DYNAMIC STOCHASTIC PROCESSES[1]

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1. Introduction and summary. The following paper introduces a new class of stochastic processes named *dynamic* stochastic processes. These processes have important applications to the study of dynamic economic systems and to the analysis of economic time series. In particular, they provide for time series analysis a theoretical basis for a systematic treatment of trend and, with only obvious modifications, of seasonal variation.

Dynamic processes belong to the class of second-order processes, the theory of which was formally developed first by Loève in 1945–46 [6]. Other notable contributions to this theory include Cramér's article on integral representations of stochastic processes [1], and Karhunen's "Über lineare Methoden der Wahrscheinlichkeitsrechnung" [5]. For a very interesting application of the general theory of Hilbert spaces to the study of stochastic processes the interested reader is also referred to Hida's article on canonical representation of Gaussian processes [3], and Parzen's recent article on time series analysis [7]. A most lucid treatment of the theory of wide sense stationary processes is found in Doob [2] for the univariate case and in Wiener and Masani [9] for the multivariate case.

The paper is divided in three sections. The first two develop systematically the theory of discrete and continuous, univariate, dynamic processes while the last section is devoted to vector-valued processes.

Before going any further I would like to express my gratitude to Professors H. P. McKean and H. Furstenberg. Their generous help and constructive criticism not only uncovered many misconceptions in earlier drafts of the paper but also guided my research in the right direction. Neither one has seen the final draft and the author alone is responsible for any remaining errors.

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2. Discrete dynamic stochastic processes. A stochastic process is a family of random variables, $\{x(t, w); t \varepsilon T\}$, defined on a probability space, $(\Omega, \mathfrak{F}, P)$, and measurable with respect to the Borel field, $\mathfrak{F}$, of w-sets in Ω. T is an index set. In the sequel $T = (\cdots, -1, 0, 1, \cdots)$ for discrete processes, and $T = (-\infty, +\infty)$ for continuous processes. The space, $L\{x(t, w); t \varepsilon T\}$, will denote the Hilbert space generated by the elements within the parenthesis. If $X_2 = L\{x(t, w); t \varepsilon T\}$, then X denotes the pre-Hilbert space of random variables of the form, $\sum_{j=1}^{n} b_j x(t_j, w)$, where $t_j \varepsilon T$ and the b_j's are complex constants.

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The norm in X_2 is denoted by $\| \|_{X_2}$. Finally the shift operators dealt with in this paper are assumed to be null preserving.[2] That a shift operator is uniquely defined will always be taken to mean that the operator is uniquely defined modulo a w-set of measure zero.

DEFINITION. Let $\{x(t); t \varepsilon T\}$ be a stochastic process such that $X_2 = L\{x(t, w); t \varepsilon T\}$ is well defined. We say that $x(t)$ is a discrete *dynamic* stochastic process if

(1) $x(t, w)$ can be represented by the relation, $x(t, w) = y(t, w) + f_x(t, w)$, where $y(t, w)$ is a wide sense stationary stochastic process, and $f_x(t, w)$ is a deterministic non-stationary stochastic process of the form,

$$f_x(t, w) = \sum_{j=1}^{a} \sum_{k=0}^{n_j - 1} A_{jk}(w)(z_j^t t^k), \quad \text{and}$$

(2) $X_2 = L_2 \oplus F_2$, where $L_2 = L\{y(t, w); t \varepsilon T\}$ and $F_2 = L\{f_x(t, w); t \varepsilon T\}$.

We assume throughout this paper that if $f_x(t, w)$ has the representation given above, then $|z_j| \neq 1, j = 1, \cdots, a$. That $f_x(t, w)$ is deterministic means that it is measurable with respect to its infinite past. Finally, the notation, $\oplus$, indicates that L_2 and F_2 are linearly independent.

