned} \tag{3.16}$$

then

$$\inf_{p \varepsilon (0, 1-l]} \sum_{\nu=0}^{[\![n_\kappa(p+l) - n_\kappa^{\frac{1}{2}}h_{1,\alpha}]\!]} \binom{n_\kappa}{\nu} p^\nu (1 - p)^{n-\nu} \leqq \begin{cases} \Phi(\chi_\kappa) + \eta(n_\kappa, p'[n_{\kappa-1}]), & \kappa = 2, 3, \cdots, \\ \Phi(\chi_1) + \eta(n_1, p'), & \kappa = 1. \end{cases}$$

For the purposes of application the following corollary is more useful than Theorem 3.2.

COROLLARY 1. *The smallest integer* $n \geqq 109$, (*satisfying* (3.15)) *and*

$$2(n^{\frac{1}{2}}l - h_{1,\alpha}) - n^{-\frac{1}{2}} \geqq \phi_{\beta - \eta(n_e, p')} , \tag{3.17}$$

satisfies (3.2), *where* p' *is defined by* (3.9), n_e *by* (3.6) *and* η *by* (3.12). This corollary follows easily from Theorem 3.2, since η is a monotone decreasing function of the first variable for fixed p'. Thus $\phi_{\beta-\eta(n,p')}$ increases as n decreases, and since n_e is no larger than any n satisfying Theorem 3.2, any n satisfying (3.18) satisfies (3.13). The sample sizes in Table 1 were computed using this corollary. It may be that $\beta - \eta(n_e, p') \leqq 0$, in which case this corollary is useless. The following corollary, proved in the same way, remedies this difficulty.

COROLLARY 2. *Choose* n^* *so that* $\beta - \eta(n^*, p') > 0$. *Then any* $n \geqq \max(n^*, 109)$ *satisfying* (3.15) *and*

$$2(n^{\frac{1}{2}}l - h_{1,\alpha}) - n^{-\frac{1}{2}} \geqq \phi_{\beta-\eta(n^*,p')} \tag{3.18}$$

satisfies (3.2).

We now outline a proof of Theorem 3.2. This outline will be broken up into several parts. For fixed n, let

$$A(p) = \sum_{\nu=0}^{[\![n(p+l)-n^{\frac{1}{2}}h_{1,\alpha}]\!]} \binom{n}{\nu} p^{\nu}(1-p)^{n-\nu}, \tag{3.19}$$

and p_o represent any value of p for which $n(p + l) - n^{\frac{1}{2}}h_{1,\alpha}$ is an integer. Let

$$c_n = n(p_o + l) - n^{\frac{1}{2}}h_{1,\alpha}. \tag{3.20}$$

LEMMA 1. *The function* A *is decreasing on* $(p_o, p_o + 1/n)$, *for* n *satisfying Theorem* 3.2.

PROOF. For $p \in (p_o, p_o + 1/n)$

$$A'(p) = -n\binom{n-1}{c_n}p^{c_n}(1-p)^{n-c_n-1} < 0. \tag{3.21}$$

Thus in order to find where A achieves its infimum, we need only examine A at points of the type p_o-. We note that

$$A(p_0 + 1/n-) - A(p_o-) = \binom{n}{c_n} p_o^{c_n}(1-p_o)^{n-c_n} - n\binom{n-1}{c_n}\int_{p_o}^{p_o+1/n} p^{c_n}(1-p)^{n-c_n-1}\,dp.$$

The sign of

$$C(p_o) = (p_o^{c_n}[1-p_o]^{n-c_n})/(n-c_n) - \int_{p_o}^{p_o+1/n} p^{c_n}(1-p)^{n-c_n-1}\,dp \tag{3.22}$$

is the same as that of $A(p_o + 1/n-) - A(p_o-)$. Using the fact that n is of the form

$$n = [(1/l)(h_{1,\alpha} + \phi_{\beta-\delta}/2)]^2 > n_e, \text{ for some } \delta > 0, \tag{3.23}$$

it can be seen that for n satisfying Theorem 3.2 the integrand in (3.22) is convex and increasing in the region of integration. Thus

