STOCHASTIC PROCESSES ON A SPHERE[1]

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0. Summary. Spectral representations of stochastic fields on a sphere are given for spherically symmetric and axially symmetric cases. Unbiased estimates of the spectral parameters are presented and the variance calculated for normally distributed fields. Time varying processes are discussed briefly with reference to terrestrial situations.

1. Introduction. Let $\xi(P)$ be a real valued random field on a unit sphere, S_2, of the three-dimensional space R_3, which has finite variance and realizations, $x(P)$, which are quadratically integrable over the surface of the sphere

$$\int_P x^2(P)\, d\Omega_P < \infty;$$

Ω_P denoting the surface element, and $\int_P$ integration over the entire surface. Because of the completeness of the spherical harmonics $Y_\nu^m(P)$, the random field may be represented

$$\xi(P) = \sum_{\nu=0}^{\infty} \sum_{m=-\nu}^{\nu} Z_{\nu m} Y_\nu^m(P). \tag{1}$$

By an isotropic field we mean one whose covariance depends only on the spherical distance between two points and whose mean $E\xi(P)$ is constant. Without loss of generality, it will be assumed that $E\xi(P) = 0$, which implies $EZ_{\nu m} = 0$. The conditions for isotropy for normalized spherical harmonics

$$\left(\int_P [Y_\nu^m(P)]^2 d\Omega_P = 1\right)$$

are

$$EZ_{\nu m} Z_{\mu n} = \delta_{\nu\mu}\delta_{mn} f_\nu \geqq 0 \tag{2}$$

δ_{ij} being the Kronecker delta. The representation (1) and conditions (2) were given by Obukhov in 1947 (Yaglom, [3]).
The covariance function is

$$r(P, Q) = E\xi(P)\xi(Q) = \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} \sum_{m=-\nu}^{\nu} \sum_{n=-\mu}^{\mu} EZ_{\nu m} Z_{\mu n} Y_\nu^m(P) Y_\mu^n(Q).$$

Calling the angle between two points P and Q, γ_{PQ}, and using the conditions for isotropy (2), $r(\gamma_{PQ}) = \sum_{\nu=0}^{\infty} f_\nu \sum_{m=-\nu}^{\nu} Y_\nu^m(P) Y_\nu^m(Q)$, which by the addition

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theorem of spherical harmonics ([1], p. 268) gives

$$r(\gamma_{PQ}) = \sum_{\nu=0}^{\infty} [(2\nu + 1)/4\pi] f_\nu P_\nu(\cos \gamma_{PQ}), \tag{3}$$

where $P_\nu(\cos \gamma_{PQ})$ is the Legendre polynomial of degree ν. This spectral representation of a positive definite function on a sphere was also given by Schoenberg [2]. Multiplying both sides of (3) by $P_k(\cos \gamma_{PQ})$ and integrating with respect to one of the points over the sphere, $f_k = \int_Q r(\gamma_{PQ}) P_k(\cos \gamma_{PQ})\, d\Omega_Q$, since $\int_Q [P_k(\cos \gamma_{PQ})]^2\, d\Omega_Q = 4\pi/(2k + 1)$.

2. Estimating the Spectrum. Using a realization, $x(P)$, of the isotropic field, a quadratic estimate whose weight function depends only on the spherical distance between two points is

$$f_k^* = \int_P \int_Q w_k(\gamma_{PQ}) x(P) x(Q)\, d\Omega_P\, d\Omega_Q,$$

where $\int_Q w_k^2(\gamma_{PQ})\, d\Omega_Q < \infty$. Writing $w_k(\gamma_{PQ}) = \sum_{\nu=0}^{\infty} a_\nu P_\nu(\cos \gamma_{PQ})$,

$$Ef_k^* = \int_P \int_Q w_k(\gamma_{PQ}) r(\gamma_{PQ})\, d\Omega_P\, d\Omega_Q = 4\pi \int_Q w_k(\gamma_{PQ}) r(\gamma_{PQ})\, d\Omega_Q$$
$$= \sum_{\nu,\mu=0}^{\infty} a_\nu (2\mu + 1) f_\mu \int_Q P_\nu\, (\cos \gamma_{PQ}) P_\mu\, (\cos \gamma_{PQ})\, d\Omega_Q = 4\pi \sum_{\nu=0}^{\infty} a_\nu f_\nu .$$

The only choice of the a's which gives an unbiased estimate ($Ef_k^* = f_k$) for any spectrum is $a_\nu = \delta_{\nu k}/4\pi$. Then $w_k(\gamma_{PQ}) = (1/4\pi) P_k(\cos \gamma_{PQ})$, so

$$f_k^* = \frac{1}{4\pi} \int_P \int_Q P_k\, (\cos \gamma_{PQ}) x(P) x(Q)\, d\Omega_P\, d\Omega_Q . \tag{4}$$