THEOREM I. *If*

(1) $\{x(t); t \varepsilon T\}$ *is a stochastic process such that* $X_2 = L\{x(t, w); t \varepsilon T\}$ *is well defined*,

(2) *the shift operator*, S, *is uniquely defined on* X *by the relation*, $Sx(t, w) = x(t + 1, w)$, *can be extended to all of* X_2, *and has a continuous inverse*, S^{-1},

(3) *the complex polynomial* $M(z) = \sum_{j=0}^{n} b_j z^{-j}$ *has no roots of modulus equal to one*, *and* $x(t, w)$ *satisfies the relation*,

$$(1) \qquad M(S)x(t, w) = \eta(t, w), \quad \text{all} \quad t \varepsilon T,$$

where $\eta(t, w)$ *is a wide sense stationary stochastic process*, *then* $x(t, w)$ *is a discrete dynamic stochastic process.*

PROOF. That $L_2 = L\{\eta(t, w); t \varepsilon T\} \subset X_2$, and that L_2 reduces S is easily obtained from our assumptions. If S_1 is the restriction of S to L_2, then using Assumption (3) we see that S_1 is uniquely defined by the relation, $S_1\eta(t, w) = \eta(t + 1, w)$. Clearly S_1 is a unitary operator with $\|S_1\| = \|S_1^{-1}\| = 1$. It follows from the general theory of linear operators[3] that if $|a| \neq 1$, then $(a - S_1)^{-1}$ is a well defined bounded operator taking L_2 into L_2. Finally by expanding $M^{-1}(z)$ in partial fractions we see immediately that $M^{-1}(S_1)$ is a well defined bounded operator taking L_2 into itself.

From this we obtain that (1) has a uniquely defined solution in L_2,

$$y(t, w) = M^{-1}(S_1)\eta(t, w).$$

Obviously $L\{y(t, w); t \varepsilon T\} \subset L_2$, and the inverse relation follows from $\eta(t, w) = \sum_{j=0}^{n} b_j y(t - j, w)$. Hence $L\{y(t, w); t \varepsilon T\} = L_2$. Let $f_x(t, w)$ be defined by

[2] That S is null preserving means that S takes sets of measure zero into sets of measure zero.

[3] See for instance Taylor [8], p. 164, Theorem 4.1-C.

the equation, $x(t, w) - y(t, w) = f_x(t, w)$. If $f_x(t, w) = 0$ a.e. for all $t \varepsilon T$, we say that $x(t, w)$ is a centered dynamic process. Suppose not. Then $f_x(t, w)$ satisfies the relation,

$$M(S)f_x(t, w) = 0, \quad \text{all} \quad t \varepsilon T.$$

Hence making use of the general theory of difference equations we deduce that

$$(1) \qquad f_x(t) = \sum_{j=1}^{a} \sum_{k=0}^{n_j-1} A_{jk}(z_j^t t^k),$$

where $\{z_j\}_{j=1}^{a}$ are distinct roots of the equation, $\sum_{j=0}^{n} b_j z^{n-j} = 0$, and n_j is the multiplicity of z_j,

(2) the system of equations,

$$\left\{ f_x(t, w) = \sum_{j=1}^{a} \sum_{k=0}^{n_j-1} A_{jk}(z_j^t\, t^k) \right\}_{t=1}^{n},$$

can be solved uniquely for each A_{jk} in terms of the $f_x(t, w)$'s.

From this it follows that the coefficients A_{jk} are indeed random variables in X_2, and that

$$x(t, w) = y(t, w) + \sum_{j=1}^{a} \sum_{k=0}^{n_j-1} A_{jk}(w)(z_j^t t^k).$$

The process, $f_x(t, w)$, is clearly deterministic since for all $t \varepsilon T$,

$$f_x(t, w) = -\sum_{j=1}^{n} (b_j/b_0) f_x(t - j, w).$$

It remains to be shown that $X_2 = L_2 \oplus F_2$.

Suppose L_2 and F_2 are two linearly independent spaces and let $Y_2 = L_2 \oplus F_2$. Since L_2 is closed and F_2 is finite dimensional, we know[4] that Y_2 is closed. Clearly $X \subset Y_2$. This gives the relation $X_2 \subset Y_2$. Since $F_2 \subset X_2$ and $L_2 \subset X_2$, clearly $Y_2 \subset X_2$. Hence $X_2 = L_2 \oplus F_2$ if L_2 and F_2 are linearly independent.