$$C(p_o) \leqq (1/[n - c_n])p_o^{c_n}(1 - p_o)^{n-c_n} - (1/n)(p_o + \tfrac{1}{2}n)^{c_n}(1 - [p_o + \tfrac{1}{2}n])^{n-c_n-1}, \tag{3.24}$$

and

$$C(p_o) \geqq (1/[n - c_n])p_o^{c_n}(1 - p_o)^{n-c_n} - \tfrac{1}{2}n \{(p_o + 1/n)^{c_n}(1 - [p_o + 1/n])^{n-c_n-1} + p_o^{c_n}(1 - p_o)^{n-c_n-1}\}. \tag{3.25}$$

Setting the right side of (3.24), ((3.25)), less (greater) than or equal to 0, transferring second term to right and dividing by it, taking logarithms, making use of the inequalities $\log(1 + x) \geqq x - x^2/2$, $\log(1 - x) \leqq -x$, $\log(1 - x) \geqq -x - x^2/2(1 - x)$, $-\log[(1 + x)/(1 - x)] \leqq -2x - 2x^3/3$ for $x \in (0, 1)$, and using $\phi_{\beta-\delta}/2n^{\frac{1}{2}} = l - h_{1,\alpha}/n^{\frac{1}{2}}$, by judiciously discarding or dominating the proper terms, we obtain

LEMMA 2.

$$C(p_o) \leqq 0 \text{ on } B_0 = \left\{p_o : \frac{\phi_\beta}{p_o} - \frac{\phi_\beta}{1 - p_o} - \frac{(l - h_{1,\alpha}/n^{\frac{1}{2}})2\phi_\beta}{(1 - p_o)(1-[l - h_{1,\alpha}/n^{\frac{1}{2}}]/[1 - p_o])} - (\phi_\beta)/(4np_o^2) - 1/(2n^{\frac{1}{2}}p_o) \geqq 0\right\}, \tag{3.26}$$

and

$$C(p_o) \geqq 0 \text{ on } B_1 = \{p_o : \phi_\beta/p_o - \phi_\beta/(1 - p_o) + (2/n^{\frac{1}{2}})/(1 - p_o) \leqq 0\}. \tag{3.27}$$

We can obtain a subset of B_0 by replacing n by ∞ in the third term, and by n_e in the last two terms of its defining expression. A subset of B_1 is obtained by replacing n by n_e. Thus $[1/n, p^*] \subset B_0$, and $[p^{**}, 1] \subset B_1$. Later we find a better upper bound for n than ∞, and this is utilized to obtain a larger subset of B_0. Thus we see that in $[1/n, 1]$, A takes on its infimum in $[p^*, p^{**}]$.

In order to complete the proof of Theorem 3.2, we quote a revised version of a theorem due to Uspensky [10], pp. 119–129. Let $S_{n,p}$ have the binomial distribution from sample size n and parameter p. Then if $np - \frac{1}{2} + \psi(np[1 - p])^{\frac{1}{2}}$ is an integer and $np(1 - p) \geqq 25$,

$$|P\{S_{n,p} \leqq np - \tfrac{1}{2} + \psi(np[1 - p])^{\frac{1}{2}}\} - \Phi(\psi)| \leqq |1 - 2p|/13.4(2\pi np[1 - p])^{\frac{1}{2}} + (.073 + .09|1 - 2p|)/(np[1 - p]). \tag{3.28}$$

Since this theorem is only to be applied on $[p^*, p^{**}]$, and since under the conditions of Theorem 3.2 $[p^*, p^{**}] \subset [.36, .64]$, in order to legitimately apply this theorem, we need only insist that $n \geqq 109$.

