DISTRIBUTION OF DEFINITE AND OF INDEFINITE QUADRATIC FORMS FROM A NON-CENTRAL NORMAL DISTRIBUTION

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1. Summary. In this paper we generalize the results of John Gurland [1] on the distribution of definite and indefinite quadratic forms to non-central normal variates.

The results of this paper may be compared with those of [4], by H. Ruben, where, by a purely geometric approach, the distribution functions of homogeneous and non-homogeneous quadratic forms are expressed as infinite linear combinations of central and non-central chi-square distribution functions.

2. Introduction. Suppose we have a quadratic form $\mathbf{y}'\mathbf{A}\mathbf{y}$ where $\mathbf{A}$ is a $p \times p$ symmetric matrix of rank $n \leqq p$, $\mathbf{y}' = (y_1, \cdots, y_p)$, and the y_i' are independent normal variates with means ν_i and variance one ($i = 1, 2, \cdots, p$). It is well known that we can make an orthogonal transformation reducing $\mathbf{y}'\mathbf{A}\mathbf{y}$ to its canonical form $\sum_{i=1}^n \lambda_i x_i^2$, where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the latent roots of the matrix $\mathbf{A}$. Under such a transformation $x_1, x_2, \cdots, x_n$ are independent normal variates with means μ_i and variance one. (The μ_i ($i = 1, \cdots, n$) are obtained from the ν_j ($j = 1, \cdots, p$) in the same manner as the x_i are obtained from the y_j). Our problem is to find the distribution function $F(x)$ of $\sum_{i=1}^n \lambda_i x_i^2$ where the λ's are real numbers and $\mathbf{x}$ has the probability density function

$$f(\mathbf{x}) = (2\pi)^{-\frac{1}{2}n} \exp\left[-\tfrac{1}{2}(\mathbf{x} - \boldsymbol{\mu})'(\mathbf{x} - \boldsymbol{\mu})\right], \tag{1}$$

where $\mathbf{x}' = (x_1, x_2, \cdots, x_n)$ and $\boldsymbol{\mu} = (\mu_1, \mu_2, \cdots, \mu_n)$.

3. Distribution of a positive-definite quadratic form. Suppose $\lambda_1, \cdots, \lambda_n$ are all positive and let

$$\alpha_j = \lambda_j - \bar{\lambda}, \tag{2}$$

where $\bar{\lambda}$ is an arbitrary number satisfying the inequality

$$\bar{\lambda} > \tfrac{1}{2} \max_j \lambda_j . \tag{3}$$

The characteristic function of $\sum_{j=1}^n \lambda_j x_j^2$ may be written as

$$\phi(t) = \exp\left(-\frac{1}{2}\sum_{j=1}^n \mu_j^2\right) \exp\left\{\sum_{j=1}^n \left[\tfrac{1}{2}\mu_j^2(1 - 2it\lambda_j)^{-1}\right]\right\} \prod_{j=1}^n (1 - 2it\lambda_j)^{-\frac{1}{2}}. \tag{4}$$

Expanding the exponential term containing t we have

$$\phi(t) = \exp\left(-\frac{1}{2}\sum_{j=1}^n \mu_j^2\right) \sum_{k=0}^\infty (k!)^{-1} \left\{\sum_{j=1}^n \tfrac{1}{2}\mu_j^2(1 - 2it\lambda_j)^{-1}\right\}^k \cdot \prod_{j=1}^n (1 - 2it\lambda_j)^{-\frac{1}{2}}.$$

Again, expanding the curly bracket by the multinomial theorem and substituting

Received February 8, 1961; revised May 28, 1962.

the values of the λ_j's from (2), we get

$$\phi(t) = \sum_{k=0}^{\infty} \sum_{\pi\text{'s}} C_k(\mu, \pi)(1 - 2it\bar{\lambda})^{-\frac{1}{2}n-k} \prod_{j=1}^{n} \{1 - 2it\alpha_j(1 - 2it\bar{\lambda})^{-1}\}^{\pi_j - \frac{1}{2}},$$

where $\sum_{\pi\text{'s}}$ means summation over π_j's such that $\sum_{j=1}^{n} \pi_j = k$, and

$$(5) \qquad C_k(\mu, \pi) = \frac{\exp\left(-\frac{1}{2}\sum_{j=1}^{n} \mu_j^2\right)(\mu_1^2)^{\pi_1} \cdots (\mu_n^2)^{\pi_n}}{k!\,\pi_1! \cdots \pi_n!\,2^k}.$$

Since $|2it\alpha_j(1 - 2it\bar{\lambda})^{-1}| < 1$ $(j = 1, 2, \cdots, n)$ for all values of t, $\phi(t)$ may be expanded as the product of n power series. Thus, we can write

$$(6) \qquad \phi(t) = \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} b_{k,p}(-2it)^p(1 - 2it\bar{\lambda})^{-\frac{1}{2}n-p-k},$$

where

$$(7) \qquad b_{k,p} = \sum_{\pi\text{'s}} C_k(\mu, \pi)a_p(\pi)$$

and $a_p(\pi)$ is the coefficient of θ^p in the expansion of

$$(8) \qquad \prod_{j=1}^{n} \sum_{l=0}^{\infty} \alpha_j^l \beta_{j,l} \theta^l, \qquad \beta_{j,l} = (-1)^l \binom{\pi_j + l - \frac{1}{2}}{l}.$$

