SELECTION OF THE BEST TREATMENT IN A PAIRED-COMPARISON EXPERIMENT[1]

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1. Introduction and summary. In the method of paired comparisons several "treatments" under investigation are presented in all possible pairwise combinations to a judge who states which member of each pair he prefers. The experiment may be repeated by the same judge or carried out by several judges acting independently. Expressions of no preference may be permitted but we shall exclude this and other complications. The method is widely used when no meaningful absolute measurements can readily be made on the "treatments," a term which may stand for "stimuli," "objects," and the like. By concentrating on just two treatments each basic comparison is free of the confusing effects which may arise when more than two treatments are compared simultaneously. There are also situations, as in testing a variety of contact lenses for irritability, when two is the only possible block size. The results of the experiment may be summed up in the total number of preferences or the "score" obtained by each treatment. We define the best treatment as the one with the highest expected score.

This paper deals with two procedures in which the emphasis is on the selection of the best treatment. Such methods have received considerable attention in many settings since the basic work by Bechhofer [2]. Making assumptions analogous to his we obtain tables giving the smallest number of replications ensuring that with at least a specified probability the best treatment will emerge with the highest score. However, it is interesting to note that in the present case the procedure does not have the usual conservative properties associated with Bechhofer's approach.

The second method is concerned with the selection of a subset of the treatments which will include the best treatment with at least a specified probability (see Gupta [7] and Gupta and Sobel [8]). In contrast to the first method which stresses the determination of experiment size prior to experimentation this approach, as we apply it, is a method of analysis which allows the elimination of inferior treatments. Further experimentation on the selected treatments may be necessary. If a more detailed grouping or ranking of the treatments is required, without special emphasis on the best treatment, other methods of analysis are more suitable [10]. In order to guarantee the inclusion of the best member in the subset at some prescribed probability level we assume that the treatments are judged on a characteristic which satisfies a linear model. Table 2 makes the

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determination of the subset immediate for a wide range of experiment sizes. The procedure is illustrated by an example.

2. Distribution theory.

2.1. *General model and assumptions.* Consider a balanced paired-comparison experiment consisting of n replications of all $\frac{1}{2}t(t-1)$ comparisons between the treatments $T_i(i = 1, 2, \cdots, t)$. Let $x_{ij\gamma}$ be a characteristic random variable corresponding to the comparison of T_i and T_j in the γth replication:

$$x_{ij\gamma} = \begin{matrix} 1 & \text{if} & T_i \rightarrow T_j \\ 0 & \text{if} & T_j \rightarrow T_i \end{matrix} \qquad \begin{matrix} (i, j = 1, 2, \cdots, t; i \neq j; \\ \gamma = 1, 2, \cdots, n), \end{matrix} \tag{2.1}$$

where $T_i \rightarrow T_j$ denotes preference of T_i over T_j. We assume that ties are not permitted, that there is no replication effect, and that all $\frac{1}{2}nt(t-1)$ comparisons are independent. Preference probabilities π_{ij} can now be defined by

$$\Pr(x_{ij\gamma} = 1) = \pi_{ij}, \qquad \Pr(x_{ij\gamma} = 0) = \pi_{ji} = 1 - \pi_{ij}. \tag{2.2}$$

The score a_i of treatment T_i is given by

$$a_i = \sum_{\gamma=1}^{n} a_{i\gamma} = \sum_{\gamma=1}^{n} \sum_{j}{}' x_{ij\gamma}, \tag{2.3}$$

where $a_{i\gamma}$ denotes the (partial) score of T_i in the γth replication, and $\sum_j'$ indicates summation over all j excluding $j = i$.

It should be noted that $\sum_{i=1}^{t} a_{i\gamma} = \frac{1}{2}t(t-1)$, $\sum_{i=1}^{t} a_i = \frac{1}{2}nt(t-1)$, showing that $a_{i\gamma}$ and $a_{j\gamma}$ are correlated, as are a_i and a_j.

2.2. *Joint distribution of the scores.* Let $\alpha_{rs}(r > s)$ be the number of times T_r is preferred to T_s in n comparisons. In view of our assumptions the α_{rs} are independent and their joint distribution is therefore given by

$$f(\alpha_{t,t-1}, \alpha_{t,t-2}, \cdots, \alpha_{21}) = \prod_{r>s}^{t} \binom{n}{\alpha_{rs}} \pi_{rs}^{\alpha_{rs}} \pi_{sr}^{n-\alpha_{rs}}. \tag{2.4}$$

Since the scores may be expressed as

$$\begin{aligned} a_t &= \alpha_{t1} + \alpha_{t2} + \cdots + \alpha_{t,t-1}, \\ a_{t-1} &= \alpha_{t-1,1} + \alpha_{t-1,2} + \cdots + \alpha_{t-1,t-2} + (n - \alpha_{t,t-1}), \cdots, \\ a_1 &= (n - \alpha_{21}) + (n - \alpha_{31}) + \cdots + (n - \alpha_{t1}), \end{aligned} \tag{2.5}$$

it follows that the joint distribution of any $u \leqq t$ scores is given by summing (2.4) subject to the restrictions on the u scores imposed by (2.5); in particular, if $u = t$ the probability function of the vector of scores, $\boldsymbol{a}$ may be written as

$$f(\boldsymbol{a}; C(\pi_{ij})) = \sum_{P_n} \prod_{r>s}^{t} \binom{n}{\alpha_{rs}} \pi_{rs}^{\alpha_{rs}} \pi_{sr}^{n-\alpha_{rs}}, \tag{2.6}$$

where $C(\pi_{ij})$ stands for the configuration of preference probabilities π_{ij}, and P_n denotes the restrictions (2.5).

If all treatments are equally good (2.6) becomes

$$f(\boldsymbol{a}; C(\tfrac{1}{2})) = 2^{-\frac{1}{2}nt(t-1)} \sum_{P_n} \prod_{r>s}^{t} \binom{n}{\alpha_{rs}}.$$

Since $f(\boldsymbol{a}; C(\frac{1}{2}))$ must be a symmetric function of the scores we have established incidentally that

$$g(\boldsymbol{a}; n) = \sum_{P_n} \prod_{r>s}^{t} \binom{n}{\alpha_{rs}} \tag{2.7}$$

is symmetric in the a_i. The function $g(\boldsymbol{a}; n)$ gives the number of ways the outcome $\boldsymbol{a}$ can be realized.

2.3. *Partition function.* From $g(\boldsymbol{a}; n)$ we can obtain immediately the *partition function* $G(\boldsymbol{a}; n)$ giving the number of permissible partitions of $\frac{1}{2}nt(t-1)$ into t scores $a_1, a_2, \cdots, a_t$ *irrespective* of order. In view of the symmetry of g we have

$$G(\boldsymbol{a}; n) = (t!/\prod_k m_k\,!)g(\boldsymbol{a}; n), \tag{2.8}$$

where m_k is the number of scores all of magnitude a_k. A program for an IBM 650 digital computer has been written giving $G(\boldsymbol{a}; n)$ for all combinations of (t, n) up to (2,66), (3,21), (4,11), (5,6), (6,4), and tables up to (3,20), (4,7), (5,3) appear in [11]. The cases $n = 1$, $t \leqq 8$ have been tabulated in [6], and closely related tables have also been prepared by Bradley and Terry [5] and Bradley [4].

2.4. *Asymptotic distribution theory.* It is clear that the exact probability function (2.6) does not lend itself to ready evaluation for larger values of t and n. In order to construct comprehensive tables for the two decision rules, we now develop some asymptotic distribution theory based on the general model of Section 2.1 and the multivariate central limit theorem.

