ON CONVERGENCE TO $+\infty$ IN THE LAW OF LARGE NUMBERS

By Leonard E. Baum[1]

University of Chicago and Institute for Defense Analyses

1. Introduction. Let $\{X_i\}$ be a sequence of identically distributed independent random variables. Denote $\sum_{i=1}^{n} X_i$ by S_n. If $\int X_i^+ = \infty$ while $\int X_i^- < \infty$, it follows from the strong law of large numbers that $S_n/n \to +\infty$ almost everywhere. In [1] Derman and Robbins prove that if $X^+ \not\varepsilon L_\alpha$ while $X^- \varepsilon L_\beta$, $0 < \alpha < \beta < 1$, then $S_n/n \to \infty$ almost everywhere, provided that for all sufficiently large t

$$P\{X^+ > t\} \geqq C/t^\alpha \tag{1}$$

for some constant C. They then asked if the part (1) of their hypothesis could be dropped without altering their conclusion: $S_n/n \to \infty$ almost everywhere. By a construction employing a highly "lacunary" atomic X^+, we show that the answer to this question of Derman and Robbins is negative.

2. The Counterexample.

Theorem 1. *Let ϕ be a continuous non-negative monotonic nondecreasing function on* $[0, \infty)$ *which is unbounded. There exists a sequence $\{Y_i\}$ of positive identically distributed independent random variables such that $Y_i \not\varepsilon L_\phi$,* (i.e., $\int \phi(Y_i) = \infty$), *and a sequence $\{n_j\}$ of positive integers, such that for all δ in the interval,* $0 < \delta < 1$,

$$P\left\{1/n_j^{1/\delta} \sum_{i=1}^{n_j} Y_i \leqq 1\right\} \to 1. \tag{2}$$

Proof. To see the theorem's content, observe that it is strongest for slowly increasing ϕ and for δ close to 1. In fact, part of the construction is unnecessary for $\delta \leqq \frac{1}{2}$.

We will construct the desired common cumulative distribution function F for a set of independent random variables $\{Y_i\}$ by choosing a very rapidly increasing sequence of non-negative integers m_j at which we will place point masses so chosen that the mass strictly beyond m_j is μ_j. Set $m_1 = 0$, $\mu_1 = \frac{1}{2}$ and define the m_j and μ_j inductively for $j \geqq 2$. After $m_1, \mu_1, m_2, \mu_2, \cdots, m_{j-1}, \mu_{j-1}$ have been chosen, we choose m_j so large that there are at least $1/\mu_{j-1}$ numbers of the form $\phi^{-1}(k)$, $k = 1, 2, \cdots$ between m_{j-1} and m_j. This is possible because ϕ is by hypothesis continuous and monotonic nondecreasing to

Received May 2, 1962.

[1] Work done at University of Chicago Mathematics Department under partial support of N.S.F. Grant G 14464. The author wishes to thank Professor Melvin Katz for calling this subject to his attention and for valuable conversations.

infinity. We then define $\mu_j = 1/j(m_j)^{m_j}$; $n_j = (m_j)^{m_j}$. Finally, we add the remaining mass $1 - \sum_{j=1}^{\infty} \mu_j$ to the mass at m_2 say.

Because of our choice of the integers m_j, the Y_i all obey the inequality

$$\sum_{k=1}^{\infty} P\{\phi(Y_i) \geqq k\} \geqq \sum_{j=2}^{\infty} (1/\mu_j)\ (\mu_j) = \infty.$$

Thus, using Abel summation, we see that the constructed Y_i are not in L_ϕ. This is the "largeness" property of the Y_i asserted in the theorem.

For any non-negative independent variables $\{Y_i\}$ which are identically distributed with common cumulative distribution function F we have the inequality

$$P\left\{\sum_{i=1}^{n} Y_i \leqq n^{1/\delta}\right\} \geqq P\left\{\max_{i=1}^{n} Y_i \leqq (1/n)n^{1/\delta}\right\} = [F(n^{(1-\delta)/\delta})]^n. \tag{3}$$

We apply this inequality to the constructed F and evaluate at the chosen integers $n_j = (m_j)^{m_j}$. For any δ in the interval, $0 < \delta < 1$, $m_j(1 - \delta)/\delta$ is eventually >1 so that $F(n_j^{(1-\delta)/\delta})$ is for all sufficiently large j greater than $F(m_j) = 1 - \mu_j$. But by choice of μ_j

$$(1 - \mu_j)^{n_j} = \left(1 - \frac{1}{j(m_j)^{m_j}}\right)^{m_j^{m_j}}$$
$$\to 1$$

as $j \to \infty$. The inequality (3) thus yields the "smallness" property (2) asserted in the theorem.