We first show that if $p \in [1/n, 1]$, then

$$\sum_{\nu=0}^{[\![n_1(p+l)-n_1^{\frac{1}{2}}h_{1,\alpha}]\!]} \binom{n_1}{\nu} p^\nu (1 - p)^{n_1-\nu} \geqq 1 - \beta, \qquad \text{for } n_1 \geqq 109.$$

That is

$$P\{S_{n_1,p} \leqq [\![n_1(p+l) - n_1^{\frac{1}{2}}h_{1,\alpha}]\!]\} \geqq 1 - \beta.$$

From the previous work we know that the infimum of the above expression in $[1/n, 1]$ is achieved at some p_o-, say $\hat{p}_{o,1} \varepsilon [p^*, p^{**}]$. Thus, in order to make use of Uspensky's Theorem we define $\hat{\psi}_1$ by letting this infimum

$$P\{S_{n_1,\hat{p}_{o,1}} \leqq n_1(\hat{p}_{o,1} + l) - n_1^{\frac{1}{2}}h_{1,\alpha} - 1\} \equiv P\{S_{n_1,\hat{p}_{o,1}} \leqq n_1\hat{p}_{o,1} - \tfrac{1}{2} + \hat{\psi}_1[n\hat{p}_{o,1}(1 - \hat{p}_{o,1})]^{\frac{1}{2}}\}.$$

Hence,

$$\hat{\psi}_1 = \frac{n_1^{\frac{1}{2}} l - h_{1,\alpha} - (\frac{1}{2})n_1^{-\frac{1}{2}}}{(\hat{p}_{o,1}[1 - \hat{p}_{o,1}])^{\frac{1}{2}}} \geqq 2[n_1^{\frac{1}{2}} l - h_{1,\alpha}] - n_1^{-\frac{1}{2}} \geqq \phi_{\beta-\eta(n_1,p')}.$$

Then by Uspensky's Theorem,

$$P\{S_{n_1,\hat{p}_{o,1}} \leqq n_1(\hat{p}_{o,1} + l) - n_1^{\frac{1}{2}}h_{1,\alpha} - 1\} \geqq \Phi(\hat{\psi}_1) - \eta(n_1, \hat{p}_{o,1}) \geqq \Phi(\phi_{\beta-\eta(n_1,p')}) - \eta(n_1, \hat{p}_{o,1}) \geqq 1 - \beta.$$

Using (3.21), we see that for $p \leqq 1/n$, $A(p) \geqq 1 - 1/c_n!$. Thus (3.2) holds for $n \geqq \max(109, n_1)$ satisfying (3.15). Using now this value of n as an upper bound for sample size, (3.25) can be improved, and using identical reasoning, (3.2) can be shown to hold for $n \geqq \max(109, n_\kappa)$, satisfying (3.15).

Finally (3.16) is proved by letting

$$\hat{\psi}_\kappa = \frac{n_\kappa^{\frac{1}{2}} l - h_{1,\alpha} - \frac{1}{2}n_\kappa^{-\frac{1}{2}}}{(\hat{p}_{o,\kappa}[1 - \hat{p}_{o,\kappa}])^{\frac{1}{2}}} \leqq \frac{n_\kappa^{\frac{1}{2}} l - h_{1,\alpha} - \frac{1}{2}n_\kappa^{-\frac{1}{2}}}{\{p'(n_{\kappa-1})[1 - p'(n_{\kappa-1})]\}^{\frac{1}{2}}},$$

where $\hat{p}_{o,\kappa}$ makes A its infimum when $n = n_\kappa$, and applying Uspensky's Theorem. The theorem is proved.

A few remarks on Theorem 3.2 are in order. The suggested procedure improves faster than Okamoto's result as l decreases (for fixed α, β) and as α decreases (for fixed l, β). Okamoto's procedure improves faster as β decreases (for fixed α, l). As an example of the use of (3.16), for $\alpha = .05$, $l = .07$, $\beta = .001$ and smallest sample size given by Theorem 3.2

$$1 - .001 \leqq \inf_{p \varepsilon (0,1-l]} \sum_{\nu=0}^{[\![n(p+l)-n^{\frac{1}{2}}h_{1,\alpha}]\!]} \binom{n}{\nu} p^\nu(1-p)^{n-\nu} \leqq 1 - .0008.$$

It does not appear that a better result is obtainable when using the Uspensky Theorem. For $n < 109$ the binomial tables may be used to find n. The usual procedure for choice of sample size includes a continuity correction, and is given by letting n be the smallest integer for which

$$\Phi\{(2[\![n/2 + nl - n^{\frac{1}{2}}h_{1,\alpha}]\!] + 1 - n)/n^{\frac{1}{2}}\} \geqq 1 - \beta. \tag{3.29}$$

Table 1 compares sample sizes obtained from the usual procedure (3.29), from Corollary 1, and from Okamoto's result (3.4) for $\alpha = .1$.