We have at once

$$\mathcal{E}(x_{ij\gamma}) = \pi_{ij}, \qquad \text{var}\,(x_{ij\gamma}) = \pi_{ij}\pi_{ji},$$

$$\mathcal{E}(a_{i\gamma}) = \sum_j^{t}{}' \pi_{ij}, \qquad \text{var}\,(a_{i\gamma}) = \sum_j^{t}{}' \pi_{ij}\pi_{ji}.$$

To find the covariance of $a_{i\gamma}$ and $a_{j\gamma}$ $(i \neq j)$, consider the variance of $a_{i\gamma} - a_{j\gamma}$, taking $i = 1$ and $j = t$ for convenience. Then

$$\begin{aligned}
\text{var}\,(a_{1\gamma} - a_{t\gamma}) &= \text{var}\left(\sum_{k=2}^{t} x_{1k\gamma} - \sum_{k=1}^{t-1} x_{tk\gamma}\right) \\
&= \text{var}\left[\sum_{k=2}^{t-1} (x_{1k\gamma} - x_{tk\gamma}) + 2x_{1t\gamma} - 1\right] \\
&= \sum_{k=2}^{t-1} (\pi_{1k}\,\pi_{k1} + \pi_{tk}\,\pi_{kt}) + 4\pi_{1t}\,\pi_{t1} \\
&= \text{var}\,(a_{1\gamma}) + \text{var}\,(a_{2\gamma}) + 2\pi_{1t}\,\pi_{t1}.
\end{aligned}$$

Thus cov $(a_{1\gamma}, a_{t\gamma}) = -\pi_{1t}\pi_{t1}$. From the independence of replications it follows that

$$\text{(2.9)} \qquad \text{var } (a_1) = n \sum_{k=2}^{t} \pi_{1k}\pi_{k1}, \qquad \text{cov } (a_1, a_t) = -n\pi_{1t}\pi_{t1}.$$

Of special interest here is the distribution of the vector of differences $d' = (d_1, d_2, \cdots, d_{t-1})$, where $d_i = a_i - a_t$ $(i = 1, 2, \cdots, t-1)$. The variate d_1 has mean

$$\text{(2.10)} \qquad \delta_1 = n\left(\sum_{k=2}^{t} \pi_{1k} - \sum_{k=1}^{t-1} \pi_{tk}\right),$$

and variance

$$\text{(2.11)} \qquad \sigma_{d_1d_1} = n\left[\sum_{k=2}^{t-1} (\pi_{1k}\,\pi_{k1} + \pi_{tk}\,\pi_{kt}) + 4\pi_{1t}\,\pi_{t1}\right].$$

The covariance of d_1 and d_2 can be written as var (a_t) − cov (a_1, a_t) − cov (a_t, a_2) + cov (a_1, a_2), giving from (2.9)

$$\text{(2.12)} \qquad \sigma_{d_1d_2} = n\left[\sum_{k=1}^{t-1} \pi_{tk}\,\pi_{kt} + (\pi_{1t}\,\pi_{t1} + \pi_{2t}\,\pi_{t2} - \pi_{12}\,\pi_{21})\right].$$

Now $d_i = \sum_{\gamma=1}^{n} (a_{i\gamma} - a_{t\gamma}) = \sum_{\gamma} d_{i\gamma}$ and the $d_{i\gamma}$ have, for any given γ means, variances and covariances δ_1/n, $\sigma_{d_1d_1}/n$ and $\sigma_{d_1d_2}/n$, respectively. It follows from the independence of replications and the multivariate central limit theorem (see e.g. Anderson [1]) that the limiting distribution as $n \to \infty$ of $(1/n^{\frac{1}{2}})(d - \delta)$ is multivariate normal $N(0, \Sigma)$ where Σ is the matrix $(1/n)(\sigma_{d_id_j})$.

2.5. *Linear Model.* Suppose that treatment T_i has true value or "merit" V_i when judged on some characteristic. The t true merits $V_1, V_2, \cdots, V_t$ can be represented by t points on a merit scale. The *observed* merit of T_i will vary from observation to observation and may be represented on the same scale by the continuous random variable y_i $(-\infty \leqq y_i \leqq \infty)$. In a paired comparison of T_i and T_j the former will be preferred if $y_i > y_j$, the latter if $y_i < y_j$. If it is possible to construct a merit scale such that the probability of preference

$$\pi_{ij} = \Pr\,(y_i - y_j > 0)$$

can be expressed for all i, j as $H(V_i - V_j)$, where $H(x)$ increases monotonically from $H(-\infty) = 0$ to $H(\infty) = 1$ and $H(-x) = 1 - H(x)$, then the preference probabilities may be said to satisfy a *linear model.* $H(x)$ is seen to be the c.d.f. of a random variable symmetrically distributed about zero.

Since for H specified the π_{ij} depend only on the differences $V_i - V_j$ it follows that all π_{ij} are expressible as functions of $t - 1$ independent differences of the V_i, with $\pi_{ij} > \frac{1}{2}, = \frac{1}{2}, < \frac{1}{2}$ according as $V_i > V_j, = V_j, < V_j$. The merits of the treatments can therefore be usefully represented by t points on a linear scale with arbitrary origin, hence the term "linear." We may also say that the *characteristic* under study satisfied a linear model or follows a linear scale if the merit of *any* treatment (not only of the t treatments T_i) can be represented on a linear scale.

The linear model is a generalization of the Thurstone-Mosteller model for which the y_i are assumed to be normal $N(V_i, \sigma^2)$ variates, equi-correlated with common correlation coefficient ρ. This corresponds to taking

$$\pi_{ij} = H(V_i - V_j) = \int_{-(V_i'-V_j')}^{\infty} \varphi(x)\, dx,$$

where $V_i' = V_i/[2\sigma^2(1-\rho)]^{\frac{1}{2}}$ and $\varphi(x) = (2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}x^2}$. Another important special case is provided by the Bradley-Terry model (Bradley, [3]) for which

$$H(V_i - V_j) = \frac{1}{4}\int_{-(V_i-V_j)}^{\infty} \operatorname{sech}^2 \frac{1}{2} y\, dy,$$

where in Bradley's notation $V_i = \log \pi_i$ ($\pi_i \geqq 0$, $\Sigma\pi_i = 1$); it follows that $\pi_{ij} = \pi_i/(\pi_i + \pi_j)$.

In the remainder of this paper it will frequently be assumed that a linear model is applicable. A theoretical ranking of the treatments may then be made according to their "merits." We associate treatment $T_{(i)}$ with merit $V_{(i)}$, where

$$V_{(1)} \leqq V_{(2)} \leqq \cdots \leqq V_{(t)}, \tag{2.13}$$

and write $a_{(i)}$ for the score of $T_{(i)}$. It is to be noted that $a_{(i)}$ is a theoretical quantity as in a practical situation we do not know which treatment to label $T_{(i)}$. For convenience of writing we will leave off brackets on the subscripts of π's and will henceforth mean by π_{ij} the preference probability Pr $\{T_{(i)} \to T_{(j)}\}$.

3. Selection of the best treatment.

3.1. *Formulation of the problem.* In many experiments designed to compare t treatments the primary interest lies in the detection of the best treatment. For a balanced paired-comparison experiment it is natural to declare as best the treatment with the highest score but this may not, in fact, be the best treatment, due to chance fluctuations. However, if $T_{(t)}$ is strictly better than $T_{(t-1)}$, and if the number n of replications is large enough, then $T_{(t)}$ should emerge with the highest score with a probability P as close to 1 as desired.

We may therefore proceed as follows:

(i) Find n corresponding to given values of t, $C(\pi_{ij})$, and P.

(ii) Perform the experiment and declare best the treatment with the highest score; if m scores tie for first place declare best one of the corresponding treatments at random.

While step (ii) is straightforward, (i) needs further discussion. Since, for fixed t and P, n depends on $C(\pi_{ij})$ we shall specialize the π_{ij} in line with the general approach due to Bechhofer [2]. We suppose that $T_{(t)}$ has probability $\pi(>\frac{1}{2})$ of being preferred to each of the other treatments and that the remaining $t-1$ treatments are of equal value. This assumption expresses in simplified form that a superior treatment is present and would appear to be reasonable as a basis for determining the number of replications of the experiment. While completely analogous to the kind of specialization employed by Bechhofer it turns out that in the present case this choice of the π_{ij} does not necessarily correspond to a least favorable case. This last rather surprising point will be further considered in Section 3.4.

