be the probability that $f_n \leqq \alpha$. Then

$$\epsilon \leqq E(f_n) \leqq \alpha p(\alpha) + 1 - p(\alpha)$$

so $p(\alpha) \leqq (1 - \epsilon)/(1 - \alpha)$. Thus,

$$\text{prob } (\textstyle\sum_1^\infty \chi_i a_i \geqq \alpha \sum_1^n a_i) \geqq \text{prob } (f_n \geqq \alpha)$$
$$\geqq 1 - p(\alpha) \geqq 1 - [(1 - \epsilon)/(1 - \alpha)].$$

Letting n go to ∞ and then α go to zero completes the proof.

THEOREM 3. *Let (θ_i) be a sequence of independent normalized Gaussian random variables and let the η_i's be independent of them. If $(\theta_i + \eta_i)$ is equivalent to (θ_i) and if m_i is any set of numbers with* $\text{prob}(\eta_i^2 \geqq m_i) \geqq \epsilon > 0$, *then $\sum m_i^2 < \infty$ and $\sum m_i \eta_i^2 < \infty$ with probability one.*

PROOF. The second assertion will follow from the first by Theorem 2. If $\sum m_i^2 = \infty$, we can choose a set of numbers β_i to satisfy $|\beta_i m_i| < 1$, $\sum \beta_i^2 m_i^2 < \infty$ and $\sum \beta_i m_i^2 = \infty$. From Theorem 2 we have $\sum \beta_i m_i \eta_i^2 < \infty$ with probability one. If χ_i is the characteristic function of $\eta_i^2 \geqq m_i$, then $\sum \beta_i m_i \eta_i^2 \geqq \sum \beta_i m_i^2 \chi_i$ and the previous lemma gives a contradiction, completing the proof.

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NOTE ON TWO BINOMIAL COEFFICIENT SUMS FOUND BY RIORDAN[1]

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In a recent paper on enumeration of graphs Riordan [7] has noted the following two combinatorial identities:

$$\sum_{k=0}^{n-1} \binom{n-1}{k} n^{n-1-k}(k+1)! = n^n \tag{1}$$

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and

$$\sum_{k=0}^{n} \binom{n}{k} (n - k)^{n-k}(k - 1)^{k-1} = -(n - 1)^n. \tag{2}$$

Of the first formula Riordan remarks that it "is one of the forms associated with Abel's generalization of the binomial formula." However, since the Abel type formula is in general of the form

$$\sum_{k=0}^{n} \binom{n}{k} (a + bk)^k c(c + b(n - k))^{n-k-1} = (a + c + bn)^n \tag{3}$$

it is evident that formula (1) is something rather different than the Abel formula, and formula (2) instead is of the Abel type. Formulas of the Abel type have appeared frequently in papers on statistics and combinatorics, [1], [2] for example. The Abel formulas are special cases of corresponding binomial coefficient convolutions as shown in [3], [5], [6]. A very valuable reference on Abel formulas is a little-known paper by Hans Salié [8]. In view of these observations, it may be of interest to show that formula (1) also follows from some more general summation relations for powers of binomial coefficients.

Using the technique outlined in [4] we note the formula

$$\sum_{k=0}^{n} \Delta f(k) = f(n + 1) - f(0), \quad \text{with} \quad \Delta f(k) = f(k + 1) - f(k). \tag{4}$$

With

$$f(k) = \binom{x}{k}^p (k!)^p y^{-kp} \tag{5}$$

it is readily found that

$$\Delta f(k) = \binom{x}{k}^p (k!)^p y^{-p(k+1)}[(x - k)^p - y^p], \tag{6}$$

so that we find the general formula involving powers of binomial coefficients

$$\begin{aligned} &\sum_{k=0}^{n} \binom{x}{k}^p (k!)^p y^{(n-k)p}[(x - k)^p - y^p] \\ &\qquad = \binom{x}{n+1}^p (n + 1)!^p - y^{(n+1)p}, \quad n \geqq 0, \end{aligned} \tag{7}$$

valid for all real x and y, and for positive or negative powers p.

It is readily verified that when $p = 1$, $x = n - 1$, $y = n$ this formula yields Riordan's relation (1). However the writer has not found any similar simple consideration which would yield (2) from the summation formula (4). It should also be observed that the basic formula in [4] is the special case of (7) above when $y = x$. Relation (7) also contains the two well known formulas

$$\sum_{k=0}^{n} k \cdot k! = (n + 1)! - 1 \tag{8}$$

and

$$\sum_{k=0}^{n} k/(k+1)! = 1 - 1/(n+1)!. \tag{9}$$

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TAIL AREAS OF THE t-DISTRIBUTION FROM A MILLS'-RATIO-LIKE EXPANSION[1]

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1. Introduction. In planning a Monte Carlo study the authors found it would be necessary to have percentage points of the t-distribution, at levels of $10^{-4}\%$ and smaller, for relatively large degrees of freedom. It seemed reasonable to look for an asymptotic expansion analogous to Mills' ratio [2] for the normal.

This note establishes the validity of the asymptotic expansion

$$\int_x^\infty \left(1 + \frac{t^2}{n}\right)^{-\frac{n+1}{2}} dt = \sum_{j=1}^{m} u_j + R_m(x), \tag{1}$$

where

$$u_1 = \frac{n}{n-1}\left(1 + \frac{x^2}{n}\right)^{-\frac{n-1}{2}} \frac{1}{x}, \qquad u_{j+1} = u_j \left(1 + \frac{n}{x^2}\right) \frac{2j-1}{2j+1-n}, \qquad j = 1, 2, \cdots, m-1,$$

$$|R_m(x)| \leqq |u_m|$$

and where $n > 2m - 1$.

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