CONVERGENCE THEOREMS FOR MULTIPLE CHANNEL LOSS PROBABILITIES[1]

By A. Joffe[2] and P. E. Ney

Cornell University

1. Introduction and summary. We are concerned in this paper with a process characterized by the arrival of units at a facility consisting of an integral number c of channels or servers, who are to process these units. Denote by the random variable t_n the arrival time at the facility of the nth unit, and by $T_n = t_n - t_{n-1}$ the inter-arrival time between the $(n - 1)$st and nth units. The random variables $\{T_n\}$, $n = 1, 2, \cdots$, are assumed to be independent and identically distributed (I.I.D.) as some typical random variable T with distribution function (d.f.) F_T. The trivial case $F_T(0) = 1$ is excluded. If an arriving unit finds channels free, it is processed by any one of them. The channels behave independently and identically, in the sense that the processing time of a unit does not depend on the particular channel doing the processing, or on the status of the other channels. If an arriving unit finds all channels busy, it departs, or is lost. It is assumed that the nth unit has associated with it a processing time R_n, whether or not it is in fact processed. The $\{R_n\}$, $n = 1, 2, \cdots$ are taken to be random variables which are I.I.D. as some typical R with d.f. F_R, and are furthermore to be independent of the $\{T_n\}$. Unless it is stated to the contrary, we assume that R has finite expectation.

In a typical example, the units might be messages or telephone calls, and the channels be lines or cables over which the messages or calls are transmitted. The $\{T_n\}$ would be the time between attempted calls. If all lines are busy when the nth call is attempted, the call or message is lost. The $\{R_n\}$ represent the length of conversation that would follow from the nth call if that call were to find a free line. Among the quantities which characterize the reliability of such a system are the probability p_n that the nth call or message is not lost, and the probability p_t that at some specified time t not all lines or cables are busy. We shall be concerned here with the convergence of $\{p_n\}$ and $\{p_t\}$ (as $n \to \infty$ and $t \to \infty$), as well as that of a larger class of probabilities.

The question of the convergence of the sequence $\{p_n\}$ was first studied for the case of one channel by F. Pollaczek in the last chapter of his book [2]. Under certain restrictions he proves the convergence by somewhat lengthy, and purely analytical methods. In Section 2 we indicate how a very elementary application of renewal theory yields a somewhat more precise result under less conditions.

It was also Pollaczek who first posed the question of the convergence of $\{p_n\}$

Received December 16, 1959; revised August 20, 1962.

[1] Research sponsored by the Office of Naval Research under contract number Nonr 401(03)—NR042-064.

[2] Now at the Université de Montréal.

Institute of Mathematical Statistics is collaborating with JSTOR to digitize, preserve, and extend access to *The Annals of Mathematical Statistics.* www.jstor.org

in the multiple channel case. The analogous problem for $\{p_t\}$ was posed to one of the authors by L. Takacs, and also by the referee of an earlier version of this paper.

The main results of this paper are to answer both these questions subject to only mild restrictions. These results are contained in Theorems 2 and 3 in Section 3, which in fact establish the convergence of a considerably larger class of probabilities.

2. The convergence of $\{p_n\}$ for a single channel. In this section it is not necessary to make any assumptions on the moments of T and R.

For any distributions F_T and F_R, define λ to be the smallest integer $\geqq 2$ such that R has no point of support in the semi-closed intervals

$$(k\lambda m,\ ((k+1)\lambda - 1)M], \qquad k = 0, 1, 2, \cdots, \tag{2.1}$$

where $m = $ ess. inf. T, and $M = $ ess. sup. T, provided that such an integer exists. If such an integer does not exist, define $\lambda = 1$. Let $\alpha_n = P\{T_1 + \cdots + T_n \geqq R\}$ and $\mu = \sum_{n=1}^{\infty} (\alpha_n - \alpha_{n-1})$.

Theorem 1. *$p_{n\lambda} \to \lambda/\mu$; $p_k = 0$ if λ does not divide k. If $\mu = \infty$ then $p_n \to 0$.*

Proof. Suppose that the $(j+1)$th unit which arrives to find the channel free is the h_jth in the sequence of arriving units. Then $p_n = \sum_{j=0}^{\infty} P\{h_j = n\}$. Let $H_j = h_j - h_{j-1}$. The sequence of random variables $\{H_j\}$ is clearly a discrete renewal process. Let $f_n = P\{h_1 = n\}$. Then $\sum_{n=1}^{k} f_n = \alpha_k \to 1$ as $k \to \infty$. Hence by the discrete renewal theorem (see e.g. pp. 386–7 of [1]), the theorem will follow if we can show that the definition of λ is equivalent to the statement that λ be the smallest integer with the property that $f_k = 0$ for every k not divisible by λ. (This is the usual definition of periodicity of $\{p_n\}$. See [1].)

Suppose there is a $\lambda \geqq 2$ satisfying (2.1). Then $P\{R \leqq k\lambda m\} = P\{R \leqq ((k+1)\lambda - 1)M\}$; $k = 0, 1, 2, \cdots$, which implies that

$$\alpha_{k\lambda} = \alpha_{(k+1)\lambda-1}\,; \qquad k = 0, 1, 2, \cdots, \tag{2.2}$$

which in turn implies that $f_{k\lambda+1} = \cdots = f_{(k+1)\lambda-1} = 0$. Thus in this case $\{p_n\}$ has period $\lambda \geqq 2$.

Conversely, suppose that $\{p_n\}$ has period $\lambda \geqq 2$. Then clearly (2.2) is satisfied. We shall show that this implies that R has no point of support in the intervals (2.1). Suppose the contrary. Then there is a point of support of R, say r, and a k such that

$$k\lambda m < r \leqq ((k+1)\lambda - 1)M. \tag{2.3}$$

Now suppose that $\tau_1, \cdots, \tau_{(k+1)\lambda-1}$ are points of support of T. Then by (2.2) we have $P\{\tau_1 + \cdots + \tau_{(k+1)\lambda-1} \geqq R\} = P\{\tau_1 + \cdots + \tau_{k\lambda} \geqq R\}$, and thus

$$r \leqq \tau_1 + \cdots + \tau_{(k+1)\lambda-1} \quad \text{implies} \quad r \leqq \tau_1 + \cdots + \tau_{k\lambda}\,. \tag{2.4}$$

Hence setting $\tau_i = M$, $i = 1, \cdots, (k+1)\lambda - 1$ we see by (2.3) that $r \leqq k\lambda M$.