The model chosen can be summed up thus:

$$\text{(3.1)} \qquad \begin{aligned} \pi_{tj} &= \pi > \tfrac{1}{2}, \quad j = 1, 2, \cdots, t-1; \\ \pi_{ij} &= \tfrac{1}{2}, \quad i, j = 1, 2, \cdots, t-1; \qquad i \neq j. \end{aligned}$$

3.2. *Exact distribution theory.* From (2.6) and (2.7) the joint distribution of scores under model (3.1) may be written

$$\text{(3.2)} \qquad f(\boldsymbol{a}_{(i)}) = 2^{-n\binom{t-1}{2}} g(\boldsymbol{a}; n) \pi^{a(t)} (1-\pi)^{n(t-1)-a(t)}.$$

Suppose that m scores tie for first place. Of the $G(\boldsymbol{a}; n)/g(\boldsymbol{a}; n)$ permutations of the scores $a_1, a_2, \cdots, a_t$, a proportion m/t must have a top score in the last place, that is, associated with $T_{(t)}$. By the randomization process referred to in 3.1 (ii) the corresponding contribution to the probability of correct selection P is

$$\text{(3.3)} \qquad \frac{1}{m} \cdot 2^{-n\binom{t-1}{2}} \frac{m}{t} G(\mathbf{a}; n) \pi^{a(t)} (1-\pi)^{n(t-1)-a(t)},$$

which is independent of m. P is then given by summing (3.3) over all $a_{(t)}$ which can be maximum scores and over all permissible values of the other scores, and may be expressed as

$$\text{(3.4)} \quad P = 2^{-n\binom{t-1}{2}} \sum_{a(t)=c}^{n(t-1)} \pi^{a(t)} (1-\pi)^{n(t-1)-a(t)} \sum (1/t) G(a_1, a_2, \cdots, a_t; n),$$

where the last summation extends over

$$\sum_{i=1}^{t-1} a_{(i)} = n\binom{t}{2} - a_{(t)}$$

and where c is the smallest integer greater than or equal to $\frac{1}{2}n(t-1)$.

EXAMPLE. If $t = 3$ and $n = 1$ (3.4) reduces to

$$P = 2^{-1}[\pi(1-\pi)\tfrac{1}{3}G(1, 1, 1; 1) + \pi^2 \tfrac{1}{3} G(0, 1, 2; 1)],$$

where $G(0, 1, 2; 1)$, the frequency of the partition [210], is 6, and $G(1, 1, 1; 1) = 2$. Thus, as is otherwise obvious in this simple case, $P = \pi^2 + \frac{1}{3}\pi(1-\pi)$. The leading term of P is always $\pi^{n(t-1)}$.

A program has been written for the IBM 650 computer from which P can be evaluated for the combination of n and t listed in Section 2.3 and for $\pi = 0.55\ (0.05)\ 0.95$.

When $t = 2$, $f(\boldsymbol{a}_{(t)})$ simply states that $a_{(2)}$ has a binomial (π, n) distribution. For n odd, say $n = 2r - 1$, the probability of choosing the better treatment is

$$P_o = \sum_{a(2)=r}^{2r-1} \binom{2r-1}{a_{(2)}} \pi^{a(2)} (1-\pi)^{2r-1-a(2)},$$

and for $n = 2r$, the probability is

$$P_e = \sum_{a(2)=r+1}^{2r} \binom{2r}{a_{(2)}} \pi^{a(2)}(1 - \pi)^{2r-a(2)} + \tfrac{1}{2}\binom{2r}{r} \pi^r(1 - \pi)^r.$$

It is interesting to note that $P_o = P_e$, as can easily be shown. Thus it is advantageous to work with odd values of n.

3.3. *Asymptotic approximation to n.* In the present case it is clear that the $t - 1$ differences $d_{(i)} = a_{(i)} - a_{(t)}$ are identically distributed equi-correlated variates. By (2.10) – (2.12) their means, variances and covariances are

$$(3.5)\qquad \delta = -nt(\pi - \tfrac{1}{2}), \qquad \sigma_d^2 = n[(t + 2)\pi(1 - \pi) + \tfrac{1}{4}(t - 2)],$$
$$\rho\sigma_d^2 = n[(t + 1)\pi(1 - \pi) - \tfrac{1}{4}].$$

Since the probability of ties tends to 0 as $n \to \infty$, the probability of selecting the best treatment is asymptotically

$$(3.6)\qquad \begin{aligned} P &= \lim_{n\to\infty} \Pr\{d_{(i)} < 0; i = 1, 2, \cdots, t - 1\} \\ &= \lim_{n\to\infty} \Pr\{v_i < \Delta; i = 1, 2, \cdots, t - 1\}, \end{aligned}$$

where $v_i = (d_{(i)} - \delta)/\sigma_d$, and $\Delta = -\delta/\sigma_d$. From the limiting multivariate normality of the v_i we have

$$(3.7)\quad P = (2\pi)^{-\frac{1}{2}(t-1)} |R|^{-\frac{1}{2}} \int_{-\infty}^{\Delta}\int_{-\infty}^{\Delta} \cdots \int_{-\infty}^{\Delta} \exp[-\tfrac{1}{2} v'R^{-1}v]\, dv_1\, dv_2 \cdots dv_{t-1},$$

R being the correlation matrix of the v_i, with elements 1 along the principal diagonal and ρ elsewhere. As is well known, for R to be positive definite $\rho > -1/(t - 2)$. It is possible to simplify (3.7) considerably as the v_i are equi-correlated. To this end we express $d_{(i)}$ as

$$(3.8)\qquad d_{(i)} = y_i - y_t \qquad i = 1, 2, \cdots, t - 1,$$

with all t y's mutually independent. The correct means of the $d_{(i)}$ are achieved if we take $\mathcal{E}(y_i) = 0$, $\mathcal{E}(y_t) = -\delta$. Also since

$$(3.9)\quad \sigma_d^2 = \text{var}\, y_i + \text{var}\, y_t, \quad \text{and} \quad \rho\sigma_d^2 \equiv \text{cov}(d_i, d_j) = \text{var}\, y_t,$$

we complete the specification of the y's by taking $\text{var}\, y_i = (1 - \rho)\sigma_d^2$, $\text{var}\, y_t = \rho\sigma_d^2$. Standardizing the y's we put $u_i = y_i/[(1 - \rho)^{\frac{1}{2}}\sigma_d]$, $u_t = (y_t + \delta)/\rho^{\frac{1}{2}}\sigma_d$ and have $v_i = (y_i - y_t - \delta)/\sigma_d = (1 - \rho)^{\frac{1}{2}}u_i - \rho^{\frac{1}{2}}u_t$. Then from (3.6)

$$P = \lim_{n\to\infty} \Pr\{u_i < [\rho/(1 - \rho)]^{\frac{1}{2}} u_t - \delta/[(1 - \rho)^{\frac{1}{2}} \sigma_d], i = 1, 2, \cdots, t - 1\}$$
$$= \int_{-\infty}^{\infty} [\Pr\{u_i < U_t \mid u_t\}]^{t-1} \varphi(u_t)\, du_t,$$

where

$$(3.10)\quad U_t = [\rho/(1 - \rho)]^{\frac{1}{2}}u_t - \delta/[(1 - \rho)^{\frac{1}{2}}\sigma_d], \qquad \varphi(u) = (2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}u^2}.$$

Thus, writing Φ for the unit normal c.d.f., we have

$$P = \int_{-\infty}^{\infty} [\Phi(U_t)]^{t-1} \varphi(u_t)\, du_t. \tag{3.11}$$

It should be noted from (3.9) that this simplification of (3.7) is possible only if $\rho > 0$, and not if $-1/(t-2) < \rho \leqq 0$. The restriction is not too serious since for $\pi \geqq \frac{1}{2}$ it is easy to see that $\rho > 0$ as long as $\pi < \frac{1}{2} + \frac{1}{2}[t/(t+1)]^{\frac{1}{2}}$. Permissible values of π therefore increase with t but even for $t = 3$, π may be as large as 0.933. For values of π larger than this sufficiently high values of P can be achieved with n quite small and the exact theory of Section 3.2 may then be applied.

3.4. *Comments on Table 1.* For an experiment involving t treatments subject to the model (3.1) the table gives the smallest number of replications n which ensure that the highest score in an experiment of size (t, n) will correspond to the best treatment with at least a pre-assigned probability P'. In the construction of the table exact theory was used for the combinations (t, n) up to (2,269), (3,18), (4,8), (5,4), (6,1), (7,1), (8,1), and asymptotic approximations elsewhere. Comparisons of exact and asymptotic values showed good agreement for the larger of these experiment sizes.

