NOTES

A GENERAL VERSION OF DOEBLIN'S CONDITION

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Let $X_0, X_1, \cdots$ be a discrete parameter Markov process on a measurable space (Ω, Σ) with stationary transition probabilities $P^k(t, A)$, $t \varepsilon \Omega$, $A \varepsilon \Sigma$. Set $P^1(t, A) = P(t, A)$. All sets to be considered will be elements of Σ. A' denotes the complement of the set A.

The classical condition of Doeblin may be stated as follows:

(1) There is a probability measure m on Σ, an integer $k \geqq 1$, and an $\epsilon > 0$, such that if $m(A) \leqq \epsilon$, then $P^k(t, A) \leqq 1 - \epsilon$ for every $t \varepsilon \Omega$.

Under condition (1), a detailed analysis of the ergodic behavior of the process may be given ([1], pp. 192 ff.); in particular Ω may be decomposed into a finite number of so-called ergodic sets, and the process always has a stationary probability measure λ, that is, $\lambda(\Omega) = 1$ and $\int P(t, A)\lambda(dt) = \lambda(A)$ for all sets A. Since many important processes do not satisfy Doeblin's condition, Doob [2] was led to consider more general conditions which could be used to analyze the ergodic behavior of the process. Doob's main hypothesis was the assertion that the process has a stationary probability measure; from this and other conditions he derived many of the ergodic properties of the Doeblin case in a generalized form. Since any process satisfying (1) does have a stationary measure, Doob's condition is more general than Doeblin's. The main purpose of this paper is to phrase a condition in terms of the transition probabilities which includes (1) as a special case, and then to show that our condition assures the existence of a stationary probability measure. Doob's results may then be employed when applicable to describe the ergodic behavior of the process.

For each set A, define the measurable set $l[A] = \{t: \lim_n \inf P^n(t, A) > 0\}$.

(2) There is a probability measure m on Σ and a δ, $0 < \delta < 1$, such that if $m(A) \geqq \delta$, then $m(l[A]) > 0$.

If (1) is satisfied, then (2) holds. For if ϵ is the positive number of (1) and $m(A) \leqq \epsilon$, then $P^k(t, A) \leqq 1 - \epsilon$ for all t, and it is easy to see that $P^n(t, A) \leqq 1 - \epsilon$ for all t and all $n \geqq k$. Thus $P^n(t, A') \geqq \epsilon$ for all t and all $n \geqq k$, and so $\lim_n \inf P^n(t, A') \geqq \epsilon > 0$ for all t, yielding $l[A'] = \Omega$. So (2) will be satisfied if $\delta = 1 - \epsilon$.

The main result is:

Theorem. *If* (2) *holds, then there exists a stationary probability measure* λ *for the process. Moreover, if* $m(A) > \delta$, *then* $\lambda(A) > 0$.

The main tools in the proof of the theorem are the notion of generalized limit

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and the use of two lemmas on finitely additive set functions. Before proceeding with the proof, we define some terms and state the necessary lemmas.

A content μ is a non-negative, extended real valued finitely additive set function defined on a field of sets such that $\mu(\phi) = 0$. A measure, in terms of content, is a countably additive content defined on a σ-field. If μ is a content on a σ-field, then μ is called purely finitely additive (p.f.a.) if the relation $0 \leqq \alpha \leqq \mu$ for α a measure implies $\alpha = 0$. (The ordering is the standard lattice ordering of set functions.) Define the transformations T^k taking the set of finite measures on Σ into itself by: $(T^k m)(\cdot) = \int P^k(t, \cdot) m(dt)$. Since a theory of integration exists for contents [3], T^k can be considered more generally as taking contents into contents. All limits will be as $n \to \infty$.

The following two lemmas are due to Yosida and Hewitt and appear in [4].

Lemma 1. *If μ is a finite content and is* p.f.a., *and if m is a finite measure and μ and m are defined on a σ-field Σ, then for every $\epsilon > 0$, there is a set $S \,\varepsilon\, \Sigma$ with $\mu(S') = 0$ and $m(S) < \epsilon$.*

Lemma 2. *If μ is a finite content on a σ-field Σ, there is a unique decomposition: $\mu = \mu_c + \mu_f$ where μ_c is a measure and μ_f is a* p.f.a. *content.*

Proof of Theorem. For every set A, put $M_n(A) = (1/n)\sum_{k=0}^{n-1} (T^k m)(A)$, where T^0 is the identity transformation. By a corollary of the Hahn-Banach theorem [3], p. 73, there exists a generalized limit, $\text{Lim}\, A_n$, for all sequences of bounded real numbers. Set $\text{Lim}\, M_n(A) = \mu(A)$ for each set A. The basic properties of this generalized limit are:

$$\mu(\Omega) = 1 \tag{3}$$

$$\mu(A) \geqq 0 \tag{4}$$

$$\text{If} \quad A \cap B = \phi, \quad \text{then} \quad \mu(A \cup B) = \mu(A) + \mu(B) \tag{5}$$

$$\text{Lim}\, M_n(A) = \text{Lim}\, M_{n+1}(A) \tag{6}$$

$$\liminf M_n(A) \leqq \mu(A) \leqq \limsup M_n(A). \tag{7}$$

It is clear from these properties that μ is a finite content and (6) easily shows that $T\mu = \mu$. It will now be shown that μ is not p.f.a. Let δ be as given in (2), and let $m(A) \geqq \delta$. By Fatou's lemma

$$\int \liminf P^n(t, A) m(dt) \leqq \liminf \int P^n(t, A) m(dt) = \liminf (T^n m)(A). \tag{8}$$