It is now sufficient to show that for all i such that $0 \leqq i \leqq k\lambda - 1$,

$$r \leqq im + (k\lambda - i)M \tag{2.5}$$

implies

$$r \leqq (i + 1)m + (k\lambda - i - 1)M, \tag{2.6}$$

since then $r \leqq k\lambda M$ implies that $r \leqq k\lambda m$, which contradicts (2.3). But (2.5) implies that $r \leqq (i + \lambda - 1)m + (k\lambda - i)M$, which by (2.4) yields $r \leqq (\lambda + i - 1)m + (k\lambda - \lambda - i + 1)M \leqq (i + 1)m + (k\lambda - i - 1)M$, which is (2.6).

3. Main results. We shall consider two processes defined on the space $\Omega = \{(x_1, \cdots, x_c): x_i \geqq 0, i = 1, \cdots, c\}$. One is the continuous parameter process $\{u_t\} = \{(u_{t1}, \cdots, u_{tc})\}$, where u_{ti} is the processing time after t remaining to the unit in channel i at time t. If channel i is empty at t, then $u_{ti} = 0$. The other process is the discrete parameter process $\{u_n\} = \{(u_{n1}, \cdots, u_{nc})\}$, where $u_{ni} \equiv u_{t_n,i}$. This process is Markovian, and typical one step transitions from $u_n = (u_{n1}, \cdots, u_{nc})$ to $u_{n+1} = (u_{n+1,1}, \cdots, u_{n+1,c})$ are

$$u_{n+1,i} = \begin{cases} (u_{ni} - T_{n+1})^+ & \text{if the } (n+1)\text{th unit is not processed by the } i\text{th channel,} \\ (R_n - T_{n+1})^+ & \text{if } u_{n,i} = 0 \text{ and if the } (n+1)\text{th unit is processed by the } i\text{th channel,} \end{cases} \tag{3.1}$$

where for any real x, $x^+ = \max(0, x)$. (We adopt the convention that if the ith channel is empty at t_n, and the nth unit is processed by this channel, then we set $u_{n,i} = 0$ rather than $u_{n,i} = R_n$.)

Analogous to the one channel case, suppose that the $(j + 1)$th unit which arrives to find all channels free is the h_jth in the sequence of arriving units, and write $g_j = t_{h_j}$. Thus g_0 is the first time at which all channels are free. We assume that at time zero the system is in an initial state $u_0 = (u_{01}, \cdots, u_{0c})$, and write $t_0 = h_{-1} = g_{-1} = 0$. Furthermore write $P\{u_t \varepsilon A \mid u_0\}$ for the probability that $u_t \varepsilon A$ if the initial state was u_0, and define $P\{u_n \varepsilon A \mid u_0\}$ similarly. Let

$$H_j = h_j - h_{j-1}; G_j = g_j - g_{j-1}, j = 0, 1, 2, \cdots;$$

$$N_H(n) = \max\{m: H_0 + \cdots + H_{m-1} \leqq n\};$$

and

$$N_G(t) = \max\{m: G_0 + \cdots + G_{m-1} \leqq t\}.$$

The instants $\{g_i\} = \{t_{h_i}\}$, $i = 0, 1, 2, \cdots$ are regeneration points (see [4]) of the $\{u_t\}$ process in the sense that at these instants any knowledge of the past history of the process has no predictive value. Similarly the points h_j are regeneration points for the discrete process $\{u_n\}$. It follows that the conditional probability that u_n or u_t respectively be in any specified Borel subset A of Ω, given (i) an initial point u_0, (ii) the fact that there has been at least one regeneration point prior to t_n or t respectively, and (iii) the time at which this regeneration occurred, is a function only of A, and time which has elapsed since this last regeneration point. Stated precisely, there exist functions $h(A;n)$ and $g(A;t)$ such that

$$P\{u_n \varepsilon A \mid u_0 ; N_H(n) > 0; h_{N_H(n)}\} = h(A; n - h_{N_H(n)}) \tag{3.2}$$

and

$$P\{u_t \varepsilon A \mid u_0 ; N_G(t) > 0; g_{N_G(t)}\} = g(A; t - g_{N_G(t)}). \tag{3.3}$$

Smith [4] calls processes with this property equilibrium processes.

Since the points $\{h_j\}$ and $\{g_j\}$ are regeneration points, the random variables $\{H_i\}$ are I.I.D. as some typical random variable H with d.f. F_H, and the random variables $\{G_i\}$ are I.I.D. as some typical G with d.f. F_G. Let $EH_i = \mu$, $EG_i = \nu$. The distributions of H_0 and G_0 will in general depend on u_0, and we denote them by F_{H_0} and F_{G_0}. It is expedient to introduce the above terminology at this time even though we do not yet know that F_H, F_G, F_{H_0} and F_{G_0} are honest d.f.'s in the sense that they go to 1 as their arguments go to ∞. That this is in fact so will be proved somewhat later below.

Our main results concern the existence of $\lim_{n\to\infty} P\{u_n \varepsilon A \mid u_0\}$ and $\lim_{t\to\infty} P\{u_t \varepsilon A \mid u_0\}$, where A is a Borel subset of Ω. In order to be able to state the conditions under which the convergence holds we must first make some additional definitions.

Denote by B^+ the class of all Borel subsets A of Ω which have the property that if $(u_1, \cdots, u_c) \varepsilon A$, then for any real $k \geqq 0$, $(u_1 + k, \cdots, u_c + k) \varepsilon A$. Similarly let B^- be the set of all A such that if $(u_1, \cdots, c_c) \varepsilon A$ then $((u_1 - k)^+, \cdots, (u_c - k)^+) \epsilon A$, and let $B^* = B^+UB^-$. For example, the set representing the event "channel i will be busy for at least u_i before becoming idle" is in B^+, and in particular so is "all servers are busy." The sets for the events "channel i will be busy at most u_i before becoming idle" and "all servers are free," are in B^-. Note that our class B^- satisfies the requirements of the class A^* discussed in paragraph 3.6 of Smith [4].