If model (3.1) were conservative the value of n obtained from Table 1 would ensure correct selection with probability $P \geqq P'$ as long as

$$\Pr\,(T_{(t)} \rightarrow T_{(t-1)}) = \pi_{t,t-1} \geqq \pi. \tag{3.12}$$

That this is not necessarily so is most readily seen from Figs. 1A and 1B which show how P increases as a function of π for $n = 1$ and 10 when model (3.1) holds. For example, take $t = 20$ and suppose that the first five treatments, considered by themselves, satisfy the model (3.1) with $t = 5$. Complete the specification by adding to these five treatments fifteen of no value, that is, having probability zero of being preferred to any of the first five. The condition (3.12) is clearly satisfied in this augmented case. Now for $n = 1$, Fig. 1A shows that P is greater for $t = 20$ than for $t = 5$ if $0.69 < \pi < 0.95$. Thus for this range of π the augmented case, which leads to the same value of P as the case $t = 5$ graphed, is actually less favorable than the case labelled $t = 20$. There is, of course, no guarantee that it is least favorable. In fact, considerations of this kind indicate that it would in general be difficult to determine the least favorable configuration and that this configuration may be far removed from a realistic situation. We have not succeeded in determining the least favorable configuration except in very special cases. Thus Table 1 can be used as a safe guide to the appropriate n only if model (3.1) holds. However, this model is of considerable importance in its own right, corresponding as it does to the situation of a single "outlier."

It may be remarked that, in view of the lack of independence among the scores, there is no contradiction here with the most-economical character of some Bechhofer and Sobel decision rules established by Hall [9].

TABLE 1

Smallest number of replications required to ensure with at least a predetermined probability P' the selection of the best treatment when

$Pr(T_{(t)} \to T_{(i)}) \geq \pi \,(i=1,2,\ldots,t-1),\; Pr(T_{(i)} \to T_{(j)}) = \frac{1}{2}\; (i \neq j;\; i,j=1,2,\ldots,t-1)$

P'	t \ π	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
0.75	2	45*	11*	5*	3*	1*	1*	1*	1*	1*
	3	69	17*	8*	4*	3*	2*	1*	1*	1*
	4	71	18	8*	5*	3*	2*	2*	1*	1*
	5	68	17	8	4*	3*	2*	2*	1*	1*
	6	65	16	7	4	3	2	2	1*	1*
	7	61	15	7	4	3	2	2	1*	1*
	8	58	15	7	4	3	2	1*	1*	1*
	9	54	14	6	4	2	2	1	1	1
	10	52	13	6	4	2	2	1	1	1
	12	47	12	5	3	2	2	1	1	1
	14	43	11	5	3	2	2	1	1	1
	16	39	10	5	3	2	1	1	1	1
	18	37	9	4	3	2	1	1	1	1
	20	34	9	4	3	2	1	1	1	1
0.90	2	163*	41*	17*	9*	7*	5*	3*	1*	1*
	3	165	41	18*	10*	6*	4*	3*	2*	1*
	4	150	37	16	9	6*	4*	3*	2*	1*
	5	135	33	15	8	5	4*	3*	2*	1*
	6	122	30	13	7	5	3	2	2	1*
	7	112	28	12	7	4	3	2	2	1*
	8	103	26	11	6	4	3	2	2	1*
	9	95	24	11	6	4	3	2	1	1
	10	89	22	10	6	4	3	2	1	1
	12	79	20	9	5	3	2	2	1	1
	14	71	18	8	5	3	2	2	1	1
	16	64	16	7	4	3	2	2	1	1
	18	59	15	7	4	3	2	1	1	1
	20	55	14	6	4	2	2	1	1	1
0.95	2	269*	67*	29*	15*	9*	7*	5*	3*	1*
	3	243	60	26	15*	9*	6*	4*	3*	2*
	4	212	52	23	12	8*	5*	4*	3*	2*
	5	186	46	20	11	7	5	3*	3*	2*
	6	166	41	18	10	6	4	3	2	2
	7	150	37	16	9	6	4	3	2	2
	8	137	34	15	8	5	4	3	2	2
	9	126	31	14	8	5	3	2	2	1
	10	117	29	13	7	5	3	2	2	1
	12	102	26	11	6	4	3	2	2	1
	14	91	23	10	6	4	3	2	1	1
	16	82	21	9	5	3	2	2	1	1
	18	75	19	9	5	3	2	2	1	1
	20	69	17	8	5	3	2	2	1	1
0.99	2	537	133*	57*	31*	19*	13*	9*	5*	3*
	3	433	106	45	24	15*	10*	7*	5*	3*
	4	358	88	38	20	12	8*	6*	4*	3*
	5	306	75	33	18	11	7	5	4*	3*
	6	267	66	29	16	10	6	4	3	2
	7	238	59	26	14	9	6	4	3	2
	8	214	53	23	13	8	5	4	3	2
	9	195	48	21	12	7	5	3	2	2
	10	180	45	20	11	7	4	3	2	2
	12	155	39	17	9	6	4	3	2	2
	14	137	34	15	8	5	4	3	2	1
	16	123	31	14	8	5	3	2	2	1
	18	112	28	12	7	4	3	2	2	1
	20	102	26	11	6	4	3	2	2	1

* Values based on exact theory.

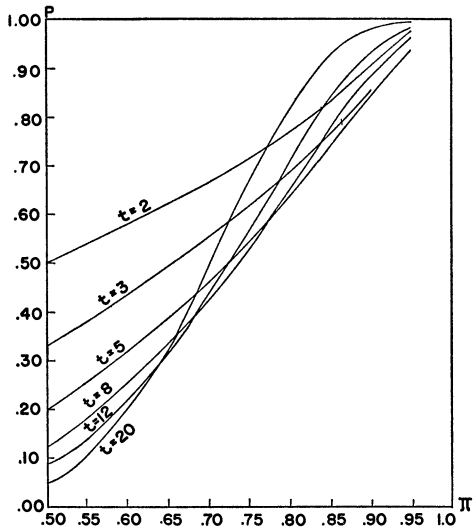

FIG. 1A. $n = 1$

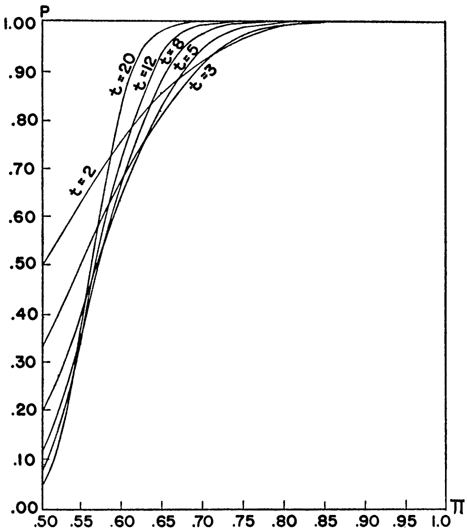

FIG. 1B. $n = 10$

Probability of correctly selecting the superior treatment $T_{(t)}$ in the presence of $t - 1$ equal others; n = number of replications,

$$\pi = \Pr \{T_{(t)} \rightarrow T_{(i)}\}, \qquad i = 1, 2, \cdots, t-1$$

4. Selection of a subset containing the best treatment.

4.1. *Formulation of the problem.* Consider the set of treatments $T = \{T_1, T_2, \cdots, T_t\}$ and let S be a subset of T consisting of those treatments with the highest scores. In this section our aim is to select S just large enough to ensure, with at least a pre-assigned probability P^*, that the best treatment $T_{(t)}$ is included in S. Following Gupta and Sobel [8] we use the decision rule $\mathcal{R}$:

Retain in S only those treatments T_i for which $a_i \geqq a_{\max} - \nu$, where $a_{\max}$ is the highest score and ν, a non-negative integer, is a function of t, n, and P^*.