Explicitly, $a_p(\pi)$ may be written as

$$a_p(\pi) = \sum_{j=1}^{n} \beta_{j,p}\alpha_j^p + \sum_{j \neq t} \beta_{j,p-1}\beta_{t,1}\alpha_j^{p-1}\alpha_t$$

$$(8.1) \qquad + \sum_{j \neq t} \beta_{j,p-2}\beta_{t,2}\alpha_j^{p-2}\alpha_t^2 + \cdots + \sum_{j \neq t \neq l} \beta_{j,p-2}\beta_{t,1}^2\alpha_j^{p-2}\alpha_t\alpha_l$$

$$+ \sum_{j \neq t \neq l} \beta_{j,p-3}\beta_{t,2}\beta_{l,1}\alpha_j^{p-3}\alpha_t^2\alpha_l + \cdots.$$

Thus,

$$a_0(\pi) = 1, \; a_1(\pi) = \sum_{j=1}^{n} \beta_{j,1}\alpha_j, \; a_2(\pi) = \sum_{t \neq j} \beta_{t,1}\beta_{j,1}\alpha_t\alpha_j + \sum_{j} \beta_{j,2}\alpha_j^2,$$

$$a_3(\pi) = \sum_{t \neq j \neq l} \beta_{t,1}\beta_{j,1}\beta_{l,1}\alpha_t\alpha_j\alpha_l$$

$$+ \sum_{t \neq j} \beta_{t,2}\beta_{j,1}\alpha_t^2\alpha_j + \sum_{j=1}^{n} \beta_{j,3}\alpha_j^3, \cdots.$$

Application of the inversion formula [2], namely,[1]

$$(9) \qquad F(x) = \tfrac{1}{2} - (2\pi i)^{-1} \oint \phi(t)t^{-1} \exp(-itx)\, dt,$$

[1] The integral $\oint$ is understood as a principle value, i.e., the limit, as $\epsilon \to 0^+$ and $T \to \infty$ of the integral over $\epsilon < |t| < T$.

to (6), which is uniformly convergent for all t, yields

$$(10)\quad F(x) = \tfrac{1}{2} - (2\pi i)^{-1} \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} b_{k,p} \oint t^{-1}(-2it)^{p}(1 - 2it\bar{\lambda})^{-\frac{1}{2}n-p-k} \exp(-itx)\, dt.$$

By using

$$\tfrac{1}{2} - (2\pi i)^{-1} \oint t^{-1} \exp(-2it\bar{\lambda}x)(1 - 2it\bar{\lambda})^{-\frac{1}{2}n-k}\, dt$$
$$= \{2^{\frac{1}{2}n+k}\,\Gamma(\tfrac{1}{2}\,n + k)\}^{-1} \int_0^{2x} v^{\frac{1}{2}n+k-1} \exp(-\tfrac{1}{2}v)\, dv,$$

and

$$(\bar{\lambda})^{p}(-2\pi i)^{-1} \oint (-2it)^{p}(1 - 2it\bar{\lambda})^{-\frac{1}{2}n-p-k} t^{-1} \exp(-2it\bar{\lambda}x)\, dt$$
$$= \Gamma(p)\{\Gamma(\tfrac{1}{2}n + p + k)\}^{-1} \exp(-x)x^{\frac{1}{2}n+k} L_{p-1}^{(\frac{1}{2}n+k)}(x),$$

where $p \geqq 1$, and [3]

$$(d/dx)^{p} \exp(-x)x^{\gamma+p} = p!\exp(-x)x^{\gamma}L_{p}^{(\gamma)}(x),\ \gamma > -1,$$

we rewrite (10) as

$$(11)\quad \begin{aligned} F(x) = \sum_{k=0}^{\infty} \Bigg\{ & b_{k,0}[2^{\frac{1}{2}n+k}\Gamma(\tfrac{1}{2}n + k)]^{-1} \int_0^{x/\bar{\lambda}} v^{\frac{1}{2}n+k-1} \exp(-\tfrac{1}{2}v)\, dv \\ & + \sum_{p=1}^{\infty} b_{k,p} \frac{\Gamma(p) \exp(-x/2\bar{\lambda})x^{\frac{1}{2}n+k}}{\Gamma(\frac{1}{2}n + p + k)2^{\frac{1}{2}n+k}\bar{\lambda}^{\frac{1}{2}n+p+k}} L_{(p-1)}^{(\frac{1}{2}n+k)}(x/2\bar{\lambda}) \Bigg\}. \end{aligned}$$

We see from (11) that Gurland's result [1] is the particular case when all $\mu_i = 0$ $(i = 1, 2, \cdots, n)$.

