EXACT AND APPROXIMATE POWER FUNCTION OF THE NON-PARAMETRIC TEST OF TENDENCY

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0. Summary. The properties of the power function of the test of tendency or the "quadrant measure of association" are studied. A formula giving a *lower bound* for the exact power function of this test with respect to normal bivariate one-tailed ($\rho > 0$) alternatives is obtained. Further, an approximate formula for this "minimum power" is suggested. Some numerical values are calculated from the exact and approximate formulae for this lower bound of the power function. A comparison with Konijn's approximation is presented.

1. Testing for tendency. In this paper there is discussed the power function of a test of independence based on the number of pairs (x_i, y_i) $(i = 1, 2, \cdots, N)$ in a two-dimensional sample, for which $(x_i - \text{Me}_x)(y_i - \text{Me}_y) > 0$, where Me_x and Me_y are the sample medians respectively.

There are several names for this measure of dependence: quadrant measure of association ϙ [7],[2] medial correlation [6] and measure of tendency [4], [5].

The first idea of this measure regarding the population was suggested by Sheppard [12], and its estimator was defined by Mosteller [9]. The distribution of this statistic and its properties have been discussed by Blomqvist[3] [1] and by the author [4] independently. A comprehensive discussion of ϙ and the relations between ϙ and Kendall's τ, as well as between ϙ and Spearman's σ_ρ is given by Kruskal [7].

The problem is the following: We wish to test the null hypothesis, H_0, that the random variables X and Y are independent. To do this we take note of the signs of the products $(x_i - \text{Me}_x)(y_i - \text{Me}_y)$. First, we assume $N = 2n$. (The case $N = 2n + 1$ can be reduced by simple modification to the case $N = 2n$; see [1], [4]).

In order to construct the test, a statistic U is introduced;

$$U = \sum_{i=1}^{2n} u_i, \tag{1}$$

where

$$u_i = \begin{cases} 1, & \text{if } (x_i - \text{Me}_x)(y_i - \text{Me}_y) > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

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[2] ϙ, old Greek letter "koppa" has been used by Kruskal (see [7]).

[3] This estimator was denoted by q' by Blomqvist. We shall use U in the remainder of this paper.

The probability under $H_0: \rho = 0$ is

$$\Pr\{U \geqq 2r \mid \rho = 0\} \frac{\sum_{i=r}^{n} \binom{n}{i}^2}{\sum_{i=0}^{n} \binom{n}{i}^2} = \frac{1}{\binom{2n}{n}} \sum_{i=r}^{n} \binom{n}{i}^2. \tag{3}$$

The test based on the statistic U is an easy and rapid non-parametric test of independence. However, its asymptotic efficiency in the normal case is only 41% [2], [1].

TABLE 1

Exact and approximate values of the "minimum power" function of the test of tendency for different sample sizes, 2n, and significance level, α, nearest to 0.05 and 0.01.

$2n$	$2R$	ρ	0.0	0.2	0.4	0.6	0.8	0.9	1.0
		p	.5000	.5641	.6310	.7048	.7952	.8564	1.0000
		$1-p$	.5000	.4359	.3690	.2952	.2048	.1436	0.0000
6	6	e*	.0500	.1021	.1934	.3496	.6108	.7936	1.0000
		a†	.0625	.1164	.2026	.3396	.5635	.7344	1.0000
10	8	e	.1032	.2092	.3731	.5967	.8497	.9545	1.0000
		a	.1133	.2179	.3738	.5837	.8287	.9394	1.0000
	10	e	.0040	.0131	.0391	.1108	.3196	.5606	1.0000
		a	.0057	.0180	.0467	.1194	.3080	.5169	1.0000
16	12	e	.0660	.1691	.3547	.6219	.8955	.9792	1.0000
		a	.0717	.1754	.3563	.6140	.8845	.9739	1.0000
	14	e	.0051	.0208	.0709	.2092	.5401	.8018	1.0000
		a	.0063	.0241	.0765	.2139	.5292	.7802	1.0000
20	16	e	.0115	.0469	.1744	.3812	.7628	.9950	1.0000
		a	.0133	.0508	.1535	.3807	.7517	.9306	1.0000
24	18	a	.0216	.0831	.2396	.5330	.8789	.9801	1.0000
30	20	a	.0748	.2340	.5157	.8241	.9849	.9993	1.0000
	22	a	.0147	.0708	.2369	.5651	.9153	.9909	1.0000
40	26	a	.0298	.2280	.5537	.8769	.9951	.9999	1.0000
60	38	a	.0198	.2132	.6081	.9358	.9994	1.0000	1.0000
100	60	a	.0230	.3074	.8091	.9935	1.0000	1.0000	1.0000
	60	k‡	.0228	.2345	.7396	.9892	1.0000	1.0000	1.0000

* Exact probability from formula (8).