For $P^* = 0.75, 0.90, 0.95, 0.975, 0.99$, and a wide range of values of t and n, the value of ν is given in Table 2. It should be noted that the size of S is a random variable which can range from 1 to t. The rule gives a correct selection if $a_{(t)} \geqq a_{\max} - \nu$. For fixed t and n, the probability of correct selection P_{CS} depends on ν and on the configuration of preference probabilities $C(\pi_{ij})$, and we have

$$P_{CS} = \Pr \{a_{(t)} \geqq a_{\max} - \nu \mid t, n, C(\pi_{ij})\}.$$

In Table 2, ν has been chosen as the smallest integer making $P_{CS} \geqq P^*$ when $C(\pi_{ij}) = C(\frac{1}{2})$, i.e. when all treatments are equivalent (we tag treatment $T_{(t)}$ so that $a_{(t)}$ is still defined). It will be shown in Section 4.4 that for fixed t, n, ν, the configuration $C(\frac{1}{2})$ leads to the infimum of P_{CS} for any $C(\pi_{ij})$ satisfying a linear model. In this case we may speak of $C(\frac{1}{2})$ as a conservative con-

TABLE 2

Values of ν for the decision rule $\mathcal{R}$

n \ t	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
									P* = 0.75										
1	1•	1•	2•	2•	2•	3•	3•	3	4	4	4	4	5	5	5	5	5	6	6
2	0•	2•	2•	3•	3	4	4	5	5	5	6	6	6	7	7	7	8	8	8
3	1•	2•	3•	4•	4	5	5	6	6	7	7	7	8	8	9	9	9	10	10
4	2•	2•	3•	4	5	5	6	7	7	8	8	9	9	10	10	10	11	11	12
5	1•	3•	4•	5	5	6	7	7	8	9	9	10	10	11	11	12	12	13	13
6	2•	3•	4•	5	6	7	7	8	9	9	10	11	11	12	12	13	13	14	14
7	1•	3•	4•	5	6	7	8	9	9	10	11	11	12	13	13	14	14	15	15
8	2•	4•	5•	6	7	8	9	9	10	11	12	12	13	14	14	15	15	16	16
9	3•	4•	5	6	7	8	9	10	11	12	12	13	14	14	15	16	16	17	17
10	2•	4•	5	7	8	9	10	10	11	12	13	14	14	15	16	16	17	18	18
11	3•	4•	6	7	8	9	10	11	12	13	14	14	15	16	17	17	18	19	19
12	2•	4•	6	7	8	9	10	11	12	13	14	15	16	17	17	18	19	19	20
13	3•	5•	6	7	9	10	11	12	13	14	15	16	16	17	18	19	20	20	21
14	2•	5•	6	8	9	10	11	12	13	14	15	16	17	18	19	19	20	21	22
15	3•	5•	7	8	9	11	12	13	14	15	16	17	18	18	19	20	21	22	23
16	2•	5•	7	8	10	11	12	13	14	15	16	17	18	19	20	21	22	22	23
17	3•	5•	7	9	10	11	12	14	15	16	17	18	19	20	21	21	22	23	24
18	2•	5•	7	9	10	12	13	14	15	16	17	18	19	20	21	22	23	24	25
19	3•	5•	7	9	10	12	13	14	16	17	18	19	20	21	22	23	24	24	25
20	4•	6•	8	9	11	12	14	15	16	17	18	19	20	21	22	23	24	25	26
25	3•	6	8	10	12	14	15	17	18	19	20	22	23	24	25	26	27	28	29
30	4•	7	9	11	13	15	17	18	20	21	22	24	25	26	27	29	30	31	32
35	3•	7	10	12	14	16	18	20	21	23	24	26	27	28	30	31	32	33	34
40	4•	8	11	13	15	17	19	21	23	24	26	27	29	30	32	33	34	36	37
45	5•	8	11	14	16	18	20	22	24	26	27	29	31	32	34	35	36	38	39
50	4•	9	12	15	17	19	21	23	25	27	29	31	32	34	35	37	38	40	41
60	6•	10	13	16	19	21	23	26	28	30	32	33	35	37	39	40	42	44	45
70	6•	10	14	17	20	23	25	28	30	32	34	36	38	40	42	44	45	47	49
80	6•	11	15	18	22	24	27	30	32	34	37	39	41	43	45	47	48	50	52
90	6•	12	16	20	23	26	29	31	34	36	39	41	43	45	47	49	51	53	55
100	6•	12	17	21	24	27	30	33	36	38	41	43	46	48	50	52	54	56	58
									P* = 0.90										
1	1•	2•	2•	3•	3•	4•	4•	4	5	5	5	6	6	6	6	7	7	7	7
2	2•	3•	3•	4•	5	5	6	6	7	7	8	8	8	9	9	9	10	10	10
3	3•	3•	4•	5•	6	6	7	8	8	9	9	10	10	11	11	12	12	12	13
4	2•	4•	5•	6	7	7	8	9	9	10	11	11	12	12	13	13	14	14	15
5	3•	4•	5•	6	7	8	9	10	11	11	12	12	13	14	14	15	15	16	16
6	4•	5•	6•	7	8	9	10	11	12	12	13	14	14	15	16	16	17	17	18
7	3•	5•	6•	8	9	10	11	12	12	13	14	15	16	16	17	18	18	19	20
8	4•	5•	7•	8	9	10	11	12	13	14	15	16	17	17	18	19	20	20	21
9	3•	6•	7	9	10	11	12	13	14	15	16	17	18	18	19	20	21	21	22
10	4•	6•	8	9	10	12	13	14	15	16	17	18	19	19	20	21	22	23	23
11	5•	6•	8	10	11	12	13	15	16	17	18	18	19	20	21	22	23	24	24
12	4•	7•	8	10	11	13	14	15	16	17	18	19	20	21	22	23	24	25	26
13	5•	7•	9	10	12	13	15	16	17	18	19	20	21	22	23	24	25	26	27
14	4•	7•	9	11	12	14	15	16	18	19	20	21	22	23	24	25	26	27	28
15	5•	8•	9	11	13	14	16	17	18	19	21	22	23	24	25	26	27	28	29
16	6•	8•	10	12	13	15	16	18	19	20	21	22	24	25	26	27	28	29	29
17	5•	8•	10	12	14	15	17	18	19	21	22	23	24	25	26	27	28	29	30
18	6•	8•	10	12	14	16	17	19	20	21	23	24	25	26	27	28	29	30	31
19	5•	8•	11	13	14	16	18	19	21	22	23	24	26	27	28	29	30	31	32
20	6•	9•	11	13	15	17	18	20	21	22	24	25	26	27	29	30	31	32	33
25	7•	10	12	15	17	18	20	22	24	25	27	28	29	31	32	33	34	36	37
30	8•	11	13	16	18	20	22	24	26	28	29	30	32	34	35	36	38	39	40
35	7•	11	15	17	20	22	24	26	28	30	31	33	35	36	38	39	41	42	44
40	8•	12	16	18	21	23	26	28	30	32	34	35	37	39	40	42	44	45	47
45	9•	13	16	19	22	25	27	29	32	34	36	37	39	41	43	45	46	48	49
50	10•	14	17	21	23	26	29	31	33	36	38	39	42	43	45	47	49	50	52
60	10•	15	19	23	26	29	31	34	37	39	41	43	46	48	50	52	53	55	57
70	12•	16	21	24	28	31	34	37	39	42	44	46	49	51	54	56	58	60	62
80	12•	17	22	26	30	33	36	39	42	45	48	50	53	55	57	60	62	64	66
90	12•	18	23	28	31	35	38	42	45	48	50	53	56	58	61	63	65	68	70
100	12•	19	25	29	33	37	41	44	47	50	53	56	59	61	64	67	69	71	74