For lack of a better phrase, we shall call x_0 a probability accumulation point (P.A.P.) of the random variable X, if for any neighborhood $W(x_0)$ of x_0, we have $P\{X \varepsilon W(x_0) - x_0\} > 0$.

Finally, for any random variable X, let $S(X)$ and $I(X)$ denote the essential supremum and infimum respectively of X; let F_T^* be the conditional distribution of T, given $T \neq 0$ (the case when $P\{T = 0\} > 0$ has not been excluded); and let T^* be the random variable with d.f. F_T^*.

We shall prove the following results.

THEOREM 2. *If* (a) $P\{T \geqq \min [R, cI(T^*)]\} > 0$, *and* (b) T *has a* P.A.P. *at zero, then for any initial point* u_0 *and any Borel subset* A *of* Ω

$$\lim_{n\to\infty} P\{u_n \varepsilon A \mid u_0\} = \frac{1}{\mu}\sum_{j=0}^{\infty} h(A;j)[1 - F_H(j)] > 0. \tag{3.4}$$

The proof will be given in Section 4.

REMARK 1. The condition (a) is also a necessary condition in the weak sense that if it is violated, then for any T we can define a random variable R and a set A for which the theorem is false. Depending on the particular choice of T, R, and A, it may then happen in some cases that the limit in (3.4) exists but is zero, or in other cases that the limit does not exist. An idea to the construction of such counter-examples can be obtained by studying the manner in which Lemma 3 (to be proved below) would fail.

REMARK 2. As a corollary of Theorem 2 we see that if the conditions of the theorem are satisfied, then $\{p_n\}$ converges. (p_n is the probability defined in Section 1).

REMARK 3. The condition (b) is not really crucial for a theorem of the above form to hold, its main purpose being to simplify the result. It assures us that the $\{u_n\}$ process is aperiodic, i.e. that there is no interger $\lambda \geqq 1$ such that $P\{u_n \varepsilon A\} = 0$ for all n which are not integral multiples of λ. If such a periodicity did exist, then an obvious modification of the proof of Theorem 2 would show that (3.4) is to be replaced by

$$\text{(3.4.1)} \qquad \lim_{n\to\infty} P\{U_{n\lambda}\, \varepsilon\, A \mid U_0\} = \frac{1}{\mu}\sum_{j=0}^{\infty} h(A; \lambda j)[1 - F_H(\lambda j)].$$

THEOREM 3. *If* (i) $P\{T \geqq \min[R, cI(T^*)]\} > 0$, *and* (ii) T *has a* P.A.P., *and* (iii) A *is in* B^*, *then for any initial point* u_0

$$\lim_{t\to\infty} P\{U_t\, \varepsilon\, A \mid U_0\} = 1/\nu \int_0^\infty g(A; t)\{1 - F_G(t)\}\, dt < \infty.$$

The proof will be given in Section 4.

REMARK 4. Condition (i) is again necessary in the sense mentioned in Remark 1. Conditions (ii) and (iii) are technical conditions with which it seems we can not dispense. Condition (ii) serves to eliminate the possibility of periodicities (note that the P.A.P. need not be at zero); while (iii) is tailored to enable us to use a theorem of Smith [4] on regenerative processes.

REMARK 5. If (i) and (ii) are satisfied then an obvious choice of A leads to the conclusion that the probabilities $\{p_t\}$, $t \geqq 0$, defined in Section 1, converge as $t \to \infty$.

4. Proofs of the main results. We shall break up the proofs into a sequence of lemmas. The first lemma gives an inequality between certain joint probabilities and product probabilities. This inequality is similar to a result given by H. Robbins in [3], and the proof can be carried out in an anologous manner.

LEMMA 1. *Let* $\{X_i\}$, $i = 1, 2, \cdots$, *be a sequence of independent random variables, and let* $S_n = \sum_{i=1}^n X_i$. *Then for any real numbers* a_i, $i = 1, 2, \cdots$, *and any integer* n,

$$\text{(4.1)} \qquad P\{S_1 \geqq a_1, \cdots, S_n \geqq a_n\} \geqq \prod_{i=1}^n P\{S_i \geqq a_i\}.$$

Now choose and fix any real k in the interval (0, 1), and let $K = I(R) -$

$kS(T)$. Also for any real b, define the set $\Delta b = \{(x_1, \cdots, x_c): 0 \leqq x_i \leqq b^+, i = 1, \cdots, c\}$.

LEMMA 2. *Given any* $0 \leqq D < \infty$, *there exists an* $\epsilon > 0$ *not depending on* D, *and and an* $N < \infty$ *such that for all* $u_0 \, \varepsilon \, \Delta D$ *and* $n > N$ *we have* (a) $P\{u_n \, \varepsilon \, \Delta K \mid u_0\} > \epsilon$; (b) *If* $P\{T \geqq R\} > 0$, *then* $P\{u_n \, \varepsilon \, \Delta \, 0 \mid u_0\} > \epsilon$.

PROOF. A sufficient condition for the event $u_n \, \varepsilon \, \Delta \, K$ is that the arrival time t_n of the nth unit should not precede by a time greater than K^+ the departure time of any previously arrived units, and that $t_n > u_{0i} - K^+$ for $i = 1, \cdots, c$.

Let
$$\sigma_i = \begin{cases} 1 \text{ if the } i\text{th arriving unit is processed,} \\ 0 \text{ if the } i\text{th arriving unit is lost.} \end{cases}$$

Then the departure time of unit i is $t_i + \sigma_i R_i$. Letting

$$P_n(u_0) = P\left\{\bigcap_{i=1}^{c} (t_n > u_{0i} - K^+)\right\}, \tag{4.2}$$

and for any random variable X

$$F_{n,T}(X) = P\{T_1 + \cdots + T_n \leqq X\}, \tag{4.3}$$

we thus obtain

$$\begin{aligned} P\{u_n \, \varepsilon \, \Delta K \mid u_0\} &\geqq P_n(u_0) \cdot P\left\{\bigcap_{i=0}^{n-1} (t_n > t_i + \sigma_i R_i - K^+)\right\} \\ &\geqq P_n(u_0) \cdot P\left\{\bigcap_{i=1}^{n} (T_i + \cdots + T_n > R_{i-1} - K^+\right\} \\ &\geqq P_n(u_0) \prod_{i=1}^{n} [1 - F_{i,T}(R - K^+)]. \end{aligned} \tag{4.4}$$

The last inequality follows from Lemma 1.