Relaxing the restriction that the quadratic estimate depends only on the spherical distance, consider the general weight function

$$f_k^* = \int_P \int_Q w(P, Q) x(P) x(Q)\, d\Omega_P\, d\Omega_Q, \tag{5}$$

where $w(P, Q) = w(Q, P)$ and $\int_P \int_Q w^2(P, Q) d\Omega_P\, d\Omega_Q < \infty$. $w(P, Q)$ may be represented

$$w(P, Q) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{m=-i}^{i} \sum_{n=-j}^{j} a_{ijmn} Y_i^m(P) Y_j^n(Q). \tag{6}$$

Equation (5) becomes (expressing the four summations in equation (6) by a single summation sign)

$$f_k^* = \sum_{i,j,m,n} a_{ijmn} \int_P \int_Q Y_i^m(P) Y_j^n(Q) x(P) x(Q)\, d\Omega_P\, d\Omega_Q . \tag{7}$$

$$Ef_k^* = \sum_{i,j,m,n} a_{ijmn} \int_P \int_Q Y_i^m(P) Y_j^n(Q) r(\gamma_{PQ})\, d\Omega_P\, d\Omega_Q$$
$$= \sum_{i,j,m,n} a_{ijmn} \sum_{\nu=0}^{\infty} \frac{2\nu + 1}{4\pi} f_\nu \int_P \int_Q Y_i^m(P) Y_j^n(Q) P_\nu\, (\cos \gamma_{PQ})\, d\Omega_P\, d\Omega_Q .$$

Using the formula for spherical harmonics ([1], p. 266)

$$\frac{2n+1}{4\pi}\int_Q Y_n^m(Q)P_\nu(\cos\gamma_{PQ})\,d\Omega_Q = \begin{cases}0, & \nu \neq n\\ Y_n^m(P), & \nu = n,\end{cases} \tag{8}$$

$Ef_k^* = \sum_{i,j,m,n} a_{ijmn} f_j \int_P Y_i^m(P)Y_j^n(P)\,d\Omega_P = \sum_{i=0}^{\infty} f_i \sum_{m=-i}^{i} a_{iimm}$. This will be unbiased for any choice of the f's if and only if

$$\sum_{m=-i}^{i} a_{iimm} = \delta_{ik}\,. \tag{9}$$

Then $Ef_k^* = f_k$.

3. The Variance. Assuming the field to be Gaussian, i.e. for any choice of N, the joint distribution of the field at any N points on the sphere is an N-dimensional normal distribution, the variance of the unbiased estimate (5) is

$$\begin{aligned} D^2 f_k^* &= E\int_P\int_Q\int_R\int_S w(P,Q)w(R,S)x(P)x(Q)\\ &\qquad\cdot x(R)x(S)\,d\Omega_P\,d\Omega_Q\,d\Omega_R\,d\Omega_S - f_k^2\\ &= \sum_{ijmn}\sum_{klpq}\int_P\int_Q\int_R\int_S Y_i^m(P)Y_j^n(Q)Y_k^p(R)Y_l^q(S)\\ &\qquad\cdot[r(\gamma_{PR})r(\gamma_{QS}) + r(\gamma_{PS})r(\gamma_{QR})]\,d\Omega_P\,d\Omega_Q\,d\Omega_R\,d\Omega_S\,. \end{aligned} \tag{10}$$

The first half of this integral may be written

$$\begin{aligned} &\sum_{ijmn}\sum_{klpq}\sum_{\nu,\mu=0}^{\infty} a_{ijmn}a_{klpq}\frac{(2\nu+1)(2\mu+1)}{(4\pi)^2} f_\nu f_\mu\\ &\qquad\cdot\int_P\int_Q\int_R\int_S Y_i^m(P)Y_j^n(Q)Y_k^p(R)Y_l^q(S)\\ &\qquad\cdot P_\nu(\cos\gamma_{PR})P_\mu(\cos\gamma_{QS})\,d\Omega_P\,d\Omega_Q\,d\Omega_R\,d\Omega_S\,. \end{aligned} \tag{11}$$