We proceed to the second portion of our construction. For any β in the interval, $0 < \beta < 1$, we can choose a sequence $\{Z_i\}$ of identically distributed non-negative independent random variables belonging to L_β such that for some δ in the interval, $\beta < \delta < 1$,

$$P\left\{1/n^{1/\delta} \sum_{i=1}^{n} Z_i > 1\right\} \to 1. \tag{4}$$

Take, for example, a sequence of independent variables Z_i with common cumulative distribution function $F(t)$ defined by:

$$F(t) = 0, \qquad t < 0,$$
$$F(t) = \tfrac{1}{2}, \qquad 0 \leqq t \leqq 2^{1/\gamma},$$
$$F(t) = 1 - (1/t^{\gamma}) \qquad 2^{1/\gamma} < t < \infty,$$

where γ is chosen in the interval $\beta < \gamma < 1$. Then $Z_i \in L_\beta$, since

$$\int_{2^{1/\gamma}}^{\infty} t^{\beta}\gamma t^{-\gamma-1}\, dt < \infty$$

for $\beta < \gamma$. Moreover, for this sequence $\{Z_i\}$ the desired relation (4) will be fulfilled for δ in the interval $\gamma < \delta \leqq 1$. In fact, since the Z_i are independent,

$$P\left\{\sum_{i=1}^{n} Z_i > n^{1/\delta}\right\} \geqq P\left\{\max_{i=1}^{n} Z_i > n^{1/\delta}\right\}$$
$$= 1 - P\{Z_i \leqq n^{1/\delta}\}^n$$
$$= 1 - \{1 - (1/n^{\gamma/\delta})\}^n$$
$$\to 1$$

for $0 < \gamma < \delta$.

For our counterexample we combine the constructions of Theorem 1 and of the above paragraph. We choose identically distributed independent random variables $\{X_i\}$ each of which is distributed like the Y_i, $(-Z_i)$, for $t \geqq 0$, $(t < 0)$, respectively; i.e.,

$$P\{X_i \leqq t\} = \begin{cases} P\{Z_i \leqq t\}, & t \leqq 0 \\ P\{Y_i \leqq t\}, & t > 0 \end{cases}.$$

This is possible because both the Y_i and Z_i were chosen to take the value 0 on sets of measure $\frac{1}{2}$. Then, by construction, $X_i^+ \not\in L_\phi$ while $X_i^- \in L_\beta$. However, for the subsequence $\{n_j\}$ of positive integers chosen in Theorem 1, and for sufficiently large $\delta < 1$,

$$P\{S_{n_j} \leqq 0\} \geqq P\left\{\left\{\sum_{i=1}^{n_j} X_i^+ \leqq n_j^{1/\delta}\right\} \cap \left\{\sum_{i=1}^{n_j} X_i^- > n_j^{1/\delta}\right\}\right\} \to 1 \text{ as } j \to \infty.$$

A fortiori, S_n/n does not converge to $+\infty$ almost everywhere or in measure. To obtain a counterexample to the specific question of Derman and Robbins [1] which was stated in the introduction of this note it is only necessary to choose, for example, the function $\log^+$ for ϕ, since $Y \not\in L_{\log^+}$ implies $Y \not\in L_\alpha$ for $0 < \alpha$.

3. An affirmative theorem. The counterexample suggests that we will have great difficulty in obtaining a positive result on convergence to $+\infty$ without some uniformity condition such as (1) on the largeness of X^+. Therefore, we state the following variant of the theorem of [1], which involves a considerable lightening of the restriction on X^- and a minor strengthening of the condition (1) on X^+ and yields a weaker conclusion (only convergence in measure).

THEOREM 2. *Let X_i be identically distributed independent random variables such that for some α in the interval $0 < \alpha < 1$, $X_i^- \in L_\alpha$ while X_i^+ obeys*

$$t^\alpha P\{X_i^+ > t\} \to \infty \quad \text{as} \quad t \to \infty.$$

Then $S_n/n \to \infty$ in measure. (i.e. *for all K, $P\{S_n/n > K\} \to 1$.*)

PROOF. (Modeled upon Derman and Robbins [1].)

$$P\left\{\frac{1}{n^{1/\alpha}} \sum_{i=1}^{n} X_i^+ \leqq 1\right\} \leqq P\left\{\max_{i=1}^{n} X_i^+ \leqq n^{1/\alpha}\right\}$$
$$= P\{X_1^+ \leqq n^{1/\alpha}\}^n$$
$$= \left\{1 - \frac{k(n)}{n}\right\}^n$$
$$\to 0,$$

for $k(n) \rightarrow \infty$ when $n \rightarrow \infty$ by hypothesis. Since the Marcinkiewicz theorem [2] yields the almost everywhere convergence of $1/n^{1/\alpha} \sum_{i=1}^{n} X_i^-$ to zero, the result follows.

The hypothesis insures $X_i^+ \not\in L_\alpha$ but we may of course have $X_i^+ \in L_{\alpha_0}$ for all $\alpha_0 < \alpha$, as the function $X^+ = |\log t|/t^{1/\alpha}$ of the variable t distributed uniformly on the unit interval demonstrates. Unlike the theorem of Derman and Robbins [1] which we stated in the introduction, our Theorem 2 is thus a theorem with conditions involving the same index α for X_i^+ and X_i^-. Necessary and sufficient conditions for convergence in measure to $+\infty$ would be desirable.

REFERENCES

[1] Derman, C. and Robbins, H. (1955). The strong law of large numbers when the first moment does not exist. *Proc. Nat. Acad. Sci. USA*, **41** 586–587.

[2] Loève, M. (1960). *Probability Theory*, 2nd Ed. Van Nostrand, Princeton. 243–244.