A sufficient condition for the product on the right side of (4.4) to converge to a positive quantity as $n \to \infty$ is that for some $\delta > 0$

$$F_{i,T}(R - K^+) < 1 - \delta \quad \text{for} \quad i = 1, 2, \cdots, \tag{4.5}$$

and

$$\sum_{i=1}^{\infty} F_{i,T}(R - K^+) < \infty. \tag{4.6}$$

If $K^+ = 0$, then $P\{T \geqq R\} > 0$, and hence (4.5) holds. If $K^+ > 0$, then $F_T(R - K^+) = P\{I(R) - kS(T) < R - T\} < 1$, since by assumption $F_T(0) < 1$ and hence $S(T) > 0$. But since the T_i are non-negative, $F_{i,T}(R - K^+) \leqq F_T(R - K^+)$ for $i = 1, 2, \cdots$, and hence (4.5) is established.

Turning to the verification of (4.6), write $\sum_{i=1}^{\infty} F_{i,T}(R - K^+) = \sum_{i=1}^{\infty} \int_{K^+}^{\infty} F_{i,T}(r - K^+) dF_R(r) = \lim_{n\to\infty} \int_0^{\infty} \sum_{i=1}^{n} F_{i,T}(r) dF_R(r + K^+)$. But from renewal theory it is well known that there exists a constant $c_1 < \infty$ such that $\sum_{i=0}^{\infty}$

$F_{i,T}(r) < c_1 r$. Hence it follows that the integrals $\int_0^\infty \sum_{i=1}^n F_{i,T}(r)\, dF_R(r + K^+)$ are uniformly bounded by $c_1 E(R) < \infty$ since we have assumed $ER < \infty$. Hence the sum (4.6) converges, implying that the product in (4.4) is uniformly bounded away from zero. Since clearly $P_n(u_0) \to 1$ uniformly for $u_0 \in \Delta D$, part (a) is proved.

Part (b) follows identically, with the replacement of K^+ by 0 throughout the proof being justified by the condition $P\{T \geqq R\} > 0$.

Following Smith [4], we shall say that the set A is an α-set with respect to the sequence $\{u_n\}$, $n = 0, 1, 2, \cdots$, if there is a real $\alpha > 0$ and an integer $N < \infty$, such that for any n,

$$P\{N_H(n + N) > N_H(n) \mid u_0 ; u_n \in A\} \geqq \alpha. \tag{4.7}$$

Note that α is independent of u_0. Similarly, we shall say that A is an α-set with respect to $\{u_t\}$, $t \geqq 0$, if there exist real numbers τ, $\alpha > 0$, such that for any t,

$$P\{N_G(t + \tau) > N_G(t) \mid u_0 ; u_t \in A\} \geqq \alpha. \tag{4.8}$$

(Smith's definitions of α-sets were weaker in the sense that he only required (4.7) and (4.8) satisfied for sufficiently large n and t.)

LEMMA 3. (a) *If $P\{T \geqq R\} > 0$, then $\Delta 0$ is an α-set with respect to $\{u_n\}$, $n = 0, 1, 2, \cdots$, and $\{u_t\}$, $t \geqq 0$.* (b) *The set ΔK is an α-set with respect to $\{u_n\}$, $n = 0, 1, 2, \cdots$, and $\{u_t\}$, $t \geqq 0$, for all R, if and only if $P\{T \geqq cI(T^*)\} > 0$.*

PROOF. Part (a) is an immediate consequence of the definition of α-set. We turn to the sufficiency part of (b).

If ΔK is an α-set with respect to $\{u_n\}$, then obviously it is one with respect to $\{u_t\}$, and hence we may limit attention to the former. Suppose first that $I(T^*) > 0$. If $P\{T < I(R)\} < 1$, then we are in case (a), and hence we need consider only the case when $P\{T < I(R)\} = 1$. Recall that the random variable $N_H(n)$ denotes the number of passages through the empty state up to time t_n (including what we have called the zeroth passage). Thus the statement $N_H(n + N) > N_H(n)$ is equivalent to saying that there is at least one passage through $\Delta 0$ in the interval (t_n, t_{n+N}). To establish the sufficiency part of the lemma for the $\{u_n\}$ sequence it is thus enough to show that there is an $\alpha_1 > 0$ and an $N < \infty$, such that for each $u_n \in \Delta K$, there is an $N'(u_n) < N$ with the property that

$$P\{u_{n+N',i} = 0, \qquad i = 1, \cdots, c\} > \alpha_1 . \tag{4.9}$$

Because of the symmetry of the problem it is no loss of generality to assume that $u_{n1} \leqq \cdots \leqq u_{nc}$, and that if several channels are free then the arriving unit enters the channel with the lowest index. It is thus sufficient to prove that there is an $\alpha_2 > 0$ and an integer $L < \infty$ with the property that if $u_n \in \Delta K$, and if $u_{n1} = \cdots = u_{nb} = 0$ for any particular integer b satisfying $1 \leqq b \leqq c$, then there is an integer $L'(u_n) < L$ such that

$$P\{u_{n+L',1} = \cdots = u_{n+L',b} = u_{n+L',b+1} = 0\} > \alpha_2 . \tag{4.10}$$

The sufficiency part of the lemma will then follow with $N = cL$.

Finally we see that it is in turn sufficient to prove the following: There exist real $\epsilon > 0$, $\alpha_3 > 0$, and an integer $M < \infty$, such that if $u_{n1} \leqq \cdots \leqq u_{nc}$, $u_{n1} = \cdots = u_{nb} = 0$, $0 < u_{n,b+1} < K$, for any $1 \leqq b \leqq c$, then there is an $M'(u_n) < M$ such that

$$(4.11)\quad P\{u_{n+M',1} = \cdots = u_{n+M',b} = 0,\ u_{n+M',b+1} \leqq [u_{n,b+1} - \epsilon]^+\} > \alpha_3 .$$

Then clearly (4.11) implies (4.10) with $L = [I(R)/\epsilon]M$, and hence (4.9) with $N = [I(R)/\epsilon]Mc$.