That L_2 and F_2 are indeed linearly independent follows from the following considerations. Let $E = F_2 \cap L_2$. E is finite dimensional and clearly reduces S. Let $\{S_1^t u(w);\, t \varepsilon T\}$ be a wide sense stationary process in E. Then $u(t)$ has the representation,

$$u(t, w) = \sum_{j=1}^{p} e^{it\lambda_j} b_j(w),$$

where the $b_j(w)$'s are mutually orthogonal and the λ_j's are real. On the other side $u(t) \varepsilon F_2$, and hence satisfies the equation,

$$M(S)u(t, w) = 0, \quad \text{for all} \quad t \varepsilon T,$$

which is impossible since $M(z)$ has no root of modulus equal to one. Q.E.D.

LEMMA I. *If* (1) $\{x(t);\, t \varepsilon T\}$ *is a dynamic stochastic process, then there exists a*

[4] See for instance Zaanen [10], p. 97, Theorem 4.

linear difference operator, $M(S)$, with constant coefficients of lowest order such that $M(S)x(t, w) \varepsilon L_2$. $M(S)$ is uniquely determined up to a multiplicative constant, and $M(z)$ has no roots of modulus equal to one.

PROOF. Let S be the shift operator defined by the relation, $Sx(t, w) = x(t + 1, w)$, $t \varepsilon T$. S is clearly uniquely defined on X and can be extended to all of X_2. We note that it is easily deduced from the definition of dynamic processes that L_2 reduces S. Furthermore, from the general representation, $f_x(t, w) = \sum_{j=1}^{a} \sum_{k=0}^{n_j-1} A_{jk}(w)(z_j^t t^k)$, we see immediately that there exists at least one finite difference operator with constant coefficients taking $x(t, w)$ into L_2. Hence there must be at least one of lowest order. Suppose there are two, $M(S)$ and $N(S)$. It is clear that $a \cdot N(S)$, where a is an arbitrary constant, is a third operator with the same properties, and that $\{M(S) - a \cdot N(S)\}x(t, w) \varepsilon L_2$. If $M(z) = \sum_{j=0}^{n} b_j z^{-j}$ and $N(z) = \sum_{j=0}^{n} v_j z^{-j}$, we can choose a such that $b_0 = a \cdot v_0$. It is now evident that $\{M(S) - a \cdot N(S)\}$ is an operator of lower order than both $M(S)$ and $N(S)$ unless $M(S) = a \cdot N(S)$.

That $M(z)$ has no root of modulus equal to one follows from the assumption that $|z_j| \neq 1$ for all j in the representation of $f_x(t, w)$.

THEOREM II. *If (1) $\{x(t); t \varepsilon T\}$ is a discrete dynamic stochastic process, then it has one and only one centering function.*

PROOF. The only thing left to prove is the uniqueness of $y(t, w)$. Suppose that we have two operators, $M(S)$ and $N(S)$, neither of which has zeros of modulus equal to one, such that $x(t, w)$ satisfies the two relations,

$$M(S)x(t, w) = \eta(t, w), \quad \text{and}$$

$$N(S)x(t, w) = \xi(t, w).$$

We have to prove that $L\{\eta(t, w); t \varepsilon T\} = L\{\xi(t, w); t \varepsilon T\}$.

Clearly $N(S)\eta(t, w) = M(S)\xi(t, w)$, and if S_1^M and S_1^N are the restrictions of S to the two spaces respectively, then $\xi(t, w) = M^{-1}(S_1^M)N(S_1^M)\eta(t, w)$, which gives

$$L\{\xi(t, w); t \varepsilon T\} \subset L\{\eta(t, w); t \varepsilon T\},$$

and

$$\eta(t, w) = N^{-1}(S_1^N)M(S_1^N)\xi(t, w),$$

which gives the converse relation.

This together with the proof of Lemma I establishes our result.

3. Continuous dynamic stochastic processes. The preceding results can be extended to continuous processes in the following way.