TABLE I

Table of Sample Sizes for = .1

Based on (3.29) Usual Procedure (nonrigorous)				Based on Corollary 1				Based on Okamoto's Result—(3.4)			
$\beta \backslash l$	.1	.05	.01	$\beta \backslash l$	.1	.05	.01	$\beta \backslash l$	.1	.05	.01
.1	349	1390	34,600	.1	359	1,400	34,600	.1	525	2,100	52,500
.05	421	1680	41,800	.05	429	1,690	41,800	.05	598	2,390	59,800
.01	560	2270	54,600	.01	585	2,300	54,600	.01	750	3,000	75,000

(At best the above sample sizes are correct to 3 significant figures.)

4. Concerning whether the choice of sample size is too conservative. Suppose n is chosen as in the previous section. A natural question to ask is whether there is a distribution function F with $\inf_{H\varepsilon\mathcal{H}_o^*} d_1(F, H) \geqq l$ such that $P_F\{\text{acc } \mathcal{H}_o^*\}$ is "close" to $\sup_{p\varepsilon(0,1-l]} [1 - A(p)] \leqq \beta$. The main results of this section are embodied in the following theorem.

THEOREM 4.1. *If either of the functions H_1 or H_2 defining $\mathcal{H}_o^*$ is both continuous and takes on the values* [.44, .52], *then for $\alpha \leqq .05$, $\beta \leqq .05$, $l \leqq .1$, with n chosen from Theorem* 3.2, *there is a distribution function $\hat{F}$ with $\inf_{H\varepsilon\mathcal{H}_o^*} d_1(\hat{F}, H) = l$, such that*

$$
\begin{aligned}
&\sup_{p\varepsilon(0,1-l]} [1 - A(p)] \geqq P_{\hat{F}}\{\text{acc } \mathcal{H}_o^*\} \\
(4.1)\quad &\geqq \sup_{p\varepsilon(0,1-l]} [1 - A(p)] \Bigg[\inf_{\substack{v\varepsilon[.44,.615] \\ nv \text{ an integer}}} P_F\{(nv)^{\frac{1}{2}} d_1(F, F_{nv}) \leqq (\tfrac{7}{9}) h_{1,\alpha}/v^{\frac{1}{2}}\} \\
&\cdot P_F\{(n[1 - v])^{\frac{1}{2}} d_1(F, F_{n(1-v)}) \leqq (\tfrac{7}{9}) h_{1,\alpha}/(1 - v)^{\frac{1}{2}}\} - \sum_{i=[\![(\frac{4}{9}) n^{\frac{1}{2}} h_{1,\alpha}+1]\!]}^{i'} P_i \Bigg],
\end{aligned}
$$

where F is continuous and

$$0 \leqq \sum_{i=[\![(\frac{4}{9}) n^{\frac{1}{2}} h_{1,\alpha}+1]\!]}^{i'} P_i \leqq .099.$$

Before outlining a proof, we make a few remarks about this theorem

$$
\begin{aligned}
&\lim_{n\to\infty} \inf_{\substack{v\varepsilon[.44,.615] \\ nv \text{ an integer}}} P_F\{(nv)^{\frac{1}{2}} d_1(F, F_{nv}) \leqq (\tfrac{7}{9}) h_{1,\alpha}/v^{\frac{1}{2}}\} \\
(4.2)\quad &\cdot P_F\{(n[1 - v])^{\frac{1}{2}} d_1(F, F_{n(1-v)}) \leqq (\tfrac{7}{9}) h_{1,\alpha}/(1 - v)^{\frac{1}{2}}\} \\
&\geqq 1 - 4 \exp[-2.4 h_{1,\alpha}^2] \geqq .94.
\end{aligned}
$$