If $b = c$ in (4.11) then the proof degenerates to a trivial situation, and hence we assume that $b < c$. Let $\sigma_{ni} = 1$ if the nth unit is processed by the ith channel, and $\sigma_{ni} = 0$ otherwise. For any positive integer m define

$$(4.12)\quad \begin{aligned} R_i(n, n+m) &= \sigma_{n,i} R_n + \cdots + \sigma_{n+m-1,i} R_{n+m-1} ,\\ I_i(n, n+m) &= \text{total idle time of channel } i \text{ during the interval } [t_n , t_{n+m}) . \end{aligned}$$

Let $B(n, m, \delta_1, \delta_2, \delta_3)$ denote the event

$$\left\{\begin{array}{l} [I(R) - I(T^*) - \delta_1 \leqq T_{n+1} + \cdots + T_{n+m-1} < I(R) - I(T^*)] \\ \cap [I(R) - \delta_1 \leqq T_{n+1} + \cdots + T_{n+m} < I(R)] \\ \cap [I(R) \leqq R_{n+j} \leqq I(R) + \delta_2 , j = 1, \cdots , m] \\ \cap [T_{n+j} \leqq I(T^*) + \delta_3 , j = 1, \cdots , m] \end{array}\right\} .$$

We shall partition the remainder of the proof of the sufficiency part into several cases.

Case (i). $I(T^*)$ does not divide $I(R)$.

Case (ii). $I(T^*)$ divides $I(R)$ and $P\{R = I(R)\} = 0$.

Case (iii). $I(T^*)$ divides $I(R)$ and $P\{R = I(R)\} > 0$ and $P\{T = I(T^*)\} = 0$.

Case (iv). $I(T^*)$ divides $I(R)$ and $P\{R = I(R)\} > 0$ and $P\{T = I(T^*)\} > 0$.

Case (i). Let j be the largest integer smaller than $I(R)/I(T^*)$, and $\delta_1 = I(R) - j\,I(T^*)$. Then $0 < \delta_1 < I(T^*)$, due to the hypothesis of Case (i). Either $j \geqq c$, or $P\{T \geqq R\} > 0$. In the latter case we are in part (a) of the lemma and are through. In the former there is an $m \geqq c$ such that

$$(4.13)\quad P\left\{\begin{array}{l} [I(R) - I(T^*) - \delta_1 \\ \qquad\qquad \leqq T_{n+1} + \cdots + T_{n+m-1} < I(R) - I(T^*)] \\ \cap [I(R) - \delta_1 \leqq T_{n+1} + \cdots + T_{n+m} < I(R)] \end{array}\right\} = \alpha_4 > 0.$$

Fix m. Choose $0 < \delta_2 < I(T^*) - \delta_1$, $0 < \delta_3 < (1/c)\{I(T^*) - \delta_1 - \delta_2\}$, and let $\delta_4 = I(T^*) - \delta_1 - \delta_2 - c\delta_3$. Due to the assumption $u_{n1} = \cdots = u_{nb} = 0$, one verifies directly that the event $B(n, m, \delta_1, \delta_2, \delta_3)$ implies that (i) channels $1, \cdots, b$ will begin to process exactly one new unit each, during the interval $[t_n , t_{n+m-1})$, and none in $[t_{n+m-1} , t_{n+m})$; (ii) if $u_{n,b+1} \leqq I(R) - I(T^*)$, then channel $b + 1$ will begin to process exactly one new unit in $[t_n , t_{n+m})$; (iii) if $u_{n,b+1} > I(R) - I(T^*)$, then channel $b + 1$ will process no new units during $[t_n , t_{n+m-1})$. Thus $B(n, m, \delta_1, \delta_2, \delta_3)$ implies that:

$$(4.14.1)\qquad R_i\,(n, n + m - 1) = R_i(n, n + m) \leqq I(R) + \delta_2 .$$

(4.14.2) $R_{b+1}(n, n+m-1) = 0, \quad \text{if} \quad u_{n,b+1} > I(R) - I(T^*),$

(4.14.3) $R_{b+1}(n, n+m) = I(R) + \delta_2, \quad \text{if} \quad u_{n,b+1} \leqq I(R) - I(T^*);$

(4.15.1) $$I_i(n, n+m-1) = I_i(n, n+m) \leqq (c-2)\,(I(T^*) + \delta_3), \quad i = 1, \cdots, b < c,$$

(4.15.2) $I_{b+1}(n, n+m-1) = 0, \quad \text{if} \quad u_{n,b+1} > I(R) - I(T^*),$

(4.15.3) $$I_{b+1}(n, n+m) \leqq [(b-1)(I(T^*) + \delta_3) - u_{n,b+1}]^+ + (I(T^*) + \delta_3), \quad \text{if} \quad u_{n,b+1} \leqq I(R) - I(T^*);$$

(4.16.1) $u_{i,n+m-1} > 0, i = 1, \cdots, b+1, \quad \text{if} \quad u_{n,b+1} > I(R) - I(T^*),$

(4.16.2) $u_{i,n+m} > 0, i = 1, \cdots, b+1, \quad \text{if} \quad u_{n,b+1} \leqq I(R) - I(T^*).$

Furthermore,

(4.17.1) $$u_{n+m,i} \leqq R_i(n, n+m) + I_i(n, n+m) - (T_{n+1} + \cdots + T_{n+m}), \quad i = 1, \cdots, b,$$

(4.17.2) $$u_{n+m-1,i} \leqq R_i(n, n+m-1) + I_i(n, n+m-1) - (T_{n+1} + \cdots + T_{n+m-1}), \; i = 1, \cdots, b,$$

(4.17.3) $$u_{n+m,b+1} \leqq u_{n,b+1} + R_{b+1}(n, n+m) + I_{b+1}(n, n+m) - (T_{n+1} + \cdots + T_{n+m}), \quad \text{if} \quad u_{n,b+1} \leqq I(R) - I(T^*),$$