• Values based on exact theory

TABLE 2 (continued)

n \ t	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
	P* = 0.95																		
1	1*	2*	3*	3*	4*	4*	5*	5	5	6	6	6	7	7	7	8	8	8	8
2	2*	3*	4*	5*	5	6	7	7	8	8	9	9	9	10	10	11	11	11	12
3	3*	4*	5*	6*	7	7	8	9	9	10	10	11	12	12	13	13	14	14	14
4	4*	5*	6*	7	8	9	9	10	11	11	12	13	13	14	15	15	16	16	17
5	3*	5*	6*	8	9	10	10	11	12	13	14	14	15	16	16	17	17	18	19
6	4*	6*	7*	8	9	10	11	12	13	14	15	16	16	17	18	18	19	20	20
7	5*	6*	8*	9	10	11	12	13	14	15	16	17	18	18	19	20	21	21	22
8	4*	7*	8*	10	11	12	13	14	15	16	17	18	19	20	21	21	22	23	24
9	5*	7*	9	10	12	13	14	15	16	17	18	19	20	21	22	23	23	24	25
10	6*	7*	9	11	12	14	15	16	17	18	19	20	21	22	23	24	25	25	26
11	5*	8*	10	11	13	14	16	17	18	19	20	21	22	23	24	25	26	27	28
12	6*	8*	10	12	13	15	16	17	19	20	21	22	23	24	25	26	27	28	29
13	5*	8*	11	12	14	15	17	18	19	21	22	23	24	25	26	27	28	29	30
14	6*	9*	11	13	14	16	18	19	20	21	23	24	25	26	27	28	29	30	31
15	7*	9*	11	13	15	17	18	20	21	22	23	25	26	27	28	29	30	31	32
16	6*	9*	12	14	15	17	19	20	22	23	24	26	27	28	29	30	31	32	33
17	7*	10*	12	14	16	18	19	21	22	24	25	26	28	29	30	31	32	33	34
18	6*	10*	12	14	16	18	20	21	23	24	26	27	28	30	31	32	33	34	35
19	7*	10*	13	15	17	19	20	22	24	25	26	28	29	30	32	33	34	35	36
20	8*	10*	13	15	17	19	21	23	24	26	27	29	30	31	32	34	35	36	37
25	9*	12	15	17	19	21	23	25	27	29	30	32	33	35	36	38	39	40	42
30	8*	13	16	19	21	23	26	28	30	31	33	35	37	38	40	41	43	44	46
35	9*	14	17	20	23	25	28	30	32	34	36	38	40	41	43	45	46	48	49
40	10*	15	18	22	24	27	30	32	34	36	38	40	42	44	46	48	49	51	53
45	11*	16	20	23	26	29	31	34	36	39	41	43	45	47	49	51	52	54	56
50	12*	17	21	24	27	30	33	36	38	41	43	45	47	49	51	53	55	57	59
60	12*	18	23	26	30	33	36	39	42	44	47	49	52	54	56	58	60	62	64
70	14*	20	24	29	32	36	39	42	45	48	51	53	56	58	61	63	65	67	70
80	14*	21	26	31	35	38	42	45	48	51	54	57	60	62	65	67	70	72	74
90	16*	22	28	32	37	41	44	48	51	54	58	60	63	66	69	71	74	76	79
100	16*	23	29	34	39	43	47	51	54	57	61	64	67	70	73	75	78	81	83
	P* = 0.975																		
1	1*	2*	3*	4*	4*	5*	5*	6	6	6	7	7	7	8	8	8	9	9	9
2	2*	4*	4*	5*	6	7	7	8	8	9	9	10	10	11	11	12	12	13	13
3	3*	5*	6*	7*	8	8	9	10	10	11	12	12	13	13	14	14	15	15	16
4	4*	5*	7*	8	9	10	10	11	12	13	13	14	15	15	16	17	17	18	18
5	5*	6*	7*	9	10	11	12	13	13	14	15	16	16	17	18	19	19	20	20
6	4*	7*	8*	9	11	12	13	14	15	16	16	17	18	19	20	20	21	22	22
7	5*	7*	9*	10	11	13	14	15	16	17	18	19	20	20	21	22	23	23	24
8	6*	8*	9*	11	12	14	15	16	17	18	19	20	21	22	23	23	24	25	26
9	5*	8*	10	12	13	14	16	17	18	19	20	21	22	23	24	25	26	27	27
10	6*	8*	11	12	14	15	17	18	19	20	21	22	23	24	25	26	27	28	29
11	7*	9*	11	13	14	16	17	19	20	21	22	23	24	26	27	28	28	29	30
12	6*	9*	12	13	15	17	18	19	21	22	23	24	26	27	28	29	30	31	32
13	7*	10*	12	14	16	17	19	20	22	23	24	25	27	28	29	30	31	32	33
14	8*	10*	12	14	16	18	20	21	22	24	25	26	28	29	30	31	32	33	34
15	7*	10*	13	15	17	19	20	22	23	25	26	27	29	30	31	32	33	34	35
16	8*	11*	13	15	17	19	21	22	24	25	27	28	30	31	32	33	34	35	37
17	7*	11*	14	16	18	20	22	23	25	26	28	29	30	32	33	34	35	37	38
18	8*	11*	14	16	18	20	22	24	25	27	28	30	31	33	34	35	36	38	39
19	9*	12*	14	17	19	21	23	25	26	28	29	31	32	34	35	36	37	39	40
20	8*	12*	15	17	19	21	23	25	27	28	30	32	33	34	36	37	38	40	41
25	9*	14	17	19	22	24	26	28	30	32	34	35	37	38	40	41	43	44	46
30	10*	15	18	21	24	26	29	31	33	35	37	39	40	42	44	45	47	49	50
35	11*	16	20	23	26	28	31	33	36	38	40	42	44	46	47	49	51	52	54
40	12*	17	21	24	27	30	33	36	38	40	42	45	47	49	51	52	54	56	58
45	13*	18	22	26	29	32	35	38	40	43	45	47	50	52	54	56	58	59	61
50	14*	19	23	27	31	34	37	40	42	45	48	50	52	54	57	59	61	63	65
60	16*	21	26	30	34	37	40	44	47	49	52	55	57	60	62	64	67	69	71
70	16*	23	28	32	36	40	44	47	50	53	56	59	62	64	67	69	72	74	77
80	18*	24	30	35	39	43	47	50	54	57	60	63	66	69	72	74	77	79	82
90	18*	26	32	37	41	46	50	53	57	60	64	67	70	73	76	79	81	84	87
100	20*	27	33	39	43	48	52	56	60	64	67	71	74	77	80	83	86	89	91

* Values based on exact theory

TABLE 2 (continued)

n \ t	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
	$P^* = 0.99$																		
1	1*	2*	3*	4*	4*	5*	6*	6	7	7	7	8	8	9	9	9	10	10	10
2	2*	4*	5*	6*	7	8	8	9	9	10	11	11	12	12	13	13	13	14	14
3	3*	5*	6*	8*	9	9	10	11	12	12	13	14	14	15	15	16	16	17	18
4	4*	6*	7*	9	10	11	12	13	13	14	15	16	16	17	18	18	19	20	20
5	5*	7*	8*	10	11	12	13	14	15	16	17	18	18	19	20	21	21	22	23
6	6*	7*	9*	11	12	13	14	15	16	17	18	19	20	21	22	23	23	24	25
7	5*	8*	10*	12	13	14	16	17	18	19	20	21	22	23	23	24	25	26	27
8	6*	9*	11*	12	14	15	17	18	19	20	21	22	23	24	25	26	27	28	29
9	7*	9*	11	13	15	16	18	19	20	21	22	24	25	26	27	28	29	29	30
10	8*	10*	12	14	16	17	19	20	21	22	24	25	26	27	28	29	30	31	32
11	7*	10*	13	15	16	18	19	21	22	24	25	26	27	28	29	30	32	33	34
12	8*	11*	13	15	17	19	20	22	23	25	26	27	28	30	31	32	33	34	35
13	9*	11*	14	16	18	19	21	23	24	26	27	28	30	31	32	33	34	35	36
14	8*	12*	14	16	18	20	22	24	25	27	28	29	31	32	33	34	36	37	38
15	9*	12*	15	17	19	21	23	24	26	28	29	30	32	33	34	36	37	38	39
16	10*	12*	15	18	20	22	23	25	27	28	30	31	33	34	35	37	38	39	40
17	9*	13*	16	18	20	22	24	26	28	29	31	32	34	35	37	38	39	40	42
18	10*	13*	16	19	21	23	25	27	28	30	32	33	35	36	38	39	40	42	43
19	9*	14*	17	19	21	24	26	27	29	31	33	34	36	37	39	40	41	43	44
20	10*	14*	17	20	22	24	26	28	30	32	33	35	37	38	40	41	43	44	45
25	11*	16	19	22	25	27	29	31	34	36	37	39	41	43	44	46	48	49	51
30	12*	17	21	24	27	30	32	34	37	39	41	43	45	47	49	50	52	54	55
35	13*	19	22	26	29	32	35	37	40	42	44	46	49	51	52	54	56	58	60
40	14*	20	24	28	31	34	37	40	42	45	47	50	52	54	56	58	60	62	64
45	15*	21	25	29	33	36	39	42	45	48	50	53	55	57	60	62	64	66	68
50	16*	22	27	31	35	38	41	45	47	50	53	55	58	60	63	65	67	69	72
60	18*	24	29	34	38	42	45	49	52	55	58	61	64	66	69	71	74	76	78
70	20*	26	32	37	41	45	49	53	56	59	63	66	69	71	74	77	80	82	85
80	20*	28	34	39	44	48	52	56	60	64	67	70	73	76	79	82	85	88	90
90	22*	30	36	42	47	51	56	60	64	67	71	74	78	81	84	87	90	93	96
100	24*	31	38	44	49	54	59	63	67	71	75	78	82	85	89	92	95	98	101

* Values based on exact theory

figuration and have

$$P_{CS}\{C(\pi_{ij})\} \geqq P_{CS}\{C(\tfrac{1}{2})\} \geqq P^*.$$

It would be of interest to know something about the distribution, particularly the expected value, of the size of S (compare [8]). It is hoped to investigate this problem in a later paper.

