NOTES

THE CONVEX HULL OF PLANE BROWNIAN MOTION

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Denote by $Z(t, \omega)$ the Brownian motion in the plane starting at the origin. Let $J(\omega)$ be the least closed convex set containing $Z(t, \omega)$, $0 \leqq t \leqq 1$, let $K(\omega)$ be the boundary of $J(\omega)$, and let $p(s, \omega)$ be the point on $K(\omega)$ at a distance s along $K(\omega)$ in the counterclockwise direction from the intersection of $K(\omega)$ with the positive x-axis. Let $\theta(s, \omega)$ be the angle made by the tangent to $K(\omega)$ at $p(s, \omega)$ and the x-axis when such a tangent exists. Define

$$\alpha(s, \omega) = [\theta(s, \omega) - \theta(0, \omega)]/2\pi$$

for points where $\theta(s, \omega)$ is defined. Elsewhere let $\alpha(s, \omega) = \lim_{t \uparrow s} \alpha(t, \omega)$. As s increases from 0^- to $l(\omega)$, the length of $K(\omega)$, $\alpha(s, \omega)$ increases from 0 to 1. Hence $\mu[E, \omega] = \int_E d\alpha(s, \omega)$ is a completely additive probability measure on Borel sets in $[0, l(\omega)]$. P. Levy [3] introduced $J(\omega)$ and showed $\mu[E, \omega]$ to be singular with respect to Lebesgue measure. The purpose of this paper is to show that this measure can almost always be concentrated on a set $T(\omega)$ which is small in the sense of Hausdorff measures.

Definition. Let $h(t)$ satisfy (A), namely, be a positive monotone continuous function with $h(0) = 0$. Let $h_\rho^*(E)$ be the greatest lower bound of $\sum_{i>0} h(\text{diam}\, O_i)$ where the greatest lower bound is taken over all sets $\{O_i\}$ of circles with diameter less than ρ covering E. Define $h^*(E)$, the h-measure of E, by $h^*(E) = \lim_{\rho \to 0} h_\rho^*(E)$.

It is known that $h^*(E)$ is an outer measure in the sense of Carathéodory. For a general discussion of the properties of these measures see [2].

Theorem. *Let $h(t)$ satisfy A and*

$$\lim_{t \to 0} h(t) \log 1/t = 0. \tag{1}$$

For almost all ω there exists a set $T(\omega)$ in $[0, l(\omega)]$ for which $\mu[T(\omega), \omega] = 1$ and $h^*(T(\omega)) = 0$.

Proof. The proof rests on the following result of Baxter [1]. Let $\{X_i\}$, $i = 1, 2, \cdots$ be independent, identically distributed, complex-valued random variables with uniform angular distribution.[2] Let $S_0 = 0$, $S_i = \sum_{k=1}^{k=i} X_k$. If H_m is the

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[2] Baxter did not use the last restriction.

number of sides in the boundary of the complex hull of $\{S_0, S_1, \cdots, S_m\}$, then

$$(2) \qquad E[H_m] = 2\sum_{i=1}^{i=m} 1/i \cong 2\log m.$$

Let $J_n(\omega)$ be the convex hull of $\{Z(i/2^n, \omega), i = 1, 2, \cdots, 2^n\}$, $K_n(\omega)$ its boundary, and $M_n(\omega)$ the number of sides in $K_n(\omega)$. The

$$\{Z(i/2^n, \omega) - Z((i-1)/2^n, \omega), i = 1, \cdots, 2^n\}$$

satisfy the conditions on the $\{X_i\}$ of the theorem of Baxter, so

$$E[M_n(\omega)] = 2\sum_{i=1}^{i=2^n} 1/i \cong 2n \log 2.$$

Let $v(n) = a(n)/h(2\pi \cdot 2^{-n/6}) \log 2^n$. By (1), $a(n)$ can so be chosen that $\lim_{n\to\infty} a(n) = \lim_{n\to\infty} v(n)/n = 0$, and $\lim_{n\to\infty} v(n) = \infty$. Let $\{n_i\}$ be a subsequence for which $\sum a(n_i) < \infty$ and $\sum 1/v(n_i) < \infty$. Since $M_n(\omega) > 0$, Prob $\{M_n(\omega) \geqq v(n)E[M_n(\omega)]\} < 1/v(n)$. By the Borel-Cantelli lemma,

$$(3) \qquad M_{n_i}(\omega) < v(n_i)E[M_{n_i}(\omega)] \cong 2n_i v(n_i) \log 2,$$

for all but a finite number of i, for almost all ω.

For linear Brownian motion, Levy [3] has shown

$$\limsup_{s\to 0} |Z(t+s) - Z(t)|/[2s \log(/s)]^{\frac{1}{2}} = 1$$

uniformly in t, with probability one. It follows from this that for plane Brownian motion $\lim_{s\to 0} |Z(t) - Z(t-s)| s^{-1/3} = 0$ uniformly in t, with probability one.

