ON NECESSARY CONDITIONS FOR THE EXISTENCE OF SOME SYMMETRICAL AND UNSYMMETRICAL TRIANGULAR PBIB DESIGNS AND BIB DESIGNS[1]

By Esther Seiden

Michigan State University

1. Summary. Consider PBIB designs based on triangular association scheme with $v = n(n-1)/2, b = (n-1)(n-2)/2, k = n, r = n-2, \lambda_1 = 1, \lambda_2 = 2$. It is established here that a necessary condition for the existence of these PBIB designs is the existence of symmetrical triangular PBIB designs with $v = b = (n-1)(n-2)/2, r = k = n-2, \lambda_1 = 1, \lambda_2 = 2$. Atiqullah [1] showed that the same condition is necessary for the existence of BIB designs with $v = (n-1)(n-2)/2, b = n(n-1)/2, k = n-2, r = n, \lambda = 2$. It is shown further that for an infinite number of values for n this condition cannot be satisfied.

2. Introduction. It is well known that for the triangular PBIB designs with $v = (n-1)(n-2)/2$

$$|NN'| = \rho_0\rho_1^{n-2}\rho_2^{(n-1)(n-4)/2} \tag{1}$$

where N denotes the incidence matrix of the design and the ρ's are the characteristic roots. If the design is symmetric $|NN'|$ has to be a perfect square. Ogava [3] showed that a necessary condition for $|NN'|$ to be a perfect square is that

$$O_p = (-1, \rho_1)_p^{(n-2)(n-3)/2}(\rho_1, n-1)_p(\rho_1, n-2)_p^{n-2}(-1, \rho_2)_p^{(n-1)(n-2)(n-3)(n-4)/8} \cdot (\rho_2, 2)_p(\rho_2, n-2)_p(\rho_2, n-3)_p^{n-2} = +1 \text{ for all primes } p,$$

where the expressions of the form $(\alpha, \beta)_p$ are the Hilbert symbols. It will be shown that, for an infinite number of values of n, $O_p = -1$.

3. Conditions for the existence of some PBIB designs.

Lemma 1. *If there exists a PBIB design based on a triangular association scheme with $v = n(n-1)/2, b = (n-1)(n-2)/2, k = n, r = n-2, \lambda_1 = 1, \lambda_2 = 2$, then each of the blocks contains exactly one pair of the $(n-1)(n-1)/2$ pairs of each of the $n-1$ mutually first associate varieties appearing in the same row of the association scheme.*

Proof. Assume without loss of generality that a row of the association scheme contains the varieties $1, 2, \cdots, n-1$. Suppose further that there is a block of the design which contains the varieties $1, 2, \cdots k, k > 2$. Then there must be $(n-3)k$ additional blocks each containing exactly one of the varieties 1 through k. Since each of the varieties has to appear $n-2$ times the minimum number of blocks to be added is equal to the sum of the integers from $n-2-k$ down to 1. This sum, added to the sum obtained from the blocks previously counted,

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which amounted to $(n-3)k+1=(n-2)+(k-1)(n-2)$, clearly exceeds the sum of the integers from $n-2$ down to 1 which is equal to the total number of available blocks. Hence we have derived a contradiction.

Let us illustrate this lemma with an example. Take $n=5$. Then $v=10$, $b=6$, $k=5$, $r=3$, $\lambda_1=1$, $\lambda_2=2$. Let the varieties 1, 2, 3, 4 be mutually first associates. Suppose that 1, 2, 3 appear in one block. Then there would have to be additional blocks which can be divided into three pairs each containing exactly one of the varieties 1, 2, 3. Hence the number of blocks would have to be seven, which contradicts the assumption.

LEMMA 2. *A necessary condition for the existence of a triangular PBIB design with* $v=n(n-1)/2$, $b=(n-1)(n-2)/2$, $k=n$, $r=n-2$, $\lambda_1=1$, $\lambda_2=2$ *is the existence of a triangular PBIB design with* $v=b=(n-1)(n-2)/2$, $r=n-2$, $\lambda_1=1$, $\lambda_2=2$.

PROOF. Suppose that the first row of the association scheme includes the varieties $1, 2, \cdots, n-1$. By Lemma 1 each of the blocks of the design must contain two elements from this row. Cross out from each block the two elements belonging to the first row of the association scheme. The remainder will clearly form the design satisfying the necessary condition of the lemma.

LEMMA 3. (Atiqullah [1]) *A necessary and sufficient condition for the existence of BIB design for* $n \geqq 5$ *with* $v=(n-1)(n-2)/2$, $b=n(n-1)/2$, $k=n-2$, $r=n$, $\lambda=2$ *is that there exists a symmetrical triangular PBIB design with* $v=b=(n-1)(n-2)/2$, $r=k=n-2$, $\lambda_1=1$, $\lambda_2=2$.