We now give the exact and asymptotic theory underlying the construction of Table 2.

4.2. *Exact evaluation of* ν. To determine ν as outlined above, we require an exact formula for $P_{CS}\{C(\frac{1}{2})\}$. A convenient expression may be obtained with the help of tables of the partition function $G(\boldsymbol{a}; n)$ referred to in Section 2.3. Rewrite $P_{CS}\{C(\frac{1}{2})\}$ as

$$\begin{aligned} P_{CS} &= \Sigma \Pr\{a_{(t)} \geqq a_{\max} - \nu \mid \boldsymbol{a}, C(\tfrac{1}{2})\} p(\boldsymbol{a}; C(\tfrac{1}{2})) \\ &= 2^{-\frac{1}{2}nt(t-1)} \Sigma [\Pr\{a_{(t)} \geqq a_{\max} - \nu \mid \boldsymbol{a}, C(\tfrac{1}{2})\} G(\boldsymbol{a}; n)], \end{aligned}$$

where $p(\boldsymbol{a}; C(\frac{1}{2}))$ is the partition probability function and where the summation extends over all the distinct partitions $\boldsymbol{a}$ of $\frac{1}{2}nt(t-1)$. To evaluate the quantity $Q(\boldsymbol{a}; n, \nu)$, say, in square brackets, consider all possible permutations $a_{i_1}, a_{i_2}, \cdots, a_{i_t}$ of $\boldsymbol{a}$ and take a_{i_t} to be the score corresponding to the best treatment $T_{(t)}$. Then $Q(\boldsymbol{a}; n, \nu)$ is the frequency associated with permutations for which $a_{i_t} \geqq a_{\max} - \nu$. The proportion of permutations for each distinct value of a_{i_t} is m_{i_t}/t, where m_{i_t} is the number of scores in $\boldsymbol{a}$ tied with a_{i_t}. Thus $Q(\boldsymbol{a}; n, \nu) = M(\boldsymbol{a}; \nu)G(\boldsymbol{a}; n)/t$, the multiple M being the number of a's in $\boldsymbol{a}$ which exceed or equal $a_{\max} - \nu$. For a given n, t, and ν, we have therefore

$$P_{CS}\{C(\tfrac{1}{2})\} = t^{-1}2^{-\frac{1}{2}nt(t-1)}\Sigma M(\boldsymbol{a}; \nu)G(\boldsymbol{a}; n).$$

We illustrate the procedure for $t = 4$, $n = 2$. In this case there are 16 permissible partitions of which a typical one (written in ascending order) is [0345] with frequency $G(0, 3, 4, 5; 2) = 144$. Also we see that for

$\nu = 0$ $\quad M = 1,$ corresponding to $a_{(t)} = 5$

$\nu = 1$ $\quad M = 2,$ corresponding to $a_{(t)} = 5, 4$

$\nu = 2, 3, 4$ $\quad M = 3,$ corresponding to $a_{(t)} = 5, 4, 3$

$\nu = 5$ $\quad M = 4,$ corresponding to $a_{(t)} = 5, 4, 3, 0.$

Thus the contribution to $P_{CS}\{C(\frac{1}{2})\}$ from this partition is, for a given ν, $36M(\nu)/2^{12}$. If the corresponding contributions for all 16 partitions are added up, the resulting values of $P_{CS}\{C(\frac{1}{2})\}$ are:

ν	0	1	2	3	4	5
$P_{CS}\{C(\frac{1}{2})\}$	0.342773	0.559570	0.769042	0.905273	0.975098	0.997070

The entries in Table 2 for $t = 4$, $n = 2$ follow at once.

The procedure here illustrated has been formalized for use on the IBM 650, details being given in [12].

4.3. *Asymptotic approximation to* ν. In the manner of Section 3.3 the asymptotic probability of correct selection under rule $\mathcal{R}$ may be written, with a continuity correction on ν, as

$$P_{CS} = \lim_{n\to\infty} \Pr\{a_{\max} - a_{(t)} < \nu + \tfrac{1}{2}\}$$
$$= \lim_{n\to\infty} \Pr\{d_{(i)} < \nu + \tfrac{1}{2}; i = 1, 2, \cdots, t-1\}$$

In the present case equations (3.5) are simply

$$\delta = 0, \qquad \sigma_d^2 = \tfrac{1}{2}nt, \qquad \rho\sigma_d^2 = \tfrac{1}{4}nt, \tag{4.1}$$

and, with $\rho = \frac{1}{2}$, P_{CS} reduces to

$$P_{CS} = \int_{-\infty}^{\infty} [\Phi(u_t + w)]^{t-1}\,\varphi(u_t)\,du_t, \tag{4.2}$$

where

$$w = 2(\nu + \tfrac{1}{2})/(nt)^{\frac{1}{2}}. \tag{4.3}$$

Values of w as solutions of (4.2) have been tabulated by Bechhofer [2] and Gupta [7] for a wide range of t and P_{CS}. We therefore obtain asymptotic approximations to ν by using the tabulated values and (4.3). These values have been found in [12] to agree well with the exact results. In Table 2 exact values are used for the range of (t, n) up to (2, 100), (3, 20), (4, 8), (5, 3), (6, 1), (7, 1), (8, 1). Elsewhere the approximate values as obtained by solving (4.2) and (4.3) have been rounded upward to the nearest integer.

It should be noted that for $t = 2$ and increasing n the values ν in Table 2 corresponding to odd and even values of n form two non-decreasing sequences consisting respectively of odd and even numbers. As a result, it would be incorrect to use, for example, $\nu = 6$ when $n = 95$ and $P^* = 0.75$. With the help of the Tables of the Binomial Probability Distribution, Harvard University Press, 1955, one finds $P_{CS}(n = 95, \nu = 5, 6) = 0.73080$, $P_{CS}(n = 95, \nu = 7, 8) = 0.79405$, and therefore concludes that for $n = 95$ and $P^* = 0.75$ the required value for ν is 7.

4.4. *Conservative nature of configuration* $C(\frac{1}{2})$. For any configuration $C(\pi_{ij})$ and a chosen value of ν the probability of correct selection P_{CS} can be written as

$$
\begin{aligned}
P_{CS} &= \sum_{a_{(t)}=0}^{n(t-1)} \Pr\{T_{(t)} \text{ scores } a_{(t)}\} \Pr\{T_{(i)}\ (i = 1, 2, \cdots, t-1) \\
&\qquad\qquad \text{scores at most } a_{(t)} + \nu\} \\
&= \sum_{a_{(t)}=0}^{n(t-1)} f(a_{(t)}) \Pr\{a_{(i)} \leqq a_{(t)} + \nu \mid a_{(t)}, \nu\},
\end{aligned} \tag{4.4}
$$

where the dependence on $C(\pi_{ij})$ and on the experiment size (t, n) is understood.

Suppose now that the characteristic on which the treatments are judged satisfies a linear model. We will show that P_{CS} is a minimum for $C(\frac{1}{2})$, that is when all V_i are equal.

(i) P_{CS} clearly increases with $V_{(t)}$ for fixed $V_{(i)}$ $(i = 1, 2, \cdots, t-1)$ so that $V_{(t)}$ must be taken as small as is compatible with (2.13), namely $V_{(t)} = V_{(t-1)}$.