† Approximate probability from formula (11).

‡ Konijn approximation.

In this paper we present two somewhat different approaches to this problem. These allow us to obtain formal expressions for the exact and approximate power function of the test.

2. Power function for one-tailed test of tendency ($\rho > 0$). Some formulae for the *asymptotic power function* for a group of non-parametric tests of independence were given by Konijn [6]. Applying his method to our case, with finite n, does not seem to give very good agreement even for $2n = 100$ (see Table 1).

To find the *power function* of the test against bivariate normal alternatives, we assume without loss of generality, that X and Y are from a standardized *normal* bivariate population with correlation $\rho > 0$, i.e., the joint distribution function is defined as

$$\Pr\{x > h, y > k \mid \rho\} = L(h, k, \rho) = \int_h^\infty dx \int_k^\infty f(x, y, \rho)\, dy, \tag{4}$$

where

$$f(x, y, \rho) = [2\pi(1 - \rho^2)^{\frac{1}{2}}]^{-1} \exp\{-[2(1 - \rho^2)]^{-1}[x^2 - 2\rho xy + y^2]\}.$$

Let us denote

$$\alpha(t) = (2\pi)^{-\frac{1}{2}} \int_t^\infty e^{-v^2/2}\, dv. \tag{5}$$

We will now denote the sample medians by Me_x', Me_y' and the statistic U by U' respectively. Let

$$\begin{aligned}
&\Pr\{x - \mathrm{Me}_x' > 0, y - \mathrm{Me}_y' > 0 \mid \rho\} = L(\mathrm{Me}_x', \mathrm{Me}_y', \rho) = p_1\\
&\Pr\{x - \mathrm{Me}_x' < 0, y - \mathrm{Me}_y' > 0 \mid \rho\}\\
&\qquad = \alpha(\mathrm{Me}_y') - p_1 = \alpha(\mathrm{Me}_y') - L(\mathrm{Me}_x', \mathrm{Me}_y', \rho) = p_2\\
(6)\quad &\Pr\{x - \mathrm{Me}_x' > 0, y - \mathrm{Me}_y' < 0 \mid \rho\}\\
&\qquad = \alpha(\mathrm{Me}_x') - p_1 = \alpha(\mathrm{Me}_x') - p_1 = \alpha(\mathrm{Me}_x') - L(\mathrm{Me}_x', \mathrm{Me}_y', \rho) = p_4\\
&\Pr\{x - \mathrm{Me}_x' < 0, y - \mathrm{Me}_y' < 0 \mid \rho\}\\
&\qquad = 1 - p_1 - p_2 - p_4 = 1 - \alpha(\mathrm{Me}_y') - \alpha(\mathrm{Me}_x') + L(\mathrm{Me}_x', \mathrm{Me}_y', \rho) = p_3
\end{aligned}$$

(see Figure 1).

The probability that the criterion U' takes the value $2r$ *and* the point (x, y) can be a median point of the sample (of size $2n$), is

$$\begin{aligned}
\Pr\{U = 2r; 2n, \mathrm{Me}_x', \mathrm{Me}_y' \mid \rho\} &= \frac{(2n)!}{(r!)^2[(n-r)!]^2} (p_1 p_3)^r (p_2 p_4)^{n-r}\\
&= \binom{2n}{n} (p_2 p_4)^n \binom{n}{r}^2 \left(\frac{p_1 p_3}{p_2 p_4}\right)^r.
\end{aligned} \tag{7}$$