DEFINITION. Let $\{x(t); t \varepsilon T\}$ be a stochastic process such that $X_2 = L\{x(t, w); t \varepsilon T\}$ is well defined, and such that $\lim_{t \to s} \|x(t, w) - x(s, w)\|_{X_2} = 0$. Let D be a closed differential operator on X_2 such that $\mathfrak{D}(D) = \{y(t); y(t) \varepsilon X_2$ and $Dy(t) \varepsilon X_2$, all $t \varepsilon T\}$. We say that $x(t)$ is a continuous dynamic stochastic process if

(1) $x(t, w)$ can be represented by the relation, $x(t, w) = y(t, w) + f_x(t, w)$, where $y(t, w)$ is a wide sense stationary stochastic process, and $f_x(t, w)$ is a deterministic non-stationary stochastic process of the form,

$$f_x(t, w) = \sum_{j=1}^{a} \sum_{k=0}^{n_j-1} A_{jk}(w)(e^{\lambda_j t} t^k),$$

(2) $x(t, w) \varepsilon \mathfrak{D}(D^{n+1})$, where n is the number of λ_j's in the representation of $f_x(t, w)$ each λ_j counted as many times as its multiplicity, and

(3) $X_2 = L_2 \oplus F_2$, where $L_2 = L\{y(t, w); t \varepsilon T\}$, and $F_2 = L\{f_x(t, w); t \varepsilon T\}$.

We will assume throughout this paper that no one of the λ_j's in the representation of $f_x(t, w)$ is purely imaginary.

THEOREM III. *If*

(1) *$\{x(t); t \varepsilon T\}$ is a stochastic process such that $X_2 = L\{x(t, w); t \varepsilon T\}$ is well defined and* $\lim_{s \to t} \|x(s, w) - x(t, w)\|_{X_2} = 0$,

(2) *the group of shift operators, $\{S^t; t \varepsilon T\}$, is uniquely defined on X by the relation, $S^t x(s, w) = x(s + t, w)$, and can be extended to all of X_2,*

(3) *D is the closure of the differential operator, D' defined by*

$$D'y = \lim_{h \to 0} \{S^h - I\}y/h,$$

where I is the identity operator on X_2, and the limit is taken in the strong operator topology on T. $\mathfrak{D}(D) = \{y(t) : y(t) \varepsilon X_2$ and $Dy(t) \varepsilon X_2\}$,

(4) *the complex polynomial, $M(z) = \sum_{j=0}^{n} b_j z^{-j}$, has no purely imaginary roots, $x(t, w) \varepsilon \mathfrak{D}(D^{n+1})$, and $x(t, w)$ satisfies the relation,*

$$M(D)x(t, w) = \eta(t, w), \; t \varepsilon T, \qquad \text{(a)}$$

where $\eta(t, w)$ is a wide sense stationary stochastic process, then $x(t, w)$ is a continuous dynamic stochastic process.

PROOF. Let $L_2 = L\{\eta(t, w); t \varepsilon T\}$. Clearly $L_2 \subset X_2$. Since S^t commutes with D and hence with $M(D)$ for any $t \varepsilon T$, we see using Assumption (4) that L_2 reduces S^t. Let S_1^t be the reduction of S^t to L_2. Using Assumption (4) again we find that $\{S_1^t; t \varepsilon T\}$ is a group of unitary operators which is continuous in the strong operator topology on T. Let D_1 be the corresponding infinitesimal generator. If λ is not purely imaginary, we know[5] that the resolvent, $R(\lambda, D_1)$, is a bounded operator taking L_2 into itself. D_1 clearly is the reduction of D to L_2, and we see immediately when expanding $M^{-1}(z)$ in partial fractions that $M^{-1}(D_1)$ is a bounded operator taking L_2 into itself. Hence (a) has a uniquely defined solution in L_2,

$$y(t, w) = M^{-1}(D_1)\eta(t, w).$$

Obviously $L\{y(t, w); t \varepsilon T\} \subset L_2$, and since clearly $y(t, w) \varepsilon \mathfrak{D}(D_1^{n+1})$, the inverse relation follows from $\eta(t, w) = \sum_{j=0}^{n} b_j D_1^{n-j} y(t, w)$. Hence $L\{y(t, w); t \varepsilon T\} = L_2$.

[5] See for instance Hille and Phillips [4], p. 601.