We let $J_n^*(\omega) = \{p \mid \text{distance } (p, J_n(\omega)) < 2^{-n/3}\}$ and call its boundary $K_n^*(\omega)$. By Levy's result, $Z_t(\omega) \subset J_n^*(\omega)$, $0 \leqq t \leqq 1$, for large n, with probability one. Hence, $J(\omega) \subset J_n^*(\omega)$. Obviously, $J(\omega) \supset J_n(\omega)$. Hence $K(\omega) \subset J_n^*(\omega) \cap cJ_n(\omega)$,[3] which is a strip about $J_n(\omega)$ of width $2^{-n/3}$. Let the vertices of $K_n(\omega)$ be $V_1, V_2, \cdots$. About V_j as center, we put the circle $C(j, n)$ of radius $2^{-n/6}$. Let $A(j, n) = \{s \mid p(s, \omega) \,\varepsilon\, C(j, n)\}$. Let the change of angle of the tangent to $K_n(\omega)$ at V_j be α_j.

Suppose $A(j, n) \cap A(j+1, n)$ and $A(j, n) \cap A(j-1, n)$ to be empty. Since $K(\omega)$ is contained in $J_n^*(\omega) \cap cJ_n(\omega)$, a strip of width $2^{-n/3}$, and $C(j, n)$ has radius $2^{-n/6}$, it is not difficult to see that $2\pi\mu[A(j, n), \omega] \geqq \alpha_j - 3 \cdot 2^{-n/6}$.

Likewise, if $A(k, n), A(k+1, n), \cdots, A(l, n)$ is a maximum sequence of successively overlapping segments, i.e., $A(k-1, n) \cap A(k, n)$ and $A(l, n) \cap A(l+1, n)$ are empty, $2\pi\mu[\bigcup_{j=k}^{j=l} A(j, n), \omega] \geqq \sum_{j=k}^{j=l} \alpha_j - 3 \cdot 2^{-n/6}$.

There are at most $M_n(\omega)$ maximum sequences of successively overlapping segments, and since $\sum_{j>0} \alpha_j = 2\pi$,

$$\mu[c\bigcup_{j>0} A(j, n), \omega] \leqq 3M_n(\omega)2^{-n/6}/2\pi.$$

[3] By $cJ_n(\omega)$ we mean the complement of $J_n(\omega)$.

Then for fixed ω not in one of the exceptional sets and large enough k,

$$\sum_{i>k} M_{n_i}(\omega)2^{-n_i/6} \leqq \sum_{i>k} 2n_i \log 2v(n_i)2^{-n_i/6} \leqq 2 \log 2 \sum_{n>n_k} n^2 2^{-n/6} < \infty.$$

We apply the Borel-Cantelli lemma with respect to the measure $\mu[E, \omega]$ to see that $\mu[T(\omega), \omega] = 1$, where $T(\omega) = \bigcap_{k=1}^{\infty} \bigcup_{i\geqq k} \bigcup_j A(j, n_i)$. Since $A(j, n_i)$ is part of the boundary of the convex set $J(\omega) \cap C(j, n_i)$, which is a subset of $C(j, n_i)$, the length, $|A(j, n_i)|$, of $A(j, n_i)$ is less than $2\pi\cdot 2^{-n_i/6}$. Take $\epsilon_k = 2\pi\cdot 2^{-n_k/6}$. We have:

$$h_{\epsilon_k}^*(T(\omega)) \leqq \sum_{i\geqq k} \sum_j h(|A(j, n_i)|) \leqq \sum_{i\geqq k} M_{n_i}(\omega)h(2\pi\cdot 2^{-n_i/6}).$$

From (3) and the properties of $v(n)$ and $a(n)$, we obtain

$$h_{\epsilon_k}^*(T(\omega)) \leqq \sum_{i>k} 2n_i v(n_i)h(2\pi\cdot 2^{-n_i/6}) \log 2$$

$$= 2 \log 2 \sum_{i>k} n_i a(n_i)h(2\pi\cdot 2^{-n_i/6})/h(2\pi\cdot 2^{-n_i/6}) \log 2^{n_i} = 2\sum_{i>k} a(n_i).$$

Since $\sum a(n_i) < \infty$, $\lim_{k\to\infty} h_{\epsilon_k}^*(T(\omega)) = 0$, so $h^*(T(\omega)) = 0$.

REMARK. From the uniformity of the Brownian motion, it would be surprising if $K(\omega)$ had actual corners. One might even suspect that if $k(t)$ satisfies (A) and $\lim_{t\to\infty} k(t) \log 1/t = \infty$, one would have $k^*(E) = \infty$ for any E such that $\mu[T(\omega) \cap E, \omega] > 0$.

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ON THE SAMPLE FUNCTIONS OF PROCESSES WHICH CAN BE ADDED TO A GAUSSIAN PROCESS

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Let $x(t)$ be a real measurable Gaussian process on an interval T with mean 0 and correlation function $R(s, t)$. We assume $\int_T \int_T R^2(s, t)dsdt < \infty$ so that $R(s, t)$ has an L_2 expansion $\sum \lambda_i\varphi_i(s)\varphi_i(t)$ with $\sum \lambda_i^2 < \infty$. We will write R for the compact integral operator gotten from $R(s, t)$. For any f satisfying $\int_T [R(t, t)]^{\frac{1}{2}} |f(t)| dt < \infty$ we can form the random variables $\theta(f, x) =$

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