Thus, if the asymptotic theory of the Kolmogorov-Smirnov test were exact, (4.1) would read

$$\sup_{p\varepsilon(0,1-l]} [1 - A(p)] \geqq P_{\hat{F}}\{\text{acc } \mathcal{H}_o^*\} \geqq .84 \sup_{p\varepsilon(0,1-l]} [1 - A(p)],$$

which would show that power computations based on one point deviation are not very wasteful in sample size. Then from (4.2) and the fact (which will become evident) that $\sum P_i$ gets small as α gets small, it would also follow that the smaller α, the less wasteful the "one point" power computations.

It is immediately clear that we can assume $\mathcal{H}_o^* = \{U\}$, where U is the uniform distribution function on [0, 1], since the $\hat{F}$ defined in the proof of Theorem 4.1 takes on its maximum vertical distance l from U at x_0 with $\hat{F}(x_0) \varepsilon [.44, .52]$. But then clearly under the hypotheses of this theorem, there is always an F, depending on $\mathcal{H}_o^*$, with $F(x_0') \varepsilon [.44, .52]$, and either $H_1(x_0') - F(x_0') = l$, or $F(x_0') - H_2(x_0') = l$ such that

$$P_F\{\text{acc } \mathcal{H}_o^*\} \geqq P_{\hat{F}}\{\text{acc }\{U\}\}.$$

The proof of Theorem 4.1 is broken into several parts. Let $p_o \leqq .52$ be such that $n(p_o + l) - n^{\frac{1}{2}}h_{1,\alpha}$ is an integer. For the moment we only specify that

$$\hat{F}(p_o + l) = p_o. \tag{4.2}$$

LEMMA 1.

$$P_{\hat{F}}\{F_n(p_o + l) > p_o + l - (\tfrac{5}{9})h_{1,\alpha}/n^{\frac{1}{2}} \mid p_o + l - h_{1,\alpha}/n^{\frac{1}{2}} \leqq F_n(p_o + l)\} \leqq .099.$$

PROOF. Let

$$P_i = P_{\hat{F}}\{nF_n(p_o + l) = n(p_o + l) - n^{\frac{1}{2}}h_{1,\alpha} + i \mid p_o + l - h_{1,\alpha}/n^{\frac{1}{2}} \leqq F_n(p_o + l)\}. \tag{4.3}$$

Then letting $p_o + l - h_{1,\alpha}/n^{\frac{1}{2}} + i'/n = 1$, it only remains to show that

$$\sum_{i=[\![(\frac{4}{9})n^{\frac{1}{2}}h_{1,\alpha}+1]\!]}^{i'} P_i \leqq .099.$$

Using (3.20) and (3.23), we see that

$$\delta_i = P_{i+1}/P_i = \frac{1 - (\phi_{\beta-\delta}/2 + i/n^{\frac{1}{2}})/n^{\frac{1}{2}}(1 - p_o)}{1 + [\phi_{\beta-\delta}/2 + (i+1)/n^{\frac{1}{2}}]/n^{\frac{1}{2}}p_o}$$

$$= \frac{n - c_n - i}{c_n + 1 + i}\, p_o/(1 - p_o) \geqq 0$$

is decreasing and convex as a function of the (continuous) variable $i \geqq 0$. Therefore,

$$P_{J+1} = P_0 \prod_{i=0}^{J} \delta_i \leqq P_0([\delta_0 + \delta_J]/2)^{J+1}$$

and

$$\sum_{i=J+1}^{i'} P_i \leqq P_{J+1} \sum_{i=0}^{\infty} \delta_{J+1}^i \leqq P_0([\delta_0 + \delta_J]/2)^{J+1}/(1 - \delta_{J+1}).$$