(ii) The probability that $a_{(t-1)} > a_{(t)} + \nu$ is greatest when all V_i are equal. To prove this we note first that under (i) $\pi_{tj} = \pi_{t-1,j}$ $(j = 1, 2, \cdots, t-2)$. Write π_j for the common value, α for $\alpha_{t,t-1}$, α_j, α_j' for α_{tj}, $\alpha_{t-1,j}$, respectively. Then

$$
a_{(t)} = \alpha + \sum_{j=1}^{t-2} \alpha_j, \qquad a_{(t-1)} = n - \alpha + \sum_{j=1}^{t-2} \alpha_j', \tag{4.5}
$$

and

$$
\begin{aligned}
\Pr\{a_{(t-1)} - a_{(t)} > \nu\} &= \textstyle\sum_U 2^{-n} \binom{n}{\alpha} \prod_{j=1}^{t-2} \left[\binom{n}{\alpha_j} \pi_j^{\alpha_j} (1 - \pi_j)^{n-\alpha_j} \right. \\
&\qquad\qquad \left. \cdot \binom{n}{\alpha_j'} \pi_j^{\alpha_j'} (1 - \pi_j)^{n-\alpha_j'} \right] \\
&= 2^{-n} \textstyle\sum_U \binom{n}{\alpha} \prod_{j=1}^{t-2} \binom{n}{\alpha_j} \binom{n}{\alpha_j'} \pi_j^{\alpha_j+\alpha_j'} (1 - \pi_j)^{2n-\alpha_j-\alpha_j'},
\end{aligned} \tag{4.6}
$$

where $\sum_U$ extends over all permissible values of the α's which give $a_{(t-1)} - a_{(t)} > \nu$.

For $t = 3$, (4.6) reduces to

$$(4.7)\quad \Pr\{a_{(2)} - a_{(3)} > \nu\} = 2^{-n} \sum_U \binom{n}{\alpha}\binom{n}{\alpha_1}\binom{n}{\alpha_1'} \pi_1^{\alpha_1+\alpha_1'}(1-\pi_1)^{2n-\alpha_1-\alpha_1'}.$$

Now consider the terms in $\sum_U$ for which α has a fixed value and $a_{(2)} - a_{(3)} = \Delta(\Delta = 1, 2, \cdots, 2n)$. Then by (4.5) $n - \alpha + \alpha_1' - \alpha - \alpha_1 = \Delta$, or $\alpha_1' - \alpha_1 = \Delta + 2\alpha - n = c$ (say). The values of α_1 and α_1' satisfying this are, if any,

$$\begin{aligned} &\alpha_1' = n, && \alpha = n - c && \text{and} \quad \alpha_1' = c, && \alpha = 0; \\ &\alpha_1' = n - 1, && \alpha_1 = n - c - 1 && \text{and} \quad \alpha_1' = c + 1, && \alpha_1 = 1; \\ & && \cdots && && \\ &\alpha_1' = n - i, && \alpha_1 = n - c - i && \text{and} \quad \alpha_1' = c + i, && \alpha_1 = i; \end{aligned}$$

continuing as long as $n - c - i \geqq 0$ (c has been taken non-negative; if $c < 0$ reverse roles of α_1' and α_1). Thus the terms in $\sum_U$ can be paired off as indicated, resulting in the typical term

$$\binom{n}{\alpha}\binom{n}{i}\binom{n}{c+i}[\pi_1^{2n-c-2i}(1-\pi_1)^{c+2i} + \pi_1^{c+2i}(1-\pi_1)^{2n-c-2i}],$$

which is easily seen to be a maximum for $\pi_1 = \frac{1}{2}$. If $i = \frac{1}{2}(n - c)$ only one term arises. This is proportional to $\pi_1^n(1 - \pi_1)^n$ and hence also a maximum for $\pi_1 = \frac{1}{2}$. Thus (ii) is true for $t = 3$.

For $t > 3$ refer to equation (4.6) and to start with fix α_k, α_k' for $k = 2, 3, \cdots, t - 2$. Then

$$(4.8)\quad \begin{aligned} \Pr\{a_{(t-1)} - a_{(t)} > \nu\} = 2^{-n} &\prod_{k=2}^{t-2} \binom{n}{\alpha_k}\binom{n}{\alpha_k'} \pi_j^{\alpha_k+\alpha_k'}(1-\pi_j)^{2n-\alpha_k-\alpha_k'} \\ &\cdot \sum \binom{n}{\alpha}\binom{n}{\alpha_1}\binom{n}{\alpha_1'} \pi_1^{\alpha_1+\alpha_1'}(1-\pi_1)^{2n-\alpha_1-\alpha_1'}, \end{aligned}$$

where the sum now extends over all permissible values of α, α_1, α_1' which give $a_{(t-1)} - a_{(t)} > \nu$. As for the case $t = 3$ this sum is a maximum for $\pi_1 = \frac{1}{2}$, a result which holds for all permissible choices of α_k and α_k'. The argument can be repeated with α_2, α_2', π_2 replacing α_1, α_1', π_1 to show that to maximize (4.8) we require also $\pi_2 = \frac{1}{2}$, and in fact $\pi_1 = \pi_2 = \cdots = \pi_{t-2} = \frac{1}{2}$. This establishes (ii).

(iii) The chance that one or more of the first $t - 2$ treatments has a score exceeding $a_{(t)} + \nu$ is improved by making the merits of these treatments as large as possible, that is, by making $V_1 = V_2 = \cdots = V_{t-1} = V_t$, which completes the proof.

4.5. *An Example.* We illustrate the application of the decision rule $\mathfrak{R}$ on data supplied by E. Jensen of Faellesforeningen for Danmarks Brugsforeninger, Copenhagen. Fifteen persons examined all possible pairings of 4 different samples for taste. The following preference table was obtained:

	T_1	T_2	T_3	T_4	a_i
T_1	—	3	2	2	7
T_2	12	—	11	3	26
T_3	13	4	—	5	22
T_4	13	12	10	—	35

We have $a_{\max} = 35$. To ensure that with at least a pre-assigned probability $P^* = 0.75$ the best sample is in the selected subset we enter Table 2 for $t = 4$, $n = 15$, $P^* = 0.75$, find $\nu = 7$ and hence retain only T_4 in the subset. For $P^* = 0.90$ we have $\nu = 9$, so that the subset consists of T_4 and T_2, and so on.

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REFERENCES

[1] ANDERSON, T. W. (1958). *An Introduction to Multivariate Statistical Analysis.* Wiley, New York.

[2] BECHHOFER, ROBERT E. (1954). A single-sample multiple decision procedure for ranking means of normal populations with known variances. *Ann. Math. Statist.* **25** 16–39.

[3] BRADLEY, RALPH ALLAN (1953). Some statistical methods in taste testing and quality evaluation. *Biometrics* **9** 22–38.

[4] BRADLEY, RALPH ALLAN (1954). Rank analysis of incomplete block designs. II. Additional tables for the method of paired comparisons. *Biometrika* **41** 502–537.

[5] BRADLEY, RALPH ALLAN and TERRY, MILTON E. (1952). Rank analysis of incomplete block designs. I. The method of paired comparisons. *Biometrika* **39** 324–345.

[6] DAVID, H. A. (1959). Tournaments and paired comparisons. *Biometrika* **46** 139–149.

[7] GUPTA, SHANTI S. (May, 1956). On a decision rule for a problem in ranking means. Institute of Statistics, Mimeograph Series Report No. 150, Chapel Hill, N. C., University of North Carolina.

[8] GUPTA, SHANTI S. and SOBEL, MILTON (1960). Selecting a subset containing the best of several binomial populations. *Contributions to Probability and Statistics,* Stanford Univ. Press 224–248.

[9] HALL, WM. JACKSON (1959). The most economical character of some Bechhofer and Sobel decision rules. *Ann. Math. Statist.* **30** 964–969.

[10] STARKS, T. H. and DAVID, H. A. (1961). Significance tests for paired-comparison experiments. *Biometrika* **48** 95–108.

[11] TRAWINSKI, B. J. (1961). Frequencies of partitions for paired-comparison experiments and related tables. Technical Report No. 52, Virginia Polytechnic Institute, Blacksburg, Va.

[12] TRAWINSKI, B. J. (1961). Selection of the best treatment in a paired-comparison experiment. Ph.D. dissertation, Virginia Polytechnic Institute Library, Blacksburg, Virginia.