TWO-SAMPLE COMPARISONS OF DISPERSION MATRICES FOR ALTERNATIVES OF INTERMEDIATE SPECIFICITY

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0. Summary. For two multivariate nonsingular normal distributions, the familiar null hypothesis of equal dispersion matrices is considered against various alternatives stated in terms of certain characteristic roots, and a physical interpretation is given for the alternatives considered. An inference procedure, which depends on similar regions and is based on one independent random sample from each of the two distributions, is proposed for the null hypothesis against each of the alternative hypotheses. Also, for three of the cases, conservative confidence bounds are obtained on one or more parametric functions which might be interpreted as measures of departure from the null hypothesis in the direction of the corresponding alternative.

1. Introduction. For two nonsingular p-variate normal distributions, $N[\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1]$ and $N[\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_2]$, we start from the familiar null hypothesis H_0: $\boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2$. The characteristic roots, all positive, of $\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}_2^{-1}$, no matter whether H_0 is true or not, will be denoted by $\gamma_1, \gamma_2, \cdots, \gamma_p$. Most often, the largest and smallest roots will be denoted respectively by γ_M and γ_m. H_0 can now be stated in the form H_0: all γ's $= 1$. As alternatives, however, the following are considered: (i) H_1: all γ's > 1; (ii) H_2 : all γ's < 1; (iii) H_3 : all γ's > 1 or all γ's < 1; (iv) H_4 : at least one $\gamma > 1$; (v) H_5 : at least one $\gamma < 1$; (vi) H_6 : at least one $\gamma > 1$ and at least one $\gamma < 1$; (vii) H_7 : at least one $\gamma > 1$ or < 1. It may be noted that (iii) is the union of (i) and (ii), (vi) is the intersection of (iv) and (v), (vii) is the union of (iv) and (v), and (vi), taken together with equality with 1 in both inequalities, is the complement of (iii). Also, while each of the alternatives forms a mutually exclusive pair with H_0, only (vii) is the complement of H_0, and it is only (vii) that has attracted attention heretofore [2, 5, 11]. The relations in logical structure between the various alternatives may be useful in understanding the forms of the inference procedures proposed in Section 2 of this paper for H_0 against each of the alternatives. Section 3 discusses some conservative confidence bounds of varying degrees of appropriateness. For the first three cases, the conservative confidence bounds are on parametric functions which are natural measures of departure from H_0 in the direction of the alternative hypothesis in question. Section 4 consists of some concluding remarks.