Let $f_x(t, w)$ be defined by the relation, $x(t, w) - y(t, w) = f_x(t, w)$. If $f_x(t, w) = 0$ a.e. for almost all $t \varepsilon T$, we say that $x(t, w)$ is a centered dynamic stochastic process. Suppose not. Then $f_x(t, w)$ satisfies the relation,

$$M(D)f_x(t, w) = 0.$$

Hence making use of the general theory of differential equations we deduce, as in the discrete case, that

$$f_x(t, w) = \sum_{j=1}^{a} \sum_{k=0}^{n_j-1} A_{jk}(w)(e^{\lambda_j t}t^k),$$

and that

$$x(t, w) = y(t, w) + \sum_{j=1}^{a} \sum_{k=0}^{n_j-1} A_{jk}(w)(e^{\lambda_j t}t^k).$$

If $F_2 = L\{f_x(t, w); t \varepsilon T\}$, and L_2 and F_2 are linearly independent, then $X_2 = L_2 \oplus F_2$. This fact is obtained using verbatim the reasoning used in the proof of Theorem I. That L_2 and F_2 are indeed independent is seen from the following considerations.

Let $E = F_2 \cap L_2$. E is finite dimensional and clearly reduces S^t. Let $\{S_1^t u(w); t \varepsilon T\}$ be a wide sense stationary process in E. Then $u(t)$ has the representation,

$$u(t, w) = \sum_{j=1}^{p} e^{it\lambda_j} b_j(w),$$

where the λ_j's are real numbers and the $b_j(w)$'s are mutually orthogonal. On the other side $u(t) \varepsilon F_2$ and hence satisfies the relation,

$$M(D)u(t, w) = 0,$$

which is impossible since $M(z)$ has no roots on the imaginary axis. Q.E.D.

The following lemma is stated without proof. The proof is entirely similar to that given for Lemma I and needs no repetition.

LEMMA II. *If* (1) $\{x(t); t \varepsilon T\}$ *is a continuous dynamic stochastic process, then there exists a linear differential operator,* $M(D)$, *with constant coefficients of lowest order such that* $M(D)x(t, w) \varepsilon L_2$. $M(D)$ is uniquely determined up to a multiplicative constant, and $M(z)$ has no purely imaginary roots.

THEOREM IV. *If* (1) $\{x(t); t \varepsilon T\}$ *is a continuous dynamic stochastic process, then it has one and only one centering function.*

PROOF. Suppose $x(t, w)$ satisfies the relations,

(1) $x(t, w) = y^{(1)}(t, w) + \sum_{j=1}^{a} \sum_{k=0}^{n_j-1} A_{jk}(w)(e^{\lambda_j t}t^k)$,

(2) $x(t, w) = y^{(2)}(t, w) + \sum_{j=1}^{b} \sum_{k=0}^{m_j-1} B_{jk}(w)(e^{\lambda_j t}t^k)$.

Suppose that λ_j is the root in (1) which has the largest positive real component. Clearly,

$$\lim_{t\to\infty} \left\| \frac{x(t, w)}{e^{\lambda_j t}t^{n_j-1}} - A_{j,n_j-1}(w) \right\|_{X_2} = 0,$$

implies that there is some $B_{lm}(w)$ in (2) such that $B_{lm}(w) = A_{j,n_j-1}(w)$ with probability one and such that $\lambda_l = \lambda_j$ and has the same multiplicity as λ_j.

Let $x^{(1)}(t, w) = x(t, w) - \sum_{k=0}^{n_j-1} A_{jk}(w)(e^{\lambda_j t}t^k)$, and suppose that λ_r is the root with the largest positive real component remaining. Then

$$\lim_{t\to\infty} \left\| \frac{x^{(1)}_{(t,w)}}{e^{\lambda_r t} t^{n_r-1}} - A_{r,n_r-1}(w) \right\|_{X_2} = 0$$

implies that there is some $B_{jm}(w)$ such that $B_{jm}(w) = A_{r,n_r-1}(w)$ with probability one and such that $\lambda_j = \lambda_r$ and has the same multiplicity as λ_r.

Continuing this process of deduction taking the roots with positive real components in decreasing order of magnitude first, and then, similarly, letting $t \to -\infty$ for the roots with negative real components we easily prove the theorem. Q.E.D.

4. Vector-valued dynamic stochastic processes. To extend the preceding results to multivariate processes we need several new concepts. The interested reader is referred to Wiener and Masani [9] for details.