Using (8) and the orthogonality relations, the integrals in (11) vanish unless $\mu = l = j$, $\nu = k = i$, $n = q$ and $m = p$, and (11) becomes $\sum_{ijmn} a_{ijmn}^2 f_i f_j$. The second half of equation (10) gives the same result with a_{ijmn}^2 replaced by $a_{ijmn}a_{jinm}$, but $a_{jinm} = a_{ijmn}$ since $w(P,Q) = w(Q,P)$ so $D^2 f_k^* = 2\sum_{ijmn} a_{ijmn}^2 f_i f_j$. Every term in this sum is non-negative since $f \geqq 0$. In order to minimize the variance under the constraints (9), all the a's except a_{kkmm}, $m = -k, -k+1, -k+2, \cdots k$, should be made equal to zero as they only increase the variance without improving the estimate. Then $D^2 f_k^* = 2f_k^2 \sum_{m=-k}^{k} a_{kkmm}^2$. This is minimized under the constraint $\sum_{m=-k}^{k} a_{kkmm} = 1$ when the terms of the sum are equal, i.e. $a_{kkmm} = 1/(2k+1)$. Substituting these coefficients into (7), and again using the addition theorem for spherical harmonics, the nonparametric estimate of minimum variance,

$$D^2(f_k^*) = [2/(2k+1)]f_k^2\,,$$

is $f_k^* = (1/4\pi) \int_P \int_Q P_k(\cos\gamma_{PQ}) x(P) x(Q)\, d\Omega_P\, d\Omega_Q$, the same result that was obtained in §2.

4. Axial Symmetry. Using spherical coordinates, the point P is determined by θ, the longitude $(0 \leqq \theta \leqq 2\pi)$, and φ the polar colatitude$(0 \leqq \varphi \leqq \pi)$. Equation (1) can be expressed in terms of associated Legendre functions,

$$\xi(P) = \sum_{\nu=0}^{\infty} \sum_{m=0}^{\nu} [Z_{\nu m} \cos m\theta + Z'_{\nu m} \sin m\theta] P_\nu^m(\cos\varphi).$$

It will be assumed here that the harmonics have been normalized so that

$$\int_P \left[P_\nu^m \,(\cos\varphi_P) \,\begin{matrix}\cos m\theta_P \\ \sin m\theta_P\end{matrix}\right]^2 d\Omega_P = 1.$$

Now

$$r(P, Q) = E\xi(P)\xi(Q) = \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} \sum_{m=0}^{\nu} \sum_{n=0}^{\mu} E[Z_{\nu m} Z_{\mu n} \cos m\theta_P \cos n\theta_Q$$
$$+ Z_{\nu m} Z'_{\mu n} \cos m\theta_P \sin n\theta_Q + Z'_{\nu m} Z_{\mu n} \sin m\theta_P \cos n\theta_Q$$
$$+ Z'_{\nu m}\, Z'_{\mu n} \sin m\theta_P \sin n\theta_Q] P_\nu^m(\cos\varphi_P) P_\mu^n(\cos\varphi_Q).$$

When $EZ_{\nu m} Z_{\mu n} = Z'_{\nu m} Z'_{\mu n} = \delta_{mn} f_{\nu\mu m}$, and $EZ'_{\nu m} Z_{\mu n} = -EZ_{\nu m} Z'_{\mu n} = \delta_{mn} f'_{\nu\mu m}$, $(f_{\nu\mu m} = f_{\mu\nu m},\, f'_{\nu\mu m} = -f'_{\mu\nu m})$,

$$(12) \quad r(P, Q) = \sum_{m=0}^{\infty} \sum_{\nu=m}^{\infty} \sum_{\mu=m}^{\infty} [f_{\nu\mu m} \cos m(\theta_P - \theta_Q) + f'_{\nu\mu m} \sin m(\theta_P - \theta_Q)]$$
$$\cdot P_\nu^m(\cos\varphi_P) P_\mu^m(\cos\varphi_Q),$$

i.e. the covariance function then depends on θ_P, θ_Q only through the difference $\theta_P - \theta_Q$. When $EZ'_{\nu m} Z_{\mu n} = \delta_{mn} f'_{\nu\mu m} = 0$, the covariance function is symmetric with respect to positive and negative rotations. That a covariance function which is axially symmetric can be represented as in equation (12), follows from the completeness of $P_m^m(\cos\varphi)$, $P_{m+1}^m(\cos\varphi)$, $P_{m+2}^m(\cos\varphi)$, $\cdots$ in the interval $0 \leqq \varphi \leqq \pi$.

Since the covariance function is non-negative definite, i.e.

$$\int_P \int_Q r(P, Q) c(P) c(Q)\, d\Omega_P\, d\Omega_Q \geqq 0,$$

where $c(P)$ is any quadratically integrable function on the sphere, the relations $a_m^T F_m a_m + 2a_m^T F'_m b_m + b_m^T F_m b_m \geqq 0$ where F_m is the symmetric matrix with elements $f_{\nu\mu m}$ and F'_m is the antisymmetric matrix with elements $f'_{\nu\mu m}$, must be satisfied for any finite vectors a_m and b_m and all values of m.