LEMMA 4. *A necessary condition for the existence of a triangular PBIB design with* $v=b=(n-1)(n-2)/2$, $r=k=n-2$, $\lambda_1=1$, $\lambda_2=2$ *is that one of the following conditions holds*:

(i)	$n-2$ is a perfect square	
(ii)	$n=4k+1$ and $(4k-1,2)_p(4k-1,-1)_p=1$	k an odd integer
(iii)	$n=4k+1$ and $(4k-1,2)_p=1$	k an even integer
(iv)	$n=4k$ and $(4k-2,2)_p(4k-2,-1)_p=1$	k an odd integer
(v)	$n=4k$ and $(4k-2,2)_p=1$.	k an even integer

PROOF. Since the design is symmetric $|NN'|$ has to be a perfect square. Moreover for the triangular PBIB design $\rho_0=rk$, $\rho_1=r+(n-5)\lambda_1-(n-4)\lambda_2$, $\rho_2=r-2\lambda_1+\lambda_2$. Thus in this case $\rho_0=r^2$, $\rho_1=1$, $\rho_2=n-2$. Hence

$$|NN'|=(n-2)^{(n-1)(n-4)/2+2}.$$

Clearly exactly one of the expressions $n-1$, $n-4$ is even. Thus if $n-2$ is not a perfect square n has to assume one of the values listed in (ii) through (iv). If $n-2$ is a perfect square then the quantity O_p calculated by Ogava is always equal to $+1$ and thus no further information is obtained. In the remaining cases the condition $O_p \equiv 1$ reduces to the corresponding conditions listed beside the values of n.

THEOREM 1. *Conditions* (i) *through* (v) *of Lemma* 4 *are necessary conditions for the existence of unsymmetrical PBIB triangular designs with* $v=n(n-1)/2$,

$b = (n-1)(n-2)/2$, $k = n$, $r = n-2$, $\lambda_1 = 1$, $\lambda_2 = 2$ *and BIB designs with* $v = (n-1)(n-2)/2$, $b = n(n-1)/2$, $k = n-2$, $r = n$, $\lambda = 2$.

PROOF. The proof follows immediately from the previously stated lemmas.

4. Construction of a PBIB design of triangular type. A PBIB symmetrical triangular design will be presented which does not seem to appear elsewhere. The parameters of this design are $v = b = 28$, $k = r = 7$, $\lambda_1 = 1$, $\lambda_2 = 2$. This design was obtained from the design T34 in [2] by the method of Lemma 1.

1	2	12	16	22	25	26	3	5	11	12	18	19	25
1	3	11	15	17	20	28	3	6	10	13	15	26	27
1	4	9	17	18	23	27	3	7	8	9	18	24	26
1	5	13	14	15	22	24	3	8	10	16	22	23	28
1	6	10	16	18	19	21	4	5	9	13	16	21	28
1	7	14	19	23	26	28	4	6	8	11	14	25	28
1	8	20	21	24	25	27	4	7	10	12	17	20	22
2	3	13	17	21	23	25	4	11	15	18	21	22	26
2	4	8	13	19	20	26	5	6	8	17	19	22	27
2	5	10	18	20	24	28	5	7	8	12	15	21	23
2	6	9	11	22	23	24	5	9	10	14	17	25	26
2	7	10	11	14	21	27	6	7	9	15	16	20	25
2	9	12	15	19	27	28	6	12	13	14	18	20	23
3	4	12	14	16	24	27	7	11	13	16	17	19	24

Examples of non-existent PBIB triangular designs and BIB designs.

(1) $v = b = 15$, $r = k = 5$, $\lambda_1 = 1$, $\lambda_2 = 2$

(2) $v = 21$, $b = 15$, $r = 5$, $k = 7$, $\lambda_1 = 1$, $\lambda_2 = 2$

(3) $v = 15$, $b = 21$, $k = 5$, $r = 7$, $\lambda = 2$

In these three cases $n = 7$; hence none of the conditions of Lemma 4 is satisfied.

(4) $v = b = 630$, $r = k = 35$, $\lambda_1 = 1$, $\lambda_2 = 2$

(5) $v = 666$, $b = 630$, $r = 35$, $k = 37$, $\lambda_1 = 1$, $\lambda_2 = 2$

(6) $v = 630$, $b = 666$, $r = 37$, $k = 35$, $\lambda = 2$

Here $n = 37$, i.e., n is of the form $4k + 1$, k an odd integer, but $(35, 2)_5(35, -1)_5 = -1$.

(7) $v = b = 120$, $r = k = 15$, $\lambda_1 = 1$, $\lambda_2 = 2$

(8) $v = 136$, $b = 120$, $r = 15$, $k = 17$, $\lambda_1 = 1$, $\lambda_2 = 2$

(9) $v = 120$, $b = 136$, $r = 17$, $k = 15$, $\lambda = 2$

In these cases $n = 17$; hence n is of the form $4k + 1$, k an even integer, but $(15, 2)_3 = -1$.

(10) $v = b = 55$, $r = k = 10$, $\lambda_1 = 1$, $\lambda_2 = 2$

(11) $v = 66$, $b = 55$, $r = 10$, $k = 12$, $\lambda_1 = 1$, $\lambda_2 = 2$

(12) $v = 55$, $b = 66$, $r = 12$, $k = 10$, $\lambda = 2$

Here $n = 12$; hence n is of the form $4k$, k an odd integer, and $(10, 2)_5(10, -1)_5 = -1$.

(13) $v = b = 21$, $r = k = 6$, $\lambda_1 = 1$, $\lambda_2 = 2$

(14) $v = 28, b = 21, r = 6, k = 8, \lambda_1 = 1, \lambda_2 = 2$

(15) $v = 21, b = 28, r = 8, k = 6, \lambda = 2$

This time $n = 8$, of the form $4k$, k even and $(6, 2)_3 = -1$. Notice that in the examples n assumes the smallest possible value which violates each of the conditions being considered.

The nonexistence of the BIB designs exemplified here could be obtained by other methods. It would be interesting to know whether one could establish the nonexistence of a BIB using the method of this paper which could not be shown by other already known methods.

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