A REMARK ON A PAPER OF TRAWINSKI AND DAVID ENTITLED: "SELECTION OF THE BEST TREATMENT IN A PAIRED-COMPARISON EXPERIMENT"[1]

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The present remark is concerned with the asymptotic behavior of the highest score in a paired-comparison experiment, if the number t of treatments is very large. It answers a question implicitly posed in Fig. 1 A of [1]. Besides, a useful inequality for the joint cumulative distribution function of the scores will be derived.

Assume that the treatments $T_i (i = 1, 2, \cdots, t)$ are all equal, with the exception of a single "outlier" T_t, which will be preferred with probability $p > \frac{1}{2}$ when compared with any other treatment.[2] Assume that each pair of treatments is compared exactly once, and declare best the treatment with the highest score a_i, that is, with the highest number of preferences. Let $P_{p,t}$ be the probability of selecting the actually best treatment T_t by this procedure.

What happens if t tends to infinity? Fig. 1 A of [1] seems to indicate that $P_{p,t}$ tends to 1 if p is near 1, and to 0 if p is near $\frac{1}{2}$.

Rather surprisingly, this conjecture is false; actually we have $\lim_{t\to\infty} P_{p,t} = 1$ for all fixed $p > \frac{1}{2}$.

Consider first the case where all treatments are equal (no outlier). Then each score a_i is binomially distributed, and the reduced score $a_i^* = \{a_i - [(t-1)/2]\}/[(t-1)/4]^{\frac{1}{2}}$ is asymptotically normal with mean 0 and variance 1. In particular (see, e.g., W. Feller, *An Introduction to Probability Theory*, 2nd ed. p. 178).

$$(1) \qquad P[a_i^* > x_{t-1}] \sim [(2\pi)^{\frac{1}{2}} x_{t-1}]^{-1} e^{-\frac{1}{2}x_{t-1}^2}$$

provided $t \to \infty$, $x_{t-1} \to \infty$, $x_{t-1}^3/[(t-1)/4]^{\frac{1}{2}} \to 0$. The sign $\sim$ denotes that the ratio of the two sides tends to 1.

In particular, let $\epsilon > 0$ and put

$$(2) \qquad x_{t-1}^{\pm} = [2\log(t-1) - (1 \pm \epsilon)\log\log(t-1)]^{\frac{1}{2}};$$

one obtains

$$(3) \qquad P[a_i^* > x_{t-1}^{\pm}] \sim \log(t-1)^{\pm\epsilon/2}/(4\pi)^{\frac{1}{2}}(t-1).$$

Now replace treatment T_t by an outlier, $p > \frac{1}{2}$. This stochastically decreases the reduced scores a_i^*, $1 \leqq i < t$, by an amount less than $1/[(t-1)/4]^{\frac{1}{2}}$. But a straightforward calculation shows that we may add a term of the order

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[2] To avoid confusion with the number $\pi = 3.14 \cdots$, I changed the π of T. and D. into p.

$o[\log(t-1)]^{-\frac{1}{2}}$ to $x_{t-1}^{\pm}$ without destroying (3). Hence, (3) holds for $1 \leqq i < t$, even if T_t is an outlier.

Let $c'' < 1/(4\pi)^{\frac{1}{2}} < c'$; if the factor $1/(4\pi)^{\frac{1}{2}}$ in (3) is replaced by c' or c'', one obtains strict inequalities for sufficiently large t, and the general inequality $P[\cup A_i] \leqq \Sigma P[A_i]$ then yields

$$P[\max_{1 \leqq i < t} a_i^* > x_{t-1}^-] < c' \cdot (\log(t-1))^{-\epsilon/2} \tag{4}$$

for sufficiently large t.

With the aid of the Lemma below, one gets

$$\begin{aligned} P[\max_{1 \leqq i < t} a_i^* < x_{t-1}^+] &\leqq \prod_{i=1}^{t-1} P[a_i^* < x_{t-1}^+] \\ &< \{1 - c''[(\log(t-1))^{\epsilon/2}/(t-1)]\}^{t-1} \leqq \exp\{-c'' \cdot (\log(t-1))^{\epsilon/2}\} \end{aligned} \tag{5}$$

for sufficiently large t.