Since

$$P_0 \leqq 1 - \sum_{i=1}^{J+2} P_i = 1 - \sum_{i=1}^{J+2} P_0 \prod_{j=0}^{i-1} \delta_j \leqq 1 - \sum_{i=1}^{J+2} P_0 \delta_{J+1}^{i}$$

thus

$$\sum_{i=J+1}^{i'} P_i \leqq ([\delta_0 + \delta_J]/2)^{J+1}/(1 - \delta_{J+1}^{J+3}). \tag{4.4}$$

Putting $J = [\![(\frac{4}{9})n^{\frac{1}{2}}h_{1,\alpha}]\!]$, then using (3.23), and the hypothesis $p_o \leqq .52$ it follows from some elementary manipulation that

$$\sum_{i=J+1}^{i'} P_i \leqq ([\delta_0' + \delta_{J^*}']/2)^{J^*+1} 1.05/(1 - \delta_{J^*}')^{J^*+3} \tag{4.5}$$

where $J^* = (\frac{4}{9})n^{\frac{1}{2}}h_{1,\alpha}$ and

$\delta_i' = [1 - (\phi_{\beta-\delta}/2 + i/n^{\frac{1}{2}})/n^{\frac{1}{2}}(1 - p_o)]/[1 + (\phi_{\beta-\delta}/2 + i/n^{\frac{1}{2}})/n^{\frac{1}{2}}p_o]$.

Let

$$r_x(p_o) = [1 - x/(1 - p_o)]/(1 + x/p_0) \tag{4.6}$$

and

$$x_1 = l\frac{\phi_{\beta-\delta}/2}{\phi_{\beta-\delta}/2 + h_{1,\alpha}}, \qquad x_2 = l\frac{\phi_{\beta-\delta}/2 + (\frac{4}{9})h_{1,\alpha}}{\phi_{\beta-\delta}/2 + h_{1,\alpha}}. \tag{4.7}$$

Then

$$x_1 < x_2 < l, \text{ and } \delta_0' = r_{x_1}(p_o),\ \delta_{J^*}' = r_{x_2}(p_o).$$

r_x has a maximum at $p_o = (1 - x)/2$, the maximum being

$$r_x([1 - x]/2) = [(1 - x)/(1 + x)]^2, \tag{4.8}$$

decreasing in x. Now considering $l \leqq .1$, $\beta \leqq .05$ and considering the separate cases $h_{1,\alpha}$ in $[\frac{9}{4}, \infty)$, $[\frac{7}{4}, \frac{9}{4})$, $[1.5, \frac{7}{4})$, $[1.42, 1.5)$, $[1.36, .42)$, using (4.7) we obtain lower bounds of the form cl (c known) for x_1 and x_2. These lower bounds using (4.8) give upper bounds for δ_0' and δ_{J^*}' which may be plugged into (4.5). With some more elementary manipulation we obtain the desired result.

Stronger results can be obtained in case we know more about x_1, x_2 and p_o; in particular, in any given case x_1 and x_2 and a lower bound for $p_o(p'(n_\kappa))$ are known, permitting smaller upper bounds for δ_0' and δ_{J^*}'.

The preceding lemma states essentially that if rejection of $\mathcal{H}_o^* = \{U\}$ does not take place because $F_n(p_o + l)$ is too small, then $F_n(p_o + l)$ with high probability took on value in the $\frac{2}{9}$'ths of the acceptance region for $\mathcal{H}_o^*$ nearest $\hat{F}(p_o + l)$. Let

$$\hat{F}(x) = \begin{cases} \{p_o/[p_o + l - (\frac{7}{9})h_{1,\alpha}/n^{\frac{1}{2}}]\}x, & 0 \leqq x \leqq p_o + l - (\frac{7}{9})(h_{1,\alpha}/n^{\frac{1}{2}}), \\ p_o, & p_o + l - (\frac{7}{9})(h_{1,\alpha}/n^{\frac{1}{2}}) \leqq x < p_o + l, \\ 1 + \dfrac{p_o + [(1 - p_o)/(1 - p_o - l - (\frac{5}{9})h_{1,\alpha}/n^{\frac{1}{2}})](\frac{7}{9})h_{1,\alpha}/n^{\frac{1}{2}} - 1}{p_o + l - 1}(x - 1), & \\ & x \geqq p_o + l, \quad x \,\varepsilon\, [0, 1], \end{cases}$$