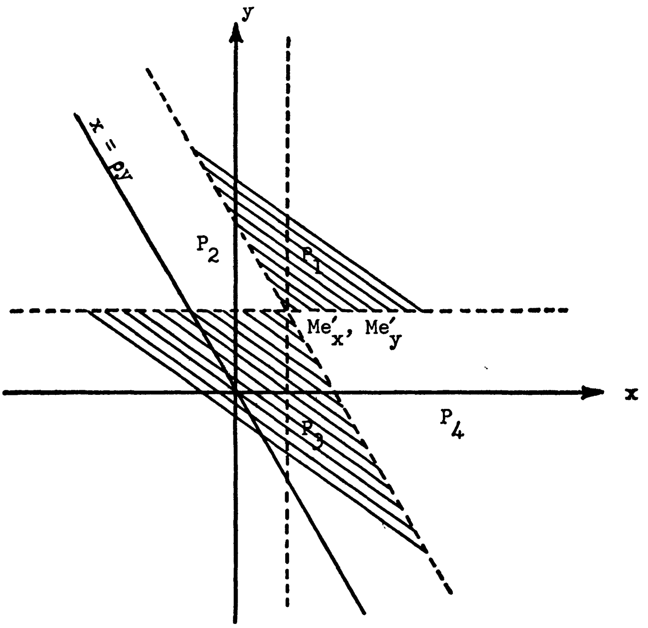

FIG. 1

Hence the *conditional* probability that $U' \geqq 2r$, *given* that (x, y) can be a median point of the sample, is

$$(8)\quad \Pr\{U' \geqq 2r \mid 2n, \mathrm{Me}'_x, \mathrm{Me}'_y, \rho\} = \left[\sum_{i=r}^{n} \binom{n}{i}^2 \left(\frac{p_1p_3}{p_2p_4}\right)^i\right] \Big/ \left[\sum_{i=0}^{n} \binom{n}{i}^2 \left(\frac{p_1p_3}{p_2p_4}\right)^i\right].$$

If $\rho = 0$, then $p_1 = \alpha(\mathrm{Me}'_x)\alpha(\mathrm{Me}'_y)$, $p_2 = \alpha(\mathrm{Me}'_y)[1 - \alpha(\mathrm{Me}'_x)]$, $p_4 = \alpha(\mathrm{Me}'_x)[1 - \alpha(\mathrm{Me}'_y)]$, $p_3 = [1 - \alpha(\mathrm{Me}'_x)][1 - \alpha(\mathrm{Me}'_y)]$, and

$$(9)\quad p_1p_3/p_2p_4 = 1$$

This means that under the null hypothesis the distribution does not depend on the position of sample medians and is equal—as it is easy to see—to the probability given by formula (2).

But if $\rho \neq 0$, the probability (8) depends on the position of $(\mathrm{Me}'_x, \mathrm{Me}'_y)$.

It appears that, when $\rho > 0$, the probability (8) is a *minimum*, when $\mathrm{Me}'_x = \mathrm{Me}'_y = 0$. So when the sample medians are equal to the population medians we obtain a *lower bound* for the power function. For convenience we will call it "*minimum power*" function.

To prove this we have to show that the function

$$(10)\quad g(\mathrm{Me}'_x, \mathrm{Me}'_y, \rho) = p_1p_3/p_2p_4$$

has its minimum at the point (0, 0), since (8) is a monotone increasing function of g. A discussion of this point is given in the Appendices I and II.

In the same way it appears that when $\rho < 0$, the probability (8) has a *maximum* at the point $\text{Me}'_x = \text{Me}'_y = 0$.

Calculated values of the "minimum power" function for the significance level α, nearest to 0.05 and 0.01, $2n = 6, 10, 16, 20$ and critical values $2r = 2R$ are given in Table 1. For $2n > 20$, the calculations are rather tedious, using the ordinary electric calculating machine. The approximation (also given in Table 1) appears to be reasonably good.