We consider one possible physical meaning of the alternatives considered in this paper. If $\mathbf{x}_i(p \times 1)$ is p-variate nonsingular $N[\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i]$, $(i = 1, 2)$, and the components (variates) of $\mathbf{x}_1$ are physically of the same nature as those of $\mathbf{x}_2$ (for example, the first element in both is amount of steel produced, the second element is total farm produce, etc.), then, if $\mathbf{a}' = (a_1, a_2, \cdots, a_p)$ is a vector of nonstochastic utilitarian "weights" that go with the p variates, the linear functions $\mathbf{a}'\mathbf{x}_1$, and $\mathbf{a}'\mathbf{x}_2$ are of utilitarian interest. It is well known that $\mathbf{a}'\mathbf{x}_i$ is univariate $N(\mathbf{a}'\boldsymbol{\mu}_i, \mathbf{a}'\boldsymbol{\Sigma}_i\mathbf{a})$, $(i = 1, 2)$. If $\mathbf{a}'$ were known then a direct comparison of $\mathbf{a}'\mathbf{x}_1$ and $\mathbf{a}'\mathbf{x}_2$, for observed values of $\mathbf{x}_1$ and $\mathbf{x}_2$, using the usual univariate techniques would be quite appropriate. Thus, for instance, one may be interested in the difference between the means $\mathbf{a}'\boldsymbol{\mu}_1$ and $\mathbf{a}'\boldsymbol{\mu}_2$, or in the ratio of the variances, $\mathbf{a}'\boldsymbol{\Sigma}_1\mathbf{a}/\mathbf{a}'\boldsymbol{\Sigma}_2\mathbf{a}$. For a known system of utilitarian weights then, one may, for instance, wish to test $H_0 : \mathbf{a}'\boldsymbol{\Sigma}_1\mathbf{a}/\mathbf{a}'\boldsymbol{\Sigma}_2\mathbf{a} = 1$, against $H_1 : \mathbf{a}'\boldsymbol{\Sigma}_1\mathbf{a}/\mathbf{a}'\boldsymbol{\Sigma}_2\mathbf{a} > 1$. The test is the well-known one-sided F-test. But now, if $\mathbf{a}'$ is not known or given, then one may want to obtain a weight-free solution by protecting oneself against the worst possible or most stringent set of weights (in a sort of minimax sense) and pose the question as a test of $H_0 : \mathbf{a}'\boldsymbol{\Sigma}_1\mathbf{a}/\mathbf{a}'\boldsymbol{\Sigma}_2\mathbf{a} = 1$ for all $\mathbf{a}$, against $H_1 : \mathbf{a}'\boldsymbol{\Sigma}_1\mathbf{a}/\mathbf{a}'\boldsymbol{\Sigma}_2\mathbf{a} > 1$ for all $\mathbf{a}$. This is exactly the null hypothesis of H_0 : all γ's $= 1$, against H_1 : all γ's > 1. Another way to express this would be to say that if σ_{1a}^2 and σ_{2a}^2 stand respectively for the variances $\mathbf{a}'\boldsymbol{\Sigma}_1\mathbf{a}$ and $\mathbf{a}'\boldsymbol{\Sigma}_2\mathbf{a}$, then H_0 means that $\sigma_{1a}^2 = \sigma_{2a}^2$ uniformly in $\mathbf{a}$ and H_1 means that $\sigma_{1a}^2 > \sigma_{2a}^2$ uniformly in $\mathbf{a}$. Of the other alternatives, H_2 and H_3 can be interpreted in exactly the same manner. According to this interpretation, H_4, H_5, H_6 and H_7 are much weaker alternatives. H_4, for example, means $\mathbf{a}'\boldsymbol{\Sigma}_1\mathbf{a}/\mathbf{a}'\boldsymbol{\Sigma}_2\mathbf{a} > 1$ for at least one $\mathbf{a}$, or in other words, that we are considering (in terms of acceptance of H_4) the most favorable kind of weights (and trying to reach in a sense a minimin solution). However, in terms of acceptance of H_0, we stay with the same worst or most stringent set of weights; similarly for H_5 to H_7. The main point in introducing H_4, H_5, H_6 and H_7 is to indicate how the customary H_7 shows up according to our interpretation.

We have always preferred the above type of interpretation of Fisher's approach to discriminant analysis and Hotelling's approach to canonical correlations to the one that is more customary. But this is a matter of opinion and we shall not press it here.

2. Inference procedures for H_0 against each of the alternatives of section 1. Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be two $(p \times p)$ matrices based on independent random samples of sizes $(n_1 + 1)$ and $(n_2 + 1)$ from the two populations. Let these denote the maximum likelihood estimators of $\boldsymbol{\Sigma}_1$ and $\boldsymbol{\Sigma}_2$ with the conventional bias correction. We assume that $p \leqq$ the smaller of n_1 and n_2, so that $\mathbf{S}_1$ and $\mathbf{S}_2$ are positive definite almost everywhere. Let c_M and c_m denote, respectively, the largest and the smallest characteristic roots of $\mathbf{S}_1\mathbf{S}_2^{-1}$. Also, let ch $(\mathbf{A})$ denote the characteristic root of any general (square) matrix $\mathbf{A}$ and $\text{ch}_m(\mathbf{A})$ and $\text{ch}_M(\mathbf{A})$ the smallest and largest roots. Then, using a heuristic argument similar to that of [5, 9], the following inference procedures, some of them three-decision procedures, are proposed, wherein $W(H)$ denotes the acceptance region for the hypothesis H, and $W(I)$, where it occurs, denotes the region of indecision or no choice between the two hypotheses in question:

(i) $W(H_0): c_M \leqq \lambda_1$; $W(H_1): c_m > \lambda_1$; $W(I): c_m \leqq \lambda_1 < c_M$,

(ii) $W(H_0): c_m \geqq \lambda_2 ; W(H_2): c_M < \lambda_2 ; W(I): c_m < \lambda_2 \leqq c_M$,