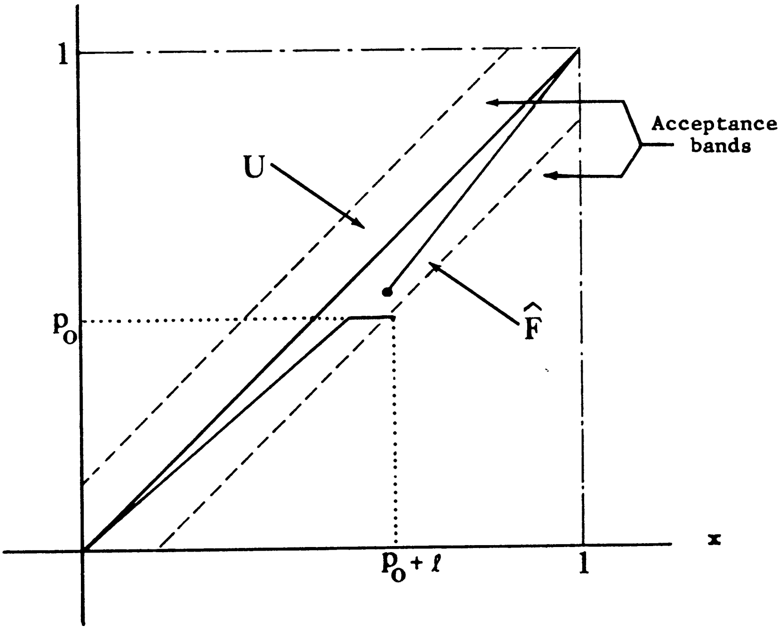

FIGURE 1

as illustrated in Figure 1. p_o is that value minimizing $A(p)$. We know $p_o \,\varepsilon\, [.44, .52]$. $\hat{F}$ is chosen so that if $F_n(p_o + l)$ is in the $\frac{2}{9}$ths of the acceptance region nearest $\hat{F}(p_o + l)$ there is small probability that $F_n(x)$ is ever outside the acceptance bands for other x. This accomplished by choosing $\hat{F}$ so that the conditional distribution function of those X_i to the left (right) of $p_o + l$ given $F_n(p_o + l)$ in the nearest $\frac{2}{9}$ths of the acceptance band lies close to U. We make use of the fact that given $F_n(p_o + l)$ the observations to the left of $p_o + l$ are conditionally independent of those to the right of $p_0 + l$. To fill in a bit more detail, let

$$A_v = \{F_n(p_o + l-) = v \,\varepsilon\, [p_o + l - h_{1,\alpha}/n^{\frac{1}{2}}, p_o + l - (\tfrac{5}{9})h_{1,\alpha}/n^{\frac{1}{2}}]\},$$

$$R_e = \{F_n(x) \not\varepsilon\, [x - (h_{1,\alpha}/n^{\frac{1}{2}}), x + (h_{1,\alpha}/n^{\frac{1}{2}})], \text{ some } x \,\varepsilon\, [0, p_o + l)\},$$

$$R_r = \{F_n(x) \not\varepsilon\, [x - (h_{1,\alpha}/n^{\frac{1}{2}}), x + (h_{1,\alpha}/n^{\frac{1}{2}})], \text{ some } x \,\varepsilon\, (p_o + l, 1]\}.$$

We compute an upper bound u_{ev} for $P_{\hat{F}}\{R_e \mid A_v\}$ by noting that given A_v, the conditional distribution function of these nv observations to the left of $p_o + l$ has an easily computed minimum vertical distance from the corresponding (re-scaled) acceptance bands. Similarly we compute an upper bound u_{rv} for $P_{\hat{F}}\{R_r \mid A_v\}$. Then $(1 - u_{ev})(1 - u_{rv})$ is a lower bound for $P_{\hat{F}}\{R_e^c \cap R_r^c \mid A_v\}$. Letting

$$(1 - u_e)(1 - u_r) = \inf_{v \text{ such that } A_v \text{ holds}} (1 - u_{ev})(1 - u_{rv}),$$

we see that

$$P_{\hat{F}}\{\text{rej } \mathcal{H}_o^* \mid A_v\} \leqq 1 - (1 - u_e)(1 - u_r).$$

