and

$$\sum_{k=0}^{n} k/(k+1)! = 1 - 1/(n+1)!. \tag{9}$$

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TAIL AREAS OF THE t-DISTRIBUTION FROM A MILLS'-RATIO-LIKE EXPANSION[1]

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1. Introduction. In planning a Monte Carlo study the authors found it would be necessary to have percentage points of the t-distribution, at levels of $10^{-4}\%$ and smaller, for relatively large degrees of freedom. It seemed reasonable to look for an asymptotic expansion analogous to Mills' ratio [2] for the normal.

This note establishes the validity of the asymptotic expansion

$$\int_x^\infty \left(1+\frac{t^2}{n}\right)^{-\frac{n+1}{2}} dt = \sum_{j=1}^{m} u_j + R_m(x), \tag{1}$$

where

$$u_1 = \frac{n}{n-1}\left(1+\frac{x^2}{n}\right)^{-\frac{n-1}{2}}\frac{1}{x}, \qquad u_{j+1} = u_j\left(1+\frac{n}{x^2}\right)\frac{2j-1}{2j+1-n},$$

$$j = 1, 2, \cdots, m-1,$$

$$|R_m(x)| \leqq |u_m|$$

and where $n > 2m - 1$.

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As n tends to infinity the above expansion approaches the Mills' ratio expansion for the normal (after allowing for the normalizing constant which has been omitted.)

Numerical examples are given to illustrate the quality of the approximation.

The density for the t-distribution is proportional to

$$\left(1 + \frac{t^2}{n}\right)^{-\frac{n+1}{2}}.$$

Since

$$\frac{\Gamma\left(\frac{n+1}{2}\right)}{\left(1 + \frac{t^2}{n}\right)^{\frac{n+1}{2}}} = \int_0^\infty u^{\frac{n-1}{2}} \exp\left[-\left(1 + \frac{t^2}{n}\right) u\right] du,$$

it follows that

$$(2) \quad \int_x^\infty \left(1 + \frac{t^2}{n}\right)^{-\frac{n+1}{2}} dt = \int_0^\infty e^{-u} \left[u^{\frac{n-1}{2}} \Big/ \Gamma\left(\frac{n+1}{2}\right)\right] du \int_x^\infty e^{-\frac{ut^2}{n}} dt.$$

The Mills' ratio expansion may be substituted for the last integral on the right of Equation (2), and Equation (1) results from integrating term by term.

The fact that $|R_m| \leqq |u_m|$ follows from the analogous property possessed by Mills' ratio [2].

Since the Mills' ratio expansion overestimates the probability when an odd number of terms is used and underestimates it for an even number, the same is true of the expansion given in Equation (1). Thus it would be unwise to take additional terms unless

$$\left|\left(1 + \frac{n}{x^2}\right) \frac{2j - 1}{2j + 1 - n}\right| < 1$$

i.e., $j \leqq (n/2)[(1 + x^2)/(n + 2x^2]$.

It should perhaps be emphasized that the expansion as given in (1) contains at most a finite number of terms since it is required that $n > 2m - 1$.

2. Numerical Examples. Federighi [1] has given a table containing extreme percentage points for the t-distribution. These values may be used for comparison with the approximation furnished by Equation (1).

Table 1 was constructed from the quantiles provided in Federighi [1] utilizing no more than the first *three* terms of Equation (1). The relative error appears to decrease as the tail decreases, as one might expect.

3. Acknowledgements. The computations for Table 1 were done partly by Miss Dorothy Spinelli, partly by Mr. Armen Fisher, and partly by Mr. Howard Weiss. Their assistance is much appreciated.

TABLE 1

Exact and Approximate Tail Areas for the t-distribution with n Degrees of Freedom

Exact(*)	Approximation		
Tail Area	$n = 7$	$n = 15$	$n = 40$
.001	.000 816	.001 06	.001 02
.000 05	.000 042 8	.000 051 5	.000 050 3
.000 01	.000 008 66	.000 010 2	.000 010 05
.000 001	.000 000 873	.000 001 02	.000 001 003
.000 000 1	.000 000 087 7	.000 000 102	.000 000 100 1

(*) These tail areas are exact to the extent that Federighi's [1] tabled quantiles are exact.

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A FINITE CRITERION FOR INDECOMPOSABLE CHANNELS[1]

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Let M be the class of all $n \times n$ Markov matrices, $n \geqq 2$, and let $I \subset M$ be the set of all indecomposable matrices. A Markov matrix is indecomposable if, [3] p. 179, it contains only one ergodic class; or equivalently, if, [4] p. 355, it contains only one irreducible set. Let $A(1)$ be a non-empty subset of M, and for $k \geqq 1$ let $A(k)$ be the set of all $m \varepsilon M$ such that m can be expressed as a product of at most k, not necessarily distinct, elements of $A(1)$. Also let $A = U_1^\infty A(k)$. The following theorem clears up a point concerning indecomposable channels [1], [2], [5] p. 74.

THEOREM. *If* $A(2^{n^2}) \subset I$ *then* $A \subset I$.

PROOF. For $m \varepsilon M$ let m' be the $n \times n$ matrix of zeroes and ones obtained by replacing every positive entry of m by a one; and for $B \subset M$ let $B' = \{m' \mid m \varepsilon B\}$. Now if $a_i, b_i \varepsilon M$; $a_i' = b_i'$; $i = 1, 2, \cdots, k$ then $(a_1a_2 \cdots a_k)' = (b_1b_2 \cdots b_k)'$ because the (i, j)th entry $(a_1a_2 \cdots a_k)_{ij}$ of $(a_1a_2 \cdots a_k)$ is positive if and only if there exists a sequence of states $i_1, i_2, \cdots, i_{k-1}$ such that $(a_1)_{ii_1}(a_2)_{i_1i_2} \cdots (a_k)_{i_{k-1}j} > 0$. Also, clearly, $B \subset I$ if and only if $B' \subset I'$; i.e., the locations of the

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