Let $\{x(t); t \varepsilon T\}$ be a p-variate column vector-valued stochastic process. Let $\mathfrak{X}$ be the linear space of all random vectors of the form, $\sum_{j=1}^{n} B_j x(t_j, w)$, where the B_j's are $p \times p$ matrices with complex entries. The ordinary inner product is not very important in the theory of multivariabe processes. Instead we introduce the Gramian matrix,

$$(x(t), x(s)) = \{(x_i(t), x_j(s))\} = \{Ex_i(t)\overline{x_j(s)}\},$$

and define the norm on $\mathfrak{X}$ by

$$\|x(t)\| = [\tau(x(t), x(t))]^{\frac{1}{2}},$$

where $\tau(x(t), x(t))$ denotes the trace of the Gramian matrix, $(x(t), x(t))$. When $\mathfrak{X}$ can be completed in this norm, we denote its completion by $\mathfrak{X}_2 = \mathfrak{L}\{x(t, w); t \varepsilon T\}$.

Definition. Let $\{x(t); t \varepsilon T\}$ be a p-variate column vector-valued stochastic process such that $\mathfrak{X}_2 = \mathfrak{L}\{x(t, w); t \varepsilon T\}$ is well defined. We say that $x(t)$ is a discrete dynamic stochastic process if,

(1) $x(t, w)$ can be represented by the relation, $x(t, w) = y(t, w) + f_x(t, w)$, where $y(t, w)$ is a p-variate column vector-valued wide sense stationary stochastic process, and $f_x(t, w)$ is a p-variate column vector-valued deterministic non-stationary stochastic process of the form,

$$f_x(t, w) = \sum_{j=1}^{a} \sum_{k=0}^{n_j-1} A_{jk}(w)(z_j^t t^k),$$

the $A_{jk}(w)$'s being p-dimensional column vectors,

(2) $\mathfrak{X}_2 = \mathfrak{L}_2 \oplus \mathfrak{F}_2$, where $\mathfrak{L}_2 = \mathfrak{L}\{y(t, w); t \varepsilon T\}$ and $\mathfrak{F}_2 = \mathfrak{L}\{f_x(t, w); t \varepsilon T\}$.

We assume again that if $f_x(t, w)$ has the representation given above, then $|z_j| \neq 1, j = 1, 2, \cdots, a$.

We note that if the "matrix-polynomial", $M(z)$, is defined by $M(z) =$

$\sum_{j=0}^{n} B_j z^{-j}$, where the B_j's are $p \times p$ matrices with complex entries, then $M(z) = \{M_{ij}(z)\}$, where the $M_{ij}(z)$'s are ordinary polynomials in z. The determinant of $M(z)$ is denoted by $\det M(z) = W(z)$. $W(z) = \sum_{j=0}^{m} w_j z^{-j}$, where the w_j's are complex constants. Whenever $W(z) \neq 0$,

$$M^{-1}(z) = V(z)/W(z) = \{V_{ij}(z)/W(z)\},$$

is well defined.

If S_1 is a unitary operator on $\mathfrak{L}_2$, and $W(z)$ has no roots of modulus $= 1$, then $M(S_1)$ has an inverse,

$$M^{-1}(S_1) = V(S_1)/W(S_1) = \{W^{-1}(S_1)V_{ij}(S_1)\} = \{V_{ij}(S_1)W^{-1}(S_1)\}.$$

THEOREM V. *If*

(1) $\{x(t); t \varepsilon T\}$ *is a p-variate column vector-valued stochastic process such that* $\mathfrak{X}_2 = \mathfrak{L}\{x(t, w); t \varepsilon T\}$ *is well defined,*

(2) *the shift operator,* S, *is uniquely defined on* $\mathfrak{X}$ *by the relation,* $Sx(t, w) = x(t + 1, w)$, *can be extended to all of* $\mathfrak{X}_2$, *and has a continuous inverse,* S^{-1},

(3) *the matrix-polynomial,* $M(z) = \sum_{j=0}^{n} B_j z^{-j}$, *where the* B_j's *are* $p \times p$ *matrices with complex entries, is such that* $\det M(z) = W(z) = \sum_{j=0}^{m} w_j z^{-j}$ *has no roots of modulus* $= 1$, *and if* $x(t, w)$ *satisfies the relation,*

$$M(S)x(t, w) = \eta(t, w), \quad \text{all} \quad t \varepsilon T,$$

where $\{\eta(t); t \varepsilon T\}$ *is a p-variate column vector-valued wide sense stationary stochastic process, then* $x(t, w)$ *is a p-variate discrete dynamic stochastic process.*