4. Distribution of an indefinite quadratic form. Suppose

$$\mathbf{y}'\mathbf{A}\mathbf{y} = \sum_{j=1}^{n_1} \lambda_j x_j^2 - \sum_{j=n_1+1}^{n} \lambda_j x_j^2$$

where $\lambda_j > 0$ for $j = 1, 2, \cdots, n$ and $n = n_1 + n_2$. We continue to assume that $\mathbf{x}$ has the probability density $f(\mathbf{x})$ of (1). Define α_j and $\bar{\lambda}$ as in (2) and (3) respectively. Then the characteristic function $\phi(t)$ can be written as

$$(12)\quad \begin{aligned} \phi(t) = \sum_{k,l=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{p} b'_{k,q}\, d_{l,p-q}(-1)^{q}(2it)^{p}(1 - 2it\bar{\lambda})^{-\frac{1}{2}n_1-k-q} \\ \cdot(1 + 2it\bar{\lambda})^{-\frac{1}{2}n_2-l-p+q}, \end{aligned}$$

where $b'_{k,q}$ is expressible as in (7) with n_1 (in place of n),

$$d_{l,p-q} = \sum_{\eta's} e_l(\mu, \eta) g_{p-q}(\eta),$$

$$e_l(\mu, \eta) = \frac{\exp\left(-\frac{1}{2} \sum_{j=n_1+1}^{n} \mu_j^2\right) (\mu_{n_1+1}^2)^{\eta_1} \cdots (\mu_n^2)^{\eta_{n_2}}}{\eta_1! \eta_2! \cdots \eta_{n_2}! l! 2^l},$$

and $g_{p-q}(\eta)$ may be expressed, similarly to (8), as

$$g_j(\eta) = \sum_{t=n_1+1}^{n} \nu_{t,j} \alpha_t^j + \sum_{t \neq t'=n_1+1}^{n} \nu_{t,j-1} \nu_{t',1} \alpha_t^{j-1} \alpha_{t'} + \cdots,$$

where $\nu_{t,j} = \binom{\eta_j + t - \frac{1}{2}}{t} (-1)^t$. Applying the inversion formula (9), the distribution function may be written as

$$(13) \quad \begin{aligned} F(x) &= \frac{1}{2} - (2\pi i)^{-1} \sum_{k,l=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{p} b'_{k,q}\, d_{l,p-q} (-1)^q \\ &\quad \cdot \oint (2it)^p t^{-1} (1 - 2it\bar{\lambda})^{-\frac{1}{2}n_1 - k - q} (1 + 2it\bar{\lambda})^{-\frac{1}{2}n_2 - l - p + q} \exp(-itx)\, dt. \end{aligned}$$

Making use of the J-polynomials and K-polynomials in the above integration (See Gurland [1]), we have the distribution function, for $x \geqq 0$.

$$(14) \quad \begin{aligned} F(x) &= \sum_{k,l=0}^{\infty} \Bigg[b'_{k,0}\, d_{l,0} \Bigg\{ K + c^{-1} \sum_{h=0}^{m+k-1} \binom{m + k - 1}{h} \Gamma\left(h + \frac{1}{2} n_2 + l\right) \\ &\quad \cdot \int_0^{x/\bar{\lambda}} \exp(-v/2) v^{m+k-h-1}\, dv \Bigg\} + c^{-1} \exp(-x/2\bar{\lambda}) \\ &\quad \cdot \sum_{p=1}^{\infty} \sum_{q=0}^{p} b'_{k,q}\, d_{l,p-q} (-1)^{p+q} \bar{\lambda}^{-p} \sum_{h=0}^{m+k+q-1} \binom{m + k + q - 1}{h} \\ &\quad \cdot 2^{m+k+q-h} \Gamma(h + \tfrac{1}{2} n_2 + l + p - q) K_{m+k+q-h-1,p-1}^{(x/2\bar{\lambda})} \Bigg], \end{aligned}$$

where n_1 is an even integer, say $2m$. For $x \leqq 0$ and $n_2 = 2m'$, we have

$$(15) \quad \begin{aligned} F(x) &= \sum_{k,l=0}^{\infty} c^{-1} \Bigg[b'_{k,0}\, d_{l,0} \sum_{h=0}^{m+k-1} \binom{m + k - 1}{h} \\ &\quad \cdot \int_{-\infty}^{x/\bar{\lambda}} \exp(-v/2) v^{m+k-h-1}\, dv \int_{-v}^{\infty} \exp(-y) y^{h+m'+l-1}\, dy \\ &\quad + \exp(-x/2\bar{\lambda}) \sum_{p=1}^{\infty} \sum_{q=0}^{p} b'_{k,q}\, d_{l,p-q} (-1)^{p+q} \bar{\lambda}^{-p} \sum_{h=0}^{m+k+q-1} \sum_{\gamma=0}^{p-1} \\ &\quad \cdot \binom{m + k + q - 1}{h} \binom{p - 1}{r} K_{m+k+q-h-1,p-1-r}^{(x/2\bar{\lambda})} J_{h+m'+l+p-q-1,r}^{(x/2\bar{\lambda})} \Bigg]. \end{aligned}$$

5. Acknowledgment. The author wishes to express his thanks to Dr. C. G. Khatri for his valuable suggestions and help.

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