Estimates of the constants $f_{\nu\mu m}$ may be formed as in §2 giving

$$f_{\nu\mu m}^* = \int_P \int_Q P_\nu^m \,(\cos\varphi_P) P_\mu^m \,(\cos\varphi_Q) \cos m(\theta_P - \theta_Q) x(P) x(Q)\, d\Omega_P\, d\Omega_Q.$$

The estimate of $f'_{\nu\mu m}$ replaces $\cos m(\theta_P - \theta_Q)$ by $\sin m(\theta_P - \theta_Q)$. $Ef_{\nu\mu m}^* = f_{\nu\mu m}$, and if the field is normal, $D^2(f_{\nu\mu m}^*) = 2f_{\nu\mu m}^2$. If the field is isotropic, $f_{\nu\mu m} = \delta_{\nu\mu} f_\nu$

and $f'_{\nu\mu m} = 0$. Writing

$$f^*_{\nu\mu m} = \left[\int_P P_\nu^m (\cos \varphi_P) \cos m\theta_P x(P) \, d\Omega_P\right]^2 + \left[\int_P P_\nu^m (\cos \varphi_P) \sin m\theta_P x(P) \, d\Omega_P\right]^2,$$

the estimate f^*_ν may be expressed

$$f^*_\nu = \frac{1}{2\nu + 1} \sum_{m=0}^{\nu} f^*_{\nu\nu m} = \sum_{m=0}^{\nu} \left[\int_P P_\nu^m(\cos \varphi_P) \cos m\theta_P \, x(P) \, d\Omega_P\right]^2 + \sum_{m=1}^{\nu} \left[\int_P P_\nu^m(\cos \varphi_P) \sin m\theta_P \, x(P) \, d\Omega_P\right]^2.$$

This is the sum of $2\nu + 1$ integrations over the sphere squared, and in most cases should be easier to handle numerically than the double integration given in equation (4).

5. Time varying processes. If a random field on a sphere varies with time, the representation (1) becomes

$$\xi(P, t) = \sum_{\nu=0}^{\infty} \sum_{m=-\nu}^{\nu} Z_{\nu m}(t) Y_\nu^m(P),$$

$Z_{\nu m}(t)$ being an ordinary one-dimensional stochastic process. The set of all $Z_{\nu m}(t)$ form a denumerably infinite dimensional stochastic process which completely defines the process on the sphere. In practice, however, a finite number of components which gives a sufficiently good representation of the process can be used. This representation has several desirable properties. If there is a tendency towards isotropy, it would be expected that the correlations between components, at any given time, would be low. When rotations about an axis enter, as when dealing with the atmosphere, the interdependence caused affects the components only in pairs. This can be seen by considering a realization of the process at a given time,

$$x(P) = \sum_{\nu=0}^{\infty} \sum_{m=0}^{\nu} [a_{\nu m} \cos m\theta + b_{\nu m} \sin m\theta] P_\nu^m(\cos \varphi).$$

When there is a change in the coordinate system, the spherical harmonics in the new coordinate system can always be expressed as a linear combination of $2\nu + 1$ independent spherical harmonics of the same order in the old coordinate system. When the change in coordinates is a rotation through an angle ψ about the axis,

$$x(P) = \sum_{\nu=0}^{\infty} \sum_{m=0}^{\nu} [a_{\nu m} \cos m(\theta - \psi) + b_{\nu m} \sin m(\theta - \psi)] P_\nu^m(\cos \varphi)$$

$$= \sum_{\nu=0}^{\infty} \sum_{m=0}^{\nu} [a'_{\nu m} \cos m\theta + b'_{\nu m} \sin m\theta] P_\nu^m(\cos \varphi)$$

where $a'_{\nu m} = a_{\nu m} \cos m\psi - b_{\nu m} \sin m\psi$, $b'_{\nu m} = a_{\nu m} \sin m\psi + b_{\nu m} \cos m\psi$. Therefore, the interdependence between components caused by rotations can be removed by this transformation.

In preliminary attempts to predict the height of a constant pressure field (500-millibar) in meteorology, it has been found that better predictions are obtained using the representation

$$x(P) = \sum_{\nu=0}^{\infty} \sum_{m=0}^{\nu} c_{\nu m} \cos(\theta + \psi_{\nu m}) P_\nu^m(\cos \varphi)$$

where $c_{\nu m}$ is the magnitude of the component and $\psi_{\nu m}$ is the phase angle.

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