Let $R_0 = \{F_n(p_o + l-) < p_o + l - h_{1,\alpha}/n^{\frac{1}{2}}\}$. Then

$$P_{\hat{F}}\{\text{rej } \mathcal{H}_o^* \mid R_o^c\} \leqq \sum_v P_{\hat{F}}\{\text{rej } \mathcal{H}_o^* \mid A_v\} P(A_v) + \sum_{i=[\![(\frac{1}{2}) n^{\frac{1}{2}} h_{1,\alpha}+1]\!]}^{i'} P_i$$

$$\leqq [1 - (1 - u_e)(1 - u_r)] \sum_v P(A_v) + \sum_{i=[\![(\frac{1}{2}) n^{\frac{1}{2}} h_{1,\alpha}+1]\!]}^{i'} P_i$$

$$\leqq u_e + u_r - u_e u_r + \sum_{i=[\![(\frac{1}{2}) n^{\frac{1}{2}} h_{1,\alpha}+1]\!]}^{i'} P_i .$$

Thus

$$P_{\hat{F}}\{\text{acc } \mathcal{H}_o^*\} = P_{\hat{F}}\{R_o^c\}[1 - P_{\hat{F}}\{\text{rej } \mathcal{H}_o^* \mid R_o^c\}]$$

$$\geqq \sup_{p \epsilon (0,1-l]} [1 - A(p)] \left[1 - \left(u_e + u_r - u_e u_r + \sum_{i=[\![(\frac{1}{2}) n^{\frac{1}{2}} h_{1,\alpha}+1]\!]}^{i'} P_i\right)\right]$$

$$= \sup_{p \epsilon (0,1-l]} [1 - A(p)] \left[(1 - u_e)(1 - u_r) - \sum_{i=[\![(\frac{1}{2}) n^{\frac{1}{2}} h_{1,\alpha}+1]\!]}^{i'} P_i\right].$$

As outlined before, this expression is that appearing in (4.1).

REFERENCES

[1] ANDERSON, T. W. and DARLING, D. A. (1952). Asymptotic theory of certain 'goodness-of-fit' criteria based on stochastic processes. *Ann. Math. Statist.* **23** 193–212.

[2] BIRNBAUM, Z. W. (1952). Numerical tabulation of the distribution of Kolmogorov's statistic for finite sample size. *J. Amer. Stat. Assn.* **47** 425–441.

[3] BIRNBAUM, Z. W. (1953). On the power of a one sided test of fit for continuous probability functions, *Ann. Math. Statist.* **24** 484–489.

[4] COCHRAN, W. G. (1952). The χ^2 test of goodness of fit. *Ann. Math. Statist.* **23** 315–345.

[5] DONSKER, M. D. (1952). Justification and extension of Doob's heuristic approach to the Kolmogorov-Smirnov theorems. *Ann. Math. Statist.* **23** 277–281.

[6] DOOB, J. L. (1949). Heuristic approach to the Kolmogorov-Smirnov limit theorems. *Ann. Math. Statist.* **20** 393–403.

[7] HODGES, JR., J. L. and LEHMANN, E. L. (1954). Testing the approximate validity of statistical hypotheses. *J. Roy. Stat. Soc., Ser. B*, **16** 261–268.

[8] MASSEY, F. J. (1951). The Kolmogorov test for goodness of fit. *J. Amer. Stat. Assn.* **46** 68–78.

[9] OKAMOTO, M. (1958). Some inequalities relating to the partial sum of binomial probabilities. *Ann. Inst. Statist. Math.* **10** 29–35.

[10] USPENSKY, J. V. (1937). *Introduction to Mathematical Probability*, McGraw Hill, New York.

[11] ROSENBLATT, JUDAH (1959). Goodness-of-fit tests for approximate hypotheses. Doctoral thesis, Columbia University, on microfilm.