If the left hand side of (8) is zero, then $\liminf P^n(t, A) = 0$ a.e. (m) or, in our notation, $m(l[A]) = 0$, contrary to (2). Therefore (8) yields that

$$\liminf (T^n m)(A) > 0.$$

(7) and the definition of $M_n(A)$ now show that $\mu(A) > 0$. Hence we have δ such that $m(A) \geqq \delta$ implies $\mu(A) > 0$ for every such set A. Lemma 1 now applies to prove that μ is not p.f.a. Now use the decomposition of Lemma 2 to write

$$\mu = \mu_c + \mu_f \quad \text{where} \quad \mu_c \neq 0.$$

$\mu = T\mu = T\mu_c + T\mu_f$ and a little reflection shows that μ_c is the maximal measure with the property: $\mu_c \leqq \mu$; one therefore obtains $T\mu_c \leqq \mu_c$. If $T\mu_c < \mu_c$, there would exist a set E with $(T\mu_c)(E) < \mu_c(E)$, but then the relation $(T\mu_c)(\Omega) = \int P(t, \Omega)\mu_c(dt) = \mu_c(\Omega)$ gives the contradiction: $(T\mu_c)(E') > \mu_c(E')$. So $T\mu_c = \mu_c$, and μ_c is a non-trivial stationary measure for the process; norming it suitably, a stationary probability measure λ is obtained. Suppose $m(A) > \delta$, say, $m(A) = \delta + \eta$, $\eta > 0$. By Lemma 1, there exists a set S, $m(S) < \eta/2$ and $\mu_f(S') = 0$. $m(A' \cup S) \leqq m(A') + m(S) < 1 - \delta - \eta + \eta/2 < 1 - \delta$. Therefore, $(A' \cup S)' = A \cap S'$ satisfies $m(A \cap S') > \delta$, and hence, as we know, $\mu(A \cap S') > 0$. But then $0 < \mu(A \cap S') = \mu_c(A \cap S') + \mu_f(A \cap S') = \mu_c(A \cap S')$ proving $\mu_c(A) > 0$ and $\lambda(A) > 0$. The proof of the theorem is complete.

As a simple example of a case covered by (2) but not satisfying (1), we consider a process cited by Doob, [2]. Let $\Omega = (-\infty, \infty)$, Σ = Borel subsets of Ω, and

$$P^n(t, A) = \frac{1}{[2\pi(1 - \rho^{2n})]^{\frac{1}{2}}} \int_A \exp - \frac{(y - \rho^n t)^2}{2(1 - \rho^{2n})} dy \tag{9}$$

where ρ is constant, $0 \leqq \rho < 1$. The process with transition probabilities given by (9) does not satisfy Doeblin's condition, and

$$\lim P^n(t, A) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_A \exp - \frac{y^2}{2} dy > 0 \tag{10}$$

whenever A has positive Lebesgue measure. If m is a probability measure equivalent to Lebesgue measure, and δ is any fixed constant $0 < \delta < 1$, (10) shows that $m(A) \geqq \delta$ implies $l[A] = \Omega$, so (2) holds. Of course, in this example, the right hand side of (10) is the unique stationary probability measure for the process.

In general, if $P^n(t, A)$ is absolutely continuous with respect to m for each n, and $f_n(t, s)$ are the respective densities such that, for each set A with $m(A) \geqq \delta$, there exists a set A^*, $m(A^*) > 0$, and a number $\epsilon(A) > 0$ with $f_n(t, s) \geqq \epsilon$ for $(t, s) \in A^* \times A$ for all n, then

$$P^n(t, A) = \int_A f_n(t, s) m(ds) \geqq \epsilon\delta > 0$$

for $t \in A^*$. So $\inf_{t \in A^*} \liminf P^n(t, A) \geqq \epsilon\delta$ and (2) is satisfied.

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A TEST FOR EQUALITY OF MEANS WHEN COVARIANCE MATRICES ARE UNEQUAL[1]

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Let $x_\alpha^{(g)}$ be an observation from the p-variate normal distribution $N(\mu^{(g)}, \Sigma_g)$, $\alpha = 1, \cdots, N_g$, $g = 1, \cdots, q$. Consider testing the null hypothesis[2]

$$H: \mu^{(1)} = \cdots = \mu^{(q)}. \tag{1}$$

When the covariance matrices Σ_g are equal, the hypothesis is a form of the so-called general linear hypothesis, and a number of tests are available. (See Chapter 8 of Anderson (1958), for example.) When $q = 2$, Bennett (1951) has extended the procedure of Scheffé (1943) to give an exact test based on Hotelling's generalized T^2. (See Section 5.6 of Anderson (1958).) In this note we extend previous procedures to $q > 2