(iii) $W(H_0): \lambda_3 \leqq c_m \leqq c_M \leqq \lambda_3' ; W(H_3): c_m > \lambda_3'$ or $c_M < \lambda_3$;

(2.1) $W(I): c_m < \lambda_3 < c_M$ and/or $c_m \leqq \lambda_3' < c_M$,

(iv) $W(H_0): c_M \leqq \lambda_4 ; W(H_4): c_M > \lambda_4$,

(v) $W(H_0): c_m \geqq \lambda_5 ; W(H_5): c_m < \lambda_5$,

(vi) $W(H_0): \lambda_6 \leqq c_m \leqq c_M \leqq \lambda_6' ; W(H_6): c_m < \lambda_6$ and $c_M > \lambda_6'$;
$W(I): c_m < \lambda_6$ and $c_M \leqq \lambda_6'$ or $c_m \geqq \lambda_6$ and $c_M > \lambda_6'$,

(vii) $W(H_0): \lambda_7 \leqq c_m \leqq c_M \leqq \lambda_7' ; W(H_7): c_m < \lambda_7$ and/or $c_M > \lambda_7'$.

For Case (i), given λ_1, the probabilities assigned to the three regions, $W(H_0)$, $W(H_1)$ and $W(I)$ under H_0 can be determined. Likewise, given the probability assigned to the region $W(H_0)$ under H_0, λ_1 can be determined by the methods described in [3, 4], and hence the probabilities assigned to $W(H_1)$ and $W(I)$ under H_0 may be determined. It should be noted that the method of evaluating the probability assigned to the region $W(I)$ under H_0, for a given λ_1, has not been explicitly considered. The authors, however, feel that this will not present any essentially new difficulty and that the methods of [3, 4] will be applicable to this problem also.

Similar remarks hold concerning the determination of the other λ's, in Cases (ii)-(vii), under (2.1). For Cases (iii), (vi) and (vii), where we have two constants to determine since the inference procedures are two-sided in each of these cases, in addition to the conditions of a given probability for $W(H_0)$ under H_0, we may impose the condition of local unbiasedness of each of these tests. These two conditions taken together will enable us to determine both constants involved uniquely. As discussed in [3, 5, 9], for Case (vii), the condition of local unbiasedness implies certain optimum power properties of the test for this case. For the other two cases, however, such implications of the condition of local unbiasedness are yet to be established. Further, regarding all the λ's in (2.1), it should be noted that, in addition to depending on the conditions discussed above, they are also functions of p, n_1 and n_2.

Case (vii), as defined in Section 1, with the test given under Case (vii) of (2.1), is the one that has been considered in great detail elsewhere [5, 6, 7, 8] and is included here merely for completeness.

Finally, it can be seen that all the probabilities (under H_0) associated with the procedures proposed under (2.1) are independent of nuisance parameters.

3. Associated confidence bounds. Given a pair (H_0, H) of composite hypothesis and alternative, disjoint but not necessarily exhaustive, we seek a parametric function that might be regarded as a measure of departure from H_0 in the direction of H, or, alternatively, some kind of a distance function between the set H_0 and the set H. We next week a confidence interval for this parametric function, one-sided (one way or the other) or two-sided, depending upon the nature of the pair (H_0, H). No claim is made at this stage that the parametric function chosen or the confidence interval proposed for it is in some sense optimal. As to the confidence coefficient, it would be very desirable if, given any permissible $1 - \alpha$, the interval could be defined such that this coefficient were equal to $1 - \alpha$. If it does not turn out that way, the next best thing would be to have a confidence coefficient $\geqq 1 - \alpha$, given any permissible α, such that the equality is attained, or, in other words, that the probability of the interval covering the parametric function, for some value of this function, is equal to $1 - \alpha$. If this does not happen, the next best thing would be, for any permissible $1 - \alpha$, to have a confidence coefficient whose greatest lower bound $\geqq 1 - \alpha$ (and might, in fact, be greater than $1 - \alpha$), provided that the interval itself is not trivial, for example, $(0, \infty)$ or $(-\infty, \infty)$, etc., but is, in fact, much better than these. We shall say that such a confidence coefficient is a conservative one, or alternatively, such a confidence region is a conservative one. For really complex problems even this may be difficult to obtain, to say nothing of intervals of the first or the second kind, and we would consider even this quite worthwhile, especially in view of the fact that we consider it more important to estimate this "distance function," pointwise or intervalwise, than to test (and accept or reject) the usual null hypothesis as such. All confidence intervals obtained in this section are conservative.