3. Approximate "minimum power" function ($\rho > 0$)**.** If the population medians, $\mathfrak{M}e_x$, $\mathfrak{M}e_y$ are *known*, we can define the statistic V, in an N-element paired sample

$$V = \sum_{i=1}^{N} v_i, \tag{11}$$

where

$$v_i = \begin{cases} 1, & \text{if } (x_i - \mathfrak{M}e_x)(y_i - \mathfrak{M}e_y) > 0 \text{ with probability } p, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\Pr\{V \geqq m \mid N, \rho\} = \sum_{i=m}^{n} \binom{N}{i} p^i (1 - p)^{N-i}, \qquad m = 0, 1, 2, \cdots, N. \tag{12}$$

In the case of the bivariate normal population

$$p = p_1 + p_3 = 1 - (1/\pi) \text{ arc cos } \rho. \tag{13}$$

But if the population medians are *unknown* (which usually happens in practice) we must estimate $\mathfrak{M}e_x$ and $\mathfrak{M}e_y$ by Me_x and Me_y respectively. By a heuristic

TABLE 2

Exact and approximate distribution under the null hypothesis ($\rho = 0$)

Prob.	2n \ 2r	0	2	4	6	8	10	12	14	16	18	20	22	24	26
P_1*	6	1.0000	.9500	.5000	.0500										
P_2†		1.0000	.9375	.5000	.0625										
$\bar{P}_1$	10	1.0000	.9960	.8968	.5000	.1032	.0040								
P_2		1.0000	.9942	.8867	.5000	.1133	.0057								
$\bar{P}_1$	16	1.0000	1.0000	.9950	.9341	.6904	.3097	.0660	.0051	.0001					
P_2		1.0000	.9999	.9936	.9432	.6854	.3145	.0717	.0063	.0001					
P_1	20	1.0000	.9999	.9994	.9884	.9105	.6718	.3281	.0894	.0115	.0005				
P_2		1.0000	1.0000	.9993	.9867	.9054	.6682	.3318	.0946	.0133	.0007				
$\bar{P}_1$	24			1.0000	.9984	.9805	.8899	.6579	.3422	.1102	.0196	.0017	.0001		
P_2				1.0000	.9980	.9784	.8852	.6550	.3450	.1147	.0216	.0020	.0001		
P_1	30			1.0000	.9999	.9986	.9866	.9285	.7670	.5000	.2330	.0715	.0134	.0014	.0001
P_2				1.0000	.9999	.9985	.9852	.9252	.7635	.5000	.2365	.0748	.0147	.0016	.0001

* Exact probability.

† Approximate probability.