The inequalities (4) and (5) constitute the main result of this note; an immediate consequence of them is, for instance, the

COROLLARY. *If* $t \to \infty$, *then* $\max_{1 \leqq i < t} a_i^* - [2 \log(t-1)]^{\frac{1}{2}} \to 0$ *in probability.*

Inequality (4) or the corollary imply that $\max_{1 \leqq i < t} a_i/(t-1)$ tends in probability to $\frac{1}{2}$. If T_t is an outlier, then $a_t/(t-1)$ tends to $p > \frac{1}{2}$, and it follows immediately that $\lim_{t \to \infty} P_{p,t} = 1$.

Furthermore, one may conclude from the corollary that $a_{\max}/(t-1) = \max_{1 \leqq i < t} a_i/(t-1)$ clusters around $m_{t-1} = \frac{1}{2} + [\log(t-1)/2(t-1)]^{\frac{1}{2}}$, with a dispersion of the order $o[1/(t-1)^{\frac{1}{2}}]$ (more precisely, there exist intervals centered at m_{t-1}, of length $o[1/(t-1)^{\frac{1}{2}}]$, which contain $a_{\max}/(t-1)$ with arbitrarily high preassigned probability). Similarly, $a_t/(t-1)$ clusters around p and has a dispersion of the order $O[1/(t-1)^{\frac{1}{2}}]$. Both dispersions are of smaller order than $m_{t-1} - \frac{1}{2}$. Now assume that p is very near to $\frac{1}{2}$; then this implies that $P_{p,t}$ will first decrease, until the value of $m_{t-1} - \frac{1}{2}$ becomes comparable with $p - \frac{1}{2}$. Already for moderate p this leads to large values of t; since, for example, $m_{1000} \approx 0.56$, $m_{50,000} \approx 0.51$, $m_{10,000,000} \approx 0.501$.

We shall now prove the inequality used in establishing (5). Let p_{ij} be the probability that T_i is preferred to T_j.

LEMMA. *For any probability matrix* (p_{ij}) *and any numbers* $(k_1, \cdots, k_m)$, $m \leqq t$, *the joint cumulative distribution function of the scores* $a_1, \cdots, a_m$ *satisfies*

$$F(k_1, \cdots, k_m) = P[a_1 < k_1, \cdots, a_m < k_m] \leqq P[a_1 < k_1] \cdot \cdots \cdot P[a_m < k_m].$$

PROOF. Any two scores a_i, a_j are dependent only through the result of the comparison between the respective treatments T_i, T_j. Put $a_i = a_i' + w_i$, $a_j = a_j' + w_j$; $w_i = 1 - w_j$ being 1 or 0, according as T_i is preferred to T_j or not. Replace w_i and w_j by independent variables without changing the marginal distributions. This destroys the dependence between a_i and a_j and changes F into a new joint distribution function F'. An explicit computation yields that F'

majorizes F:

$$F'(k_1, \cdots, k_m) - F(k_1, \cdots, k_m)$$
$$= \sum_{c_i, c_j} P[a_1 < k_1, \cdots, a_i' < k_i - c_i, \cdots, a_j' < k_j - c_j, \cdots]$$
$$\cdot \{P[w_i = c_i] \cdot P[w_j = c_j] - P[w_i = c_i, w_j = c_j]\}$$
$$= p_{ij} \cdot p_{ji} \cdot \{P[\cdots, a_i' < k_i, \cdots, a_j' < k_j, \cdots]$$
$$- P[\cdots, a_i' < k_i, \cdots, a_j' < k_j - 1, \cdots]$$
$$- P[\cdots, a_i' < k_i - 1, \cdots, a_j' < k_j, \cdots]$$
$$+ P[\cdots, a_i' < k_i - 1, \cdots, a_j' < k_j - 1, \cdots]\}$$
$$= p_{ij} p_{ji} P[a_1 < k_1, \cdots, k_i - 1 \leqq a_i' < k_i, \cdots,$$
$$k_j - 1 \leqq a_j' < k_j, \cdots, a_m < k_m] \geqq 0.$$

One repeats now this procedure for other pairs of treatments; eventually one obtains the distribution function corresponding to independent scores, thus proving that it majorizes the original distribution function.

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REFERENCE

[1] Trawinski, B. J. and David, H. A. (1963). Selection of the best treatment in a paired-comparison experiment. *Ann. Math. Statist.* **34** 75–91.