For Case (i), (H_0, H_1), we have a lower bound on γ_m, for Case (ii), (H_0, H_2), an upper bound on γ_M and for Case (iii) a lower bound on γ_m or an upper bound on γ_M. The techniques used are the same as those in [8, 9]; in fact, the bounds are only modifications or adaptations of the bounds given there, with a different interpretation relative to these new situations. Hence the results will be stated without any proofs. These techniques are based essentially on a certain type of "inversion" of tests or inference procedures, appropriate to different (H_0, H_i) pairs, extensively used by workers in this area, including the authors.

For Cases (iv)-(vii), that is, for (H_0, H_i) $(i = 4, 5, 6, 7)$, using the same tools, we have attempted but have failed so far to obtain a lower bound on γ_M for Case (iv), an upper bound on γ_m for Case (v), a lower bound on γ_M and an upper bound on γ_m for Case (vi). The trouble seems to stem from the difficulty in obtaining a lower bound on γ_M that is not also a lower bound on γ_m or an upper bound on γ_m that is not also an upper bound on γ_M. However, we find that, if instead of γ_m and γ_M we consider, respectively, $\gamma_m^* = \mathrm{ch}_m(\mathbf{\Sigma}_1)/\mathrm{ch}_M(\mathbf{\Sigma}_2)$ and $\gamma_M^* = \mathrm{ch}_M(\mathbf{\Sigma}_1)/\mathrm{ch}_m(\mathbf{\Sigma}_2)$, then bounds on γ_m^* and γ_M^* become feasible by using the techniques of [1]. The question now is, how are these intervals related to (H_0, H_i) $(i = 4, 5, 6, 7)$? For example, how is $[\gamma_M^* \geqq \mu]$ related to (H_0, H_4)? In our sense, it is not a natural associate of (H_0, H_4). If we consider $H_0^*: \mathbf{\Sigma}_1 = \mathbf{\Sigma}_2 = \delta\mathbf{I}$ (a diagonal matrix with all diagonal elements equal to δ) and $H_4^*: \gamma_M^* > 1$, we observe that $H_0^* \subset H_0$ and $H_4^* \supset H_4$, and $[\gamma_M^* \geqq \mu]$ is really a natural associate of (H_0^*, H_4^*). The appropriate bounds on the γ^*'s relevant to the pairs (H_0^*, H_i^*), $(i = 4, 5, 6, 7)$, are given below in summary form. The reader must keep in mind that these bounds are not the natural associates of the inference procedures, given under (2.1), for the Cases (iv)-(vii). These bounds are given just to inform the reader what to expect if one is tempted to try, in the first instance, the techniques of [1].

With regard to the "inversion" of tests or inference procedures mentioned above, we have realized relatively recently that for obtaining the confidence bounds that are the natural associates of any particular (H_0, H_i), the "inversion" may not necessarily be the best means (according to some interpretation of "best" at any rate). It may not even be a feasible means. For certain (H_0, H_i) pairs, other techniques might be better in some sense, or even might provide bounds when the techniques we have been using, fail. In this paper, however, we stay with the techniques of [1, 8, 9].

For Cases (i)-(iii), we can use either pp. 107–109 of [9] or the formula (2.3) of [8] (with a correction to be presently indicated), fill in some details of the proof from [10] and obtain the conservative confidence bounds in the respective forms

$$
\text{(3.1)} \qquad
\begin{aligned}
&\text{(i)} && \gamma_m \geqq c_m/\lambda_1, \\
&\text{(ii)} && \gamma_M \leqq c_M/\lambda_2, \\
&\text{(iii)} && \gamma_m \geqq c_m/\lambda_3, \quad \text{or} \quad \gamma_M \leqq c_M/\lambda_3', \qquad (\lambda_3 < \lambda_3').
\end{aligned}
$$

Here λ_1, λ_2, λ_3 and λ_3' are constants defined under (2.1) and depend on the conservative confidence coefficient and on p, n_1 and n_2.