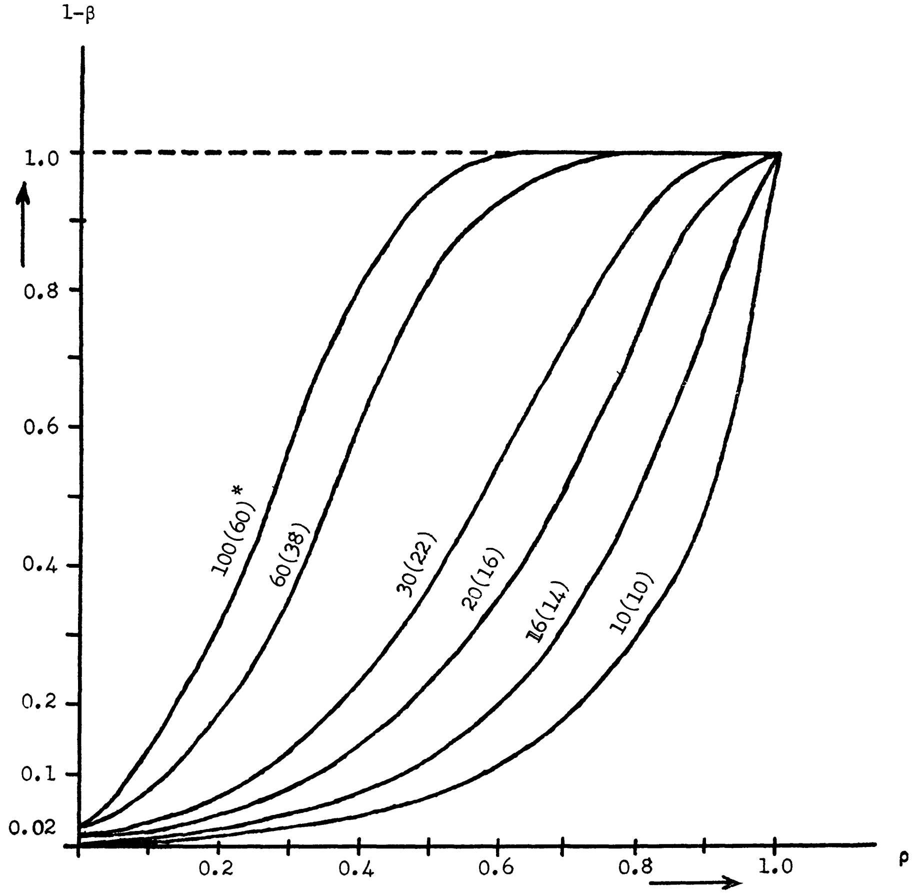

FIG. 2. Approximate "minimum power" functions of the test of tendency
* The numbers over the curves are $2n(2R)$.

TABLE 3

Approximate sample sizes, $2n$, and critical values, $2R$, for various $\rho > 0$ and $\alpha \simeq \beta \simeq 0.05$; 0.01 for one-tailed test

ρ	$\alpha \simeq \beta \simeq 0.05$		$\alpha \simeq \beta \simeq 0.01$	
	$2n$	$2R$	$2n$	$2R$
0.2	654	350	1436	764
0.4	154	88	306	174
0.6	68	38	118	72
0.8	26	18	52	36
0.9	20	16	30	22

argument (in its idea similar to the problem of estimation of a median in grouped data) we are led to the following approximate formula

$$\text{(14)}\quad \Pr\{V \geqq m \mid N, \rho\} \simeq p \sum_{i=m-1}^{N} \binom{N}{i} p^i (1-p)^{N-i} + (1-p) \sum_{i=m}^{N} \binom{N}{i} p^i (1-p)^{N-i}.$$

Hence, the power (8) of the test based on the statistic U with sample size $2n$ and critical value $2R$ can be approximately expressed

$$\text{(15)}\quad \begin{aligned} 1 - \beta &= \Pr\{U \geqq 2R \mid \rho\} \simeq \Pr\{V \geqq 2R \mid \rho\} \\ &\simeq p \sum_{i=2R-1}^{2n} \binom{2n}{i} p^i (1-p)^{2n-i} + (1-p) \sum_{i=2R}^{2n} \binom{2n}{i} p^i (1-p)^{2n-i}. \end{aligned}$$

If the null hypothesis ($\rho = 0$) is true, then $p = \frac{1}{2}$ and the probability (3) takes the form

$$\text{(16)}\quad \Pr\{U \geqq 2r \mid \rho = 0\} \simeq \Pr\{V \geqq 2r \mid \rho = 0\} = \frac{1}{2^{2n}} \sum_{i=r}^{n} \binom{2n}{i}.$$

Table 2 shows the exact and approximate values of the probabilities under the null hypothess and Table 1 compares the exact and approximate "minimum power" function for $2n = 6, 10, 16, 20$ for the one-tailed test ($\rho > 0$). The other values of the "minimum power" function ($2n = 24, 30, 40, 60, 100$) are calculated from the approximate formula (15). Some of the approximate "minimum power" functions are presented graphically in Figure 2.

It would be interesting to find how large the sample should be, if we wish the probabilities of the errors of both, first and second kinds—α and β—to be approximately the same, i.e., $\alpha \simeq \beta$. Some results of calculations on this point are shown in Table 3. The calculations are based on the central limit theorem. The sample size for $2n > 30$ is evaluated as the nearest even integer larger than

$$\text{(17)}\quad z_\alpha^2 \cdot \frac{4p(1-p) + 4[p(1+p)]^{\frac{1}{2}} + 1}{(1-2p)^2},$$

where

$$(2\pi)^{-\frac{1}{2}} \int_{z_\alpha}^{\infty} e^{-v^2/2}\, dv = \alpha.$$

APPENDIX I

We have to prove that

$$\text{(1)}\quad \Psi(g) = \left[\sum_{i=r}^{n} \binom{n}{i}^2 g^i\right] \Big/ \left[\sum_{i=0}^{n} \binom{n}{i}^2 g^i\right],$$

where $g > 0$ is a monotone increasing function of g, i.e., $(d/dg)\Psi(g) > 0$.