(4.17.4) $$u_{n+m-1,b+1} \leqq u_{n,b+1} - (T_{n+1} + \cdots + T_{n+m-1}), \quad \text{if} \quad u_{n,b+1} > I(R) - I(T^*).$$

Substituting (4.14), (4.15), (4.16) in (4.17), we see that $B(n, m, \delta_1, \delta_2, \delta_3)$ implies that

(4.18.1) $0 < u_{n+m,i} \leqq (c-1)\, I(T^*), i = 1, \cdots, b,$

(4.18.2) $0 < u_{n+m-1,i} \leqq c\, I(T^*), i = 1, \cdots, b,$

(4.18.3) $$0 < u_{n+m,b+1} \leqq \begin{cases} u_{n,b+1} + 2I(T^*) - \delta_4, \text{ if } (b-1)(I(T^*) + \delta_3) \\ \qquad \leqq u_{n,b+1} \leqq I(R) - I(T^*), \\ (b+1)\, I(T^*) - \delta_4, \text{ if } u_{n,b+1} \\ \qquad \leqq (b-1)\,(I(T^*) + \delta_3), \end{cases}$$

$$\leqq \max\{u_{n,b+1} + 2I(T^*) - \delta_4,\; (b+1)I(T^*) - \delta_4\}, \text{ if } u_{n,b+1} \leqq I(R) - I(T^*),$$

(4.18.4) $$0 < u_{n+m-1,b+1} \leqq u_{n,b+1} - I(R) + I(T^*) - \delta_1, \text{ if } u_{n,b+1} > I(R) - I(T^*).$$

Since $P\{B(n, m, \delta_1, \delta_2, \delta_3)\} > \alpha_5$ for some $\alpha_5 > 0$, we see that

$$(4.19)\quad P\left\{\begin{array}{l} 0 < u_{n+m,i} \leqq (c-1)I(T^*),\ i = 1, \cdots, b \\ 0 < u_{n+m,b+1} \leqq \max[u_{n,b+1} + 2I(T^*) - \delta_4,\ (b+1)I(T^*) - \delta_4] \end{array}\right| \quad u_{n,b+1} \leqq I(R) - I(T^*)\Bigg\} > \alpha_5,$$

and

$$(4.20)\quad P\left\{\begin{array}{l} 0 < u_{n+m-1,i} \leqq cI(T^*),\ i = 1, \cdots, b \\ 0 < u_{n+m-1,b+1} \leqq u_{n,b+1} - I(R) + I(T^*) + \delta_1 \end{array}\right| \quad u_{n,b+1} > I(R) - I(T^*)\Bigg\} > \alpha_5 .$$

Finally, let $\epsilon = \min[\delta_4, I(R) - I(T) - \delta_1]$, and note that by hypothesis there is an $\alpha_6 > 0$ such that

$$(4.21)\qquad P\{T_{n+m} \geqq cI(T^*)\} = P\{T_{n+m+1} \geqq cI(T^*)\} > \alpha_6 .$$

Then (4.19), (4.20) and (4.21) imply that for some $\alpha_7 > 0$

$$(4.22)\qquad P\{u_{n+m+1,i} = 0,\ i = 1, \cdots, b;\ u_{n+m+1,b+1} \leqq [u_{n,b+1} - \epsilon]^+ \mid u_{n,b+1} \leqq I(R) - I(T^*)\} > \alpha_7,$$

and

$$(4.23)\qquad P\{u_{n+m} = 0,\ i = 1, \cdots, b;\ u_{n+m,b+1} \leqq u_{n,b+1} - \epsilon \mid u_{n,b+1} > I(R) - I(T^*)\} > \alpha_7 .$$

Letting M be any integer greater than $m + 1$ and $\alpha_3 = \alpha_7$ we thus have proved (4.11).

Case (ii). Since by hypothesis $P\{R = I(R)\} = 0$, there is a $\delta > 0$ which is small compared to $I(T^*)$, $I(R)$ and c, and such that $I(R) + \delta$ is a point of support of R. Then $I(T^*)$ does not divide $I(R) + \delta$, and the proof of Case (i) goes through identically with $I(R)$ replaced by $I(R) + \delta$.

Case (iii). Let $j = I(R)/I(T^*)$. By hypothesis j is an integer. If $j \leqq c$ then $P\{T \geqq cI(T^*)\} \geqq P\{T \geqq I(R)\} > 0$. Since $P\{R = I(R)\} > 0$ it follows that $P\{T \geqq R\} > 0$ and we are in part (a) of the lemma. It thus remains to consider the case $j \geqq c + 1$. Since $P\{T = I(T^*)\} = 0$, there is a $\delta > 0$ which is small compared to $I(T^*)$, $I(R)$ and c, and has the properties that (a) $I(T^*) + \delta$ is a point of support of T, (b) $I(T^*) + \delta$ does not divide $I(R)$, and (c) $I(R)/(I(T^*) + \delta) > c$. The argument of Case (i) now goes through identically with $I(T^*)$ replaced by $I(T^*) + \delta$.

Case (iv). By assumption, $\{I(R)/I(T^*)\} = j$ is an integer. As in Case (iii), if $j \leqq c$, then part (a) of the lemma applies. Assume that $j \geqq c + 1$. Suppose first that u_n is one of the lattice points $\{i_1 I(T^*), \cdots, i_n I(T^*)\}$, where $i_1, \cdots,$

i_n are integers such that $0 \leqq i_n \leqq j$. Clearly there is an integer $m \geqq c$ such that

$$P\left\{\begin{matrix}[T_{n+1} + \cdots + T_{n+m-1} = I(R) - 2I(T^*)] \\ \cap [T_{n+1} + \cdots + T_{n+m} = I(R) - I(T^*)]\end{matrix}\right\} = \alpha_9 > 0.$$