The correction to be made in (2.3) of [8] is to replace the prefactor $\mathbf{S}_1$, in each expression where it occurs, by a matrix prefactor $\mathbf{T}'$ and add a matrix postfactor $\mathbf{T}$, where $\mathbf{S}_1 = \mathbf{TT}'$ and $\mathbf{T}$ is a lower triangular matrix. Thus (2.3) of [8] should read as

$$
\frac{\lambda_1 \mathbf{a}'(\mathbf{T}'\mathbf{S}_2^{-1}\mathbf{T})\mathbf{a}}{\mathbf{a}'\mathbf{a}} \geqq \frac{\mathbf{a}'(\mathbf{T}'\mathbf{D}_{\gamma_i^{\frac{1}{2}}}\mathbf{S}_1^{-1}\mathbf{D}_{\gamma_i^{\frac{1}{2}}}\mathbf{T})\mathbf{a}}{\mathbf{a}'\mathbf{a}} \geqq \frac{\lambda_2 \mathbf{a}'(\mathbf{T}'\mathbf{S}_2^{-1}\mathbf{T})\mathbf{a}}{\mathbf{a}'\mathbf{a}},
$$

for all nonnull $\mathbf{a}$.

The confidence bounds we obtain while attempting (and failing) to find the natural associates of (H_0, H_i), $(i = 4, 5, 6, 7)$, are of the respective forms

$$
\begin{aligned}
&\text{(iv)} && \gamma_M^* \geqq \nu_1 \operatorname{ch}_m(\mathbf{S}_1)/\operatorname{ch}_M(\mathbf{S}_2), \\
&\text{(v)} && \gamma_m^* \leqq \nu_2 \operatorname{ch}_M(\mathbf{S}_1)/\operatorname{ch}_m(\mathbf{S}_2), \\
\text{(3.2)}\quad &\text{(vi)} && \gamma_m^* \leqq \nu_3 \operatorname{ch}_M(\mathbf{S}_1)/\operatorname{ch}_m(\mathbf{S}_2) \quad \text{and} \quad \gamma_M^* \geqq \nu_3' \operatorname{ch}_m(\mathbf{S}_1)/\operatorname{ch}_M(\mathbf{S}_2), \\
& && \qquad (\nu_3' > \nu_3), \\
&\text{(vii)} && \gamma_m^* \leqq \nu_4 \operatorname{ch}_M(\mathbf{S}_1)/\operatorname{ch}_m(\mathbf{S}_2) \quad \text{and/or} \quad \gamma_M^* \geqq \nu_4' \operatorname{ch}_m(\mathbf{S}_1)/\operatorname{ch}_M(\mathbf{S}_2), \\
& && \qquad (\nu_4' > \nu_4).
\end{aligned}
$$

Here again, the ν's depend on the conservative confidence coefficient and on p, n_1 and n_2, and the techniques used in deriving the bounds under (3.2) are those of [1]. Also, γ_m^* and γ_M^* have been defined and discussed earlier.

The "partials" or "truncated" versions of (3.1), in the sense of [8, 9, 10], are easily obtained exactly as in [8, 9]. The "partials" for (3.2) are not so available.

4. Concluding remarks. The procedures proposed here are heuristic, and investigations are underway as to the properties of these procedures, as, for example, unbiasedness, monotonicity and admissibility for the two-decision procedures and analogous properties of the three-decision procedures. Such properties have already been established for some of the two-decision procedures, including Case (vii) of (2.1). Also under consideration is a generalization to the case of more than two dispersion matrices.

However, the more urgent and immediate problems are, if possible, to obtain (a) the meaningful bounds on γ_m and γ_M (for Cases (iv-(vii)) that we sought but could not present in this paper and (b) the greatest lower bound on the conservative confidence coefficients obtained so far.

Another problem of some statistical interest is one in which, with the same setup as the one considered in this paper, we are interested in comparing two mean vectors instead of two dispersion matrices. This has wide applications, including some in genetics, and is under the active consideration of a number of people including the authors.

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