Let us present (1) in more convenient form

$$(2)\qquad \Psi(g) = \left[\sum_{i=r}^{n}\binom{n}{i}^2 g^i\right] \Big/ \left[\sum_{i=0}^{n}\binom{n}{i}^2 g^i\right] = \left[\sum_{i=r}^{n} a_i g^i\right] \Big/ \left[\sum_{i=0}^{n} a_i g^i\right],$$

where

$$(3)\qquad \binom{n}{i}^2 = a_i > 0, \qquad i = 0, 1, \cdots, n.$$

The derivative with respect to g is

$$(4)\qquad \frac{d}{dg}\Psi(g) = \frac{\left(\sum_{i=r}^{n} ia_i g^{i-1}\right)\left(\sum_{i=0}^{n} a_i g^i\right) - \left(\sum_{i=1}^{n} ia_i g^{i-1}\right)\left(\sum_{i=r}^{n} a_i g^i\right)}{\left[\sum_{i=0}^{n} a_i g^i\right]^2}$$

It is sufficient to show that the numerator of (4) is positive.

$$\begin{aligned} S &= \left(\sum_{i=r}^{n} ia_i g^{i-1}\right)\left(\sum_{i=0}^{n} a_i g^i\right) - \left(\sum_{i=1}^{n} ia_i g^{i-1}\right)\left(\sum_{i=r}^{n} a_i g^i\right) \\ (5)\qquad &= \left(\sum_{i=r}^{n} ia_i g^{i-1}\right)\left(\sum_{i=0}^{r-1} a_i g^i\right) - \left(\sum_{i=r}^{n} a_i g^i\right)\left(\sum_{i=1}^{r-1} ia_i g^{i-1}\right) \\ &= \sum_{i=r}^{n}\sum_{j=o}^{r-1} a_i\, a_j\, g^{i+j-1}(i-j) > 0. \end{aligned}$$

Hence $(d/dg)\Psi(g) > 0$, so $\Psi(g)$ is the monotone increasing function of g.

APPENDIX II

We introduce the notations

$$z(t) = (2\pi)^{-\frac{1}{2}} e^{-t^2/2}; \qquad \alpha(t) = \int_t^\infty z(t)\, dt,$$

$$\begin{aligned} L(h, k, \rho) &= \frac{1}{2\pi(1-\rho^2)^{\frac{1}{2}}} \int_h^\infty \int_k^\infty \exp\left\{-\frac{1}{2(1-\rho^2)}[x^2 - 2\rho xy + y^2]\right\} dy\, dx \\ &= \frac{1}{2\pi}\int_h^\infty e^{-x^2/2} \int_{(k-\rho x)/(1-\rho^2)^{\frac{1}{2}}}^\infty e^{-w^2/2}\, dw\, dx \\ &= \frac{1}{2\pi}\int_k^\infty e^{-y^2/2} \int_{(h-\rho y)/(1-\rho^2)^{\frac{1}{2}}}^\infty e^{-w^2/2}\, dw\, dy. \end{aligned}$$

Let

$p_1 = L(h, k, \rho) = L$, $p_2 = \alpha(k) - L$, $p_4 = \alpha(h) - L$, $p_3 = 1 - \alpha(h) - \alpha(k) + L$.

We wish to find an extremum of the function

$$(1)\qquad g(h, k, \rho) = p_1 p_3 / p_2 p_4 .$$

We take the function $G = \log g$ and find the roots of the equations

$$(2) \qquad \partial G/\partial h = 0, \qquad \partial G/\partial k = 0$$

We have $\partial L/\partial h = -z(h)\alpha[(k - \rho h)/(1 - \rho^2)^{\frac{1}{2}}]$. Then,

$$\frac{\partial G}{\partial h} = z(h)\left\{-\alpha\left(\frac{k - \rho h}{(1 - \rho^2)^{\frac{1}{2}}}\right)\left[\frac{1}{L} + \frac{1}{\alpha(k) - L}\right]\right.$$
$$\left.+\left(1 - \alpha\left(\frac{k - \rho h}{(1 - \rho^2)^{\frac{1}{2}}}\right)\right)\left[\frac{1}{1 - \alpha(h) - \alpha(k) + L} + \frac{1}{\alpha(h) - L}\right]\right\},$$

and similarly, changing k into h, and vice-versa, we get

$$\frac{\partial G}{\partial k} = z(k)\left\{-\alpha\left(\frac{h - \rho k}{(1 - \rho^2)^{\frac{1}{2}}}\right)\left[\frac{1}{L} + \frac{1}{\alpha(h) - L}\right]\right.$$
$$\left.+\left(1 - \alpha\left(\frac{h - \rho k}{(1 - \rho^2)^{\frac{1}{2}}}\right)\right)\left[\frac{1}{1 - \alpha(k) - \alpha(h) + L} + \frac{1}{\alpha(k) - L}\right]\right\}.$$