Fix m. The event $B(n, m, I(T^*), 0, 0)$ implies (4.14), (4.15), (4.16) with $\delta_2 = \delta_3 = 0$. Furthermore it implies that the last inequality in (4.15) can be strengthened to read

$$I_{b+1}(n, n + m) \leqq [(b - 1)I(T^*) - u_{n,b+1}]^+. \tag{4.24}$$

Analogously to (4.18), $B(n, m, I(T^*), 0, 0)$ implies

$$\begin{aligned} 0 &< u_{n+m,i} \leqq (c - 1)I(T^*), \; i = 1, \cdots, b, \\ 0 &< u_{n+m-1,i} \leqq cI(T^*), \; i = 1, \cdots, b, \\ 0 &< u_{n+m,b+1} \leqq \max\{u_{n,b+1} + I(T^*), bI(T^*)\}, \\ &\qquad\qquad \text{if} \quad u_{n,b+1} \leqq I(R) - I(T^*), \\ 0 &< u_{n+m-1,b+1} \leqq u_{n,b+1} - I(R) + 2I(T^*), \\ &\qquad\qquad \text{if} \quad u_{n,b+1} > I(R) - I(T^*). \end{aligned} \tag{4.25}$$

Since clearly there is an $\alpha_{10} > 0$ such that $P\{B(n, m, I(T^*), 0, 0)\} > \alpha_{10}$, it follows that there is an $\alpha_{11} > 0$ such that

$$P\left\{\begin{matrix}0 < u_{n+m,i} \leqq (c - 1)I(T^*), \; i = 1, \cdots, b \\ 0 < u_{n+m,b+1} \leqq \max\,[u_{n,b+1} + I(T^*), bI(T^*)]\end{matrix} \;\middle|\; u_{n,b+1} \leqq I(R) - I(T^*)\right\} > \alpha_{11}, \tag{4.26}$$

and

$$P\left\{\begin{matrix}0 < u_{n+m-1,i} \leqq cI(T^*), \; i = 1, \cdots, b \\ 0 < u_{n+m-1,b+1} \leqq u_{n,b+1} - I(R) + 2I(T^*)\end{matrix} \;\middle|\; u_{n,b+1} > I(R) - I(T^*)\right\} > \alpha_{11}. \tag{4.27}$$

Then (4.21), (4.26), (4.27) imply (4.22) and (4.23) with $\epsilon = I(T^*)$, and α_7 replaced by some $\alpha_{12} > 0$.

Finally we note that if u_n is any point in ΔK, but is not one of the lattice points defined in the previous paragraph, then there is an integer $N_1 < \infty$ and a real $\alpha_{13} > 0$ (neither depending on u_n), such that with probability α_{13} the system will be at such a lattice point within N_1 steps. This is a trivial consequence of the assumptions of Case (iv). Thus setting $N \geqq N_1 + [I(R)/I(T^*)]\, c(m + 1)$, and $\alpha_3 = (\alpha_{12})(\alpha_{13})$, we have once again proved (4.11). This proves Case (iv).

It remains to remark that if $I(T^*) = 0$, then we can find an $I'(T^*) > 0$ such that $I'(T^*) < S(T)/c$ and $P\{I'(T^*) \leqq T \leqq I'(T^*) + d\} > 0$ for all $d > 0$. Then the proof goes through as above with $I(T^*)$ replaced by $I'(T^*)$.

To prove necessity, suppose that $P\{T < cI(T^*)\} = 1$, and take $R \equiv cI(T^*)$. Then with probability one, all units are processed, the $(nc + k)$th unit being served by channel k. On the other hand each arriving unit finds its predecessor still being processed, and hence with probability one the system is never empty.

This completes the proof of the lemma.

LEMMA 4. *If* $P\{T \geqq \min [R, cI(T^*)]\} > 0$, *then* $F_{H_0}(\infty) = 1$ *and* $F_{G_0}(\infty) = 1$.

PROOF. Let $W[u_n, N]$ denote the event that if the process is in state u_n at t_n then $u_{n+j} = \Delta 0$ for some $1 \leqq j \leqq N$; and let $\bar{W}$ be the complement of W. We shall show that given any $\delta > 0$, there exists an $\epsilon > 0$, and a sequence of numbers $\{N_i^*\}$, $i = 1, 2, \cdots$, such that

$$P\{\bar{W}[u_0, N_i^*]\} \leqq (1 - \epsilon)^i + \delta/(1 - \delta). \tag{4.28}$$

This will prove that $F_{H_0}(\infty) = 1$, and hence (since $F_T(\infty) = 1$) also that $F_{G_0}(\infty) = 1$.

Choose any $\delta > 0$. To prove (4.28) we shall construct a sequence of real numbers $D_0, D_1, \cdots$, and an associated sequence $N(D_0), N(D_1), \cdots$ of integers, such that if $N_i^* = N(D_0) + \cdots + N(D_i)$, then

$$P\{\bar{W}[u_0, N_i^*]\} \leqq (i - \epsilon)^i + \delta + \delta^2 + \cdots + \delta^i. \tag{4.29}$$

Note that it follows from Lemmas 2 and 3 that there is a fixed $\epsilon > 0$ with the property that given any $D < \infty$, there is an $N(D) < \infty$ such that $P\{W[u_n, N(D)]\} > \epsilon$ for any $u_n \varepsilon N(D)$, regardless of the history of the process up to t_n. Hence letting $D_0 = \max(u_{01}, \cdots, u_{0c})$, we may choose an $N(D_0)$ such that $P\{W[u_0, N(D_0)]\} > \epsilon$. Next choose D_1 so that $P\{u_{N(D_0)} \varepsilon \Delta D_1\} > 1 - \delta$, and then in turn pick $N(D_1)$ such that $P\{W[u_{N(D_0)}, N(D_1)]\} > \epsilon$. Having chosen $D_0, \cdots, D_{i-1}, N(D_0), \cdots, N(D_{i-1})$, pick D_i so that $P\{u_{N_{i-1}^*} \varepsilon \Delta D_i\} > 1 - \delta^i$, and $N(D_i)$ so that $P\{W[u_{N_{i-1}^*}, N(D_i)]\} > \epsilon$.