PROOF. That $\mathfrak{L}_2 = \mathfrak{L}\{\eta(t, w); t \varepsilon T\} \subset \mathfrak{X}_2$, and that $\mathfrak{L}_2$ reduces S is easily obtained from our assumptions. It is also straight forward to show that if S_1 is the restriction of S to $\mathfrak{L}_2$, then S_1 is uniquely defined by $S_1\eta(t, w) = \eta(t + 1, w)$, and is a unitary operator. Using the proof of Theorem I it is, therefore, clear that the equation, $M(S)x(t, w) = \eta(t, w)$, has a uniquely defined solution in $\mathfrak{L}_2$,

$$y(t, w) = M^{-1}(S_1)\eta(t, w) = \{W^{-1}(S_1)V_{ij}(S_1)\}\eta(t, w),$$

and that $\mathfrak{L}\{y(t, w); t \varepsilon T\} = \mathfrak{L}_2$.

Let $f_x(t, w)$ be defined by the relation, $x(t, w) - y(t, w) = f_x(t, w)$. If $f_x(t, w) = 0$ a.e., we say that $x(t, w)$ is a centered dynamic process. We note that some of the components may be centered, but this is of no importance as far as the rest of the proof goes.

It is clear that $f_x(t, w)$ satisfies the difference relation,

$$M(S)f_x(t, w) = 0;$$

but then in particular it satisfies the difference equation,

$$W(S)f_x(t, w) = 0.$$

Hence again as in the univariate case we can write

$$f_x(t) = \sum_{j=1}^{a} \sum_{k=0}^{n_j - 1} A_{jk}(z_j^t t^k),$$

where the A_{jk}'s are now p-dimensional column vectors. Finally the system of equations,

$$\left\{ f_x(t, w) = \sum_{j=1}^{a} \sum_{k=0}^{n_j-1} A_{jk}(z_j^t t^k) \right\}_{t=1}^{m},$$

can be solved uniquely for the A_{jk}'s in terms of the $f_x(t, w)$'s, which proves that the A_{jk}'s are indeed random vectors in $\mathfrak{X}_2$. Hence

$$x(t, w) = y(t, w) + \sum_{j=1}^{a} \sum_{k=0}^{n_j-1} A_{jk}(w)(z_j^t t^k).$$

It is clear from the representation of $f_x(t, w)$ that the proof of Theorem I carries over to the multivariate case with only obvious modifications. We have, therefore, proved that $x(t, w)$ is a p-variate dynamic stochastic process.

The order of the matrix-operator, $M(S)$, is defined by the order of its determinant-operator, $W(S)$. Since the finite-dimensionality of $\mathfrak{F}_2$ implies that for some sequence of complex matrices, $\{B_j\}$, and some sequence $\{t_j\}$,

$$f_x(t, w) = \sum_{j=1}^{n} B_j f_x(t_j, w), \quad \text{all} \quad t \varepsilon T,$$

the existence of a finite ordered matrix-valued operator taking $x(t, w)$ into $\mathfrak{L}_2$ is immediate. That we can find a matrix-valued operator such that the characteristic polynomial of the corresponding determinant-operator has no zeros of modulus equal to one follows from the representation of $f_x(t, w)$. Furthermore as far as this determinant-operator goes, Lemma I carries over directly to the multivariate case. Finally using the reasoning of the proof of Theorem II the following theorem is easily obtained.

THEOREM VI. *If* (1) $\{x(t); t \varepsilon T\}$ *is a p-variate discrete dynamic stochastic process, then it has one and only one centering function.*

This ends the discussion of dynamic stochastic processes. The preceding results are easily extended to continuous p-variate processes. The reasoning is essentially the same as used in Sections 3 and 4 with only obvious modifications. We only note that, while $M^{-1}(S_1) = \{W^{-1}(S_1)V_{ij}(S_1)\} = \{V_{ij}(S_1)W^{-1}(S_1)\}$ in the discrete case, in the continuous case the corresponding inverse operator is $M^{-1}(D_1) = \{V_{ij}(D_1)W^{-1}(D_1)\}$.

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