The equations (2) are satisfied for $h = k = 0$, since we have

$$\frac{\partial G}{\partial h} = \frac{\partial G}{\partial k} = \frac{1}{2}\left\{-\frac{1}{2}\left[\frac{1}{L_0} + \frac{1}{\frac{1}{2} - L_0}\right] + \frac{1}{2}\left[\frac{1}{L_0} + \frac{1}{\frac{1}{2} - L_0}\right]\right\} = 0,$$

where $L_0 = L(0, 0, \rho)$.

We evaluate $\partial^2 G/\partial h^2$; $\dfrac{\partial^2 G}{\partial h^2}$

$$= \left\{-\frac{(\partial L/\partial h)^2}{L^2} - \frac{[z(h) + (\partial L/\partial h)]^2}{1 - \alpha(h) - \alpha(k) + L]^2} + \frac{(\partial L/\partial h)^2}{[\alpha(k) - L]^2} + \frac{[z(h) + (\partial L/\partial h)]^2}{[\alpha(h) - L]^2}\right\}$$
$$+\left\{\frac{\partial^2 L/\partial h^2}{L} + \frac{(\partial z/\partial h) + (\partial^2 L/\partial h^2)}{1 - \alpha(h) - \alpha(k) + L} + \frac{\partial^2 L/\partial h^2}{\alpha(k) - L} + \frac{(\partial z/\partial h) + (\partial^2 L/\partial h^2)}{\alpha(h) - L}\right\},$$

where

$$\partial^2 L/\partial h^2 = (\partial z/\partial h)\alpha((k - \rho h)/(1 - \rho^2)^{\frac{1}{2}})$$
$$- z(h)z((k - \rho h)/(1 - \rho^2)^{\frac{1}{2}})\cdot\rho/(1 - \rho^2)^{\frac{1}{2}}.$$

$$(3) \qquad \left.\frac{\partial^2 G}{\partial h^2}\right|_{h=k=0} = \frac{1}{4\pi}\left\{\left[\frac{1}{(\frac{1}{2} - L_0)^2} - \frac{1}{L_0^2}\right] - 4\frac{\rho}{(1 - \rho^2)^{\frac{1}{2}}}\left[\frac{1}{\frac{1}{2} - L_0} + \frac{1}{L_0}\right]\right\}$$
$$= \frac{1}{4\pi L_0(\frac{1}{2} - L_0)}\left[\frac{L_0 - \frac{1}{4}}{L_0(\frac{1}{2} - L_0)} - \frac{2\rho}{(1 - \rho^2)^{\frac{1}{2}}}\right].$$

We obtain the same value for $(\partial^2 G/\partial k^2)_{h=k=0}$.

The mixed derivative is

$$(4) \qquad \left.\frac{\partial^2 G}{\partial h \partial k}\right|_{h=k=0} = \frac{1}{4\pi}\left\{\left[-\frac{1}{(\frac{1}{2} - L_0)^2} - \frac{1}{L_0^2}\right] + \frac{4}{(1 - \rho^2)^{\frac{1}{2}}}\left[\frac{1}{\frac{1}{2} - L_0} + \frac{1}{L_0}\right]\right\}.$$

Denote

$$\left.\frac{\partial^2 G}{\partial h^2}\right|_{h=k=0} = \left.\frac{\partial^2 G}{\partial k^2}\right|_{h=k=0} = A; \qquad \left.\frac{\partial^2 G}{\partial h \partial k}\right|_{h=k=0} = B.$$

We have

$$\begin{aligned} A + B &= \frac{1}{4\pi}\left\{-\frac{2}{L_0^2} + \frac{2}{(1-\rho^2)^{\frac{1}{2}}}\cdot\frac{1}{L_0(\frac{1}{2}-L_0)}(1-\rho)\right\} \\ &= \frac{1}{2\pi L_0(\frac{1}{2}-L_0)}\left[-\frac{\frac{1}{2}-L_0}{L_0} + \left(\frac{1-\rho}{1+\rho}\right)^{\frac{1}{2}}\right]. \end{aligned} \tag{5}$$

Similarly

$$A - B = \frac{1}{2\pi L_0(\frac{1}{2}-L_0)}\left[\frac{L_0}{\frac{1}{2}-L_0} - \left(\frac{1+\rho}{1-\rho}\right)^{\frac{1}{2}}\right]. \tag{6}$$

Denoting $L_0/\frac{1}{2} - L_0 = a > 0$, $[(1-\rho)/(1+\rho)]^{\frac{1}{2}} = b > 0$, and multiplying (5) by (6) we obtain

$$A^2 - B^2 = \left(\frac{1}{2\pi L_0(\frac{1}{2}-L_0)}\right)^2 (a-b)\left(\frac{1}{b}-\frac{1}{a}\right).$$

$$\text{If } a > b \quad \frac{1}{b} > \frac{1}{a} \quad A^2 - B^2 > 0,$$

$$\text{if } a < b \quad \frac{1}{b} < \frac{1}{a} \quad A^2 - B^2 > 0.$$

Hence, there is an extremum at the point $h = k = 0$.

We now have to show that it is a *minimum*, i.e.,

$$\partial^2 G/\partial h^2 > 0.$$

From formula (3) we can see that it is sufficient to show that

$$\frac{L_0 - \frac{1}{4}}{L_0(\frac{1}{2}-L_0)} - \frac{2\rho}{(1-\rho^2)^{\frac{1}{2}}} > 0. \tag{7}$$

It is known that

$$L_0 = L(0, 0, \rho) = \tfrac{1}{4} + (2\pi)^{-1} \operatorname{arc\,sin} \rho = \tfrac{1}{4} + \delta. \tag{8}$$

The inequality (7) can be written

$$\delta^2 + [(1-\rho^2)^{\frac{1}{2}}/2\rho]\delta - \tfrac{1}{16} > 0. \tag{9}$$

The roots of the left side are

$$\delta_{1,2} = (4\rho)^{-1}[-(1-\rho^2)^{\frac{1}{2}} \pm 1].$$

Since $\rho > 0$, $\delta = (2\pi)^{-1} \operatorname{arc\,sin} \rho > 0$, then only $\delta_2 = (4\rho)^{-1}[1 - (1-\rho^2)^{\frac{1}{2}}]$ has a meaning.

The inequality (9) is satisfied, if $\delta > \delta_2$, i.e.,

$$\varphi(\rho) = \pi^{-1} \text{arc sin } \rho - (2\rho)^{-1}[1 - (1 - \rho^2)^{\frac{1}{2}}] > 0. \tag{10}$$

We put arc sin $\rho = \theta$. Then (10) can be written in the form

$$\Psi(\theta) = (\theta/\pi) - (2 \sin \theta)^{-1}(1 - \cos \theta) > 0, \tag{11}$$

since $0 \leqq \rho \leqq 1$, $0 \leqq \theta \leqq \frac{1}{2}\pi$. We can see that

$$\lim_{\rho\to 0} \varphi(\rho) = \lim_{\theta\to 0} \Psi(\theta) = 0, \qquad \varphi(1) = \Psi(\tfrac{1}{2}\pi) = 0.$$

The inequality (11) will be satisfied, if, further $\lim_{\theta\to 0} \Psi'(\theta) > 0$ and $\Psi''(\theta) < 0$ for $0 \leqq \theta \leqq \frac{1}{2}\pi$. Now

$$\Psi'(\theta) = \frac{1}{\pi} - \frac{1}{2(1 + \cos \theta)}, \qquad \Psi'(0) = \frac{1}{\pi} - \frac{1}{4} > 0.$$

$$\Psi''(\theta) = -\frac{\sin \theta}{2(1 - \cos \theta)^2} < 0 \qquad \text{for } 0 \leqq \theta \leqq \tfrac{1}{2}\pi.$$

This shows that the function (1) has a minimum at the point (0, 0).

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