We can now prove (4.29) by induction. The first step in the induction is implicit in the definition of D_0 and $N(D_0)$. Suppose that (4.29) is true for $i = k - 1$. Then

$$\begin{aligned} P\{\bar{W}[u_0, N_k^*]\} &= P\{\bar{W}[u_0, N_k^*], u_{N_{k-1}^*} \varepsilon \Delta D_k - \Delta 0\} \\ &\quad + P\{\bar{W}[u_0, N_k^*], u_{N_{k-1}^*} \varepsilon \overline{\Delta D_k}\} \\ &\leqq P\{\bar{W}[u_{N_{k-1}^*}, N(D_k)] \mid \bar{W}[u_0, N_{k-1}^*], u_{N_{k-1}^*} \varepsilon \Delta D_k - \Delta 0\} \\ &\quad \cdot P\{\bar{W}[u_0, N_{k-1}^*], u_{N_{k-1}^*} \varepsilon \Delta D_k - \Delta 0\} + \delta^k \end{aligned} \tag{4.30}$$

But by the induction hypothesis $P\{\bar{W}[u_0, N_{k-1}^*], u_{N_{k-1}^*} \varepsilon \Delta D_k - \Delta 0\} \leqq P\{\bar{W}[u_0, N_{k-1}^*]\} \leqq (1 - \epsilon)^{k-1} + \delta + \delta^2 + \cdots + \delta^{k-1}$. Furthermore we have seen that the probability of $\bar{W}[u_{N_{k-1}^*}, N(D_k)]$ is less than $1 - \epsilon$ regardless of the history of the process up to $t_{N_{k-1}^*}$. Applying these two facts to (4.30) we obtain (4.29) with $i = k$. This proves the lemma.

LEMMA 5. *If* $P\{T \geqq \min [R, cI(T^*)]\} > 0$, *then* $\{H_i\}$ *and* $\{G_i\}$ *are renewal process with finite means* $\mu = EH_i$ *and* $\nu = EG_i$.

PROOF. We have already seen that $\{H_i\}$ and $\{G_i\}$ are each a sequence of positive I.I.D. random variables. That $\nu = EG_i < \infty$ follows from our Lemmas 2 and 3, and Theorem 4 of Smith [4]. This together with the assumption that $F_T(0) < 1$, implies that $\mu = EH_i < \infty$.

The proofs of Theorems 2 and 3 will now follow very easily.

PROOF OF THEOREM 2. Hypothesis (a) implies the conclusions of Lemmas 4 and 5.

Furthermore it follows from Lemma 4 that $P\{u_N \varepsilon \Delta 0\} > 0$ for some $N < \infty$. If $P\{T \geqq R\} > 0$, then $P\{u_{N+1} \varepsilon \Delta 0\} > 0$. If $P\{T \geqq R\} = 0$, then there is an $\epsilon > 0$ such that $P\{u_{N-1,i} > \epsilon, i = 1, \cdots, c\} > 0$. If T has a P.A.P. at zero then $P\{0 < T_n < \epsilon/2\} > 0$, and hence $P\{u_{N,i} > \epsilon/2, i = 1, \cdots, c\} > 0$. Thus $P\{u_N \not\varepsilon \Delta 0, u_{N+1} \varepsilon \Delta 0\} > 0$. Hence we have shown that there is always an N with the property that $P\{u_N \varepsilon \Delta 0\} > 0$ and $P\{u_{N+1} \varepsilon \Delta 0\} > 0$, and therefore by definition H_i has period one. Applying this fact and the conclusions of Lemmas 4 and 5 to Smith's Theorem 3 yields our result.

REMARK. Note that if the aperiodicity argument is dropped, then the conclusions of our Lemmas 4 and 5 and Smith's Theorem 3 are still sufficient to guarantee (3.4.1) for some $\lambda \geqq 1$.

PROOF OF THEOREM 3. Our result will follow from the conclusion of Theorem 2 of Smith [4]. We proceed to verify his hypotheses. We have seen that $\{u_t\}$ is an equilibrium process. Our hypothesis (i) implies the conclusion of Lemma 5, and hence there is finite mean recurrence time between regeneration points. Next we note that our hypothesis implies the conclusion of Lemma 4, and hence that with probability one we will eventually be at a regeneration point.

We must verify that the random variables G_i are aperiodic. But our hypothesis (ii) says that the inter-arrival time distribution of successive units has distinct but arbitrarily close points of support. Furthermore G_i is a sum of a positive number of independent random variables each distributed as T. Hence clearly G_i also has a P.A.P. and is aperiodic.

Finally, it remains to verify a technical condition in Smith's theorem, namely that $g(A; t) \cdot \{1 - F_G(t)\}$ is of bounded variation. To do so we make use of his Lemma 2 [4]. Suppose that $A \varepsilon B^+$. For any interval $I = (\tau', \tau'')$, define

$$\delta_I^+ = \begin{array}{r} 0 \text{ if } u_{\tau'} \varepsilon A \text{ and } u_{\tau''} \varepsilon A \\ 0 \text{ if } u_{\tau'} \not\varepsilon A \text{ and } u_{\tau''} \not\varepsilon A \\ -1 \text{ if } u_{\tau'} \not\varepsilon A \text{ and } u_{\tau''} \varepsilon A \\ +1 \text{ if } u_{\tau'} \varepsilon A \text{ and } u_{\tau''} \not\varepsilon A. \end{array}$$

Let n_I be the number of units arriving at the system during any time interval I, and let $y_I = 2n_I + \delta_I^+$. Then it is straight forward to verify that y_I satisfies the conditions of Smith's Lemma 2, and that the required function is therefore of bounded variation. If $A \varepsilon B^{-1}$, set $\delta_I^- = -\delta_I^+$, and define $y_I = 2n_I + \delta_I^-$. This completes the proof of the theorem.

REFERENCES

[1] FELLER, WILLIAM (1957). *An Introduction to Probability Theory and its Applications*, **1** (2nd ed.). Wiley, New York.

[2] POLLACZEK, F. (1957). *Problèmes Stochastiques Posés par le Phénomène de Formation d'Une Queue d'Attente à un Guichet et par des Phénomènes Apparentés*. Gauthier-Villars, Paris.

[3] ROBBINS, H. (1954). A remark on the joint distribution of cumulative sums. *Ann. Math. Statist.* **25** 614–616.

[4] SMITH, W. L. (1955). Regenerative stochastic processes. *Proc. Roy. Soc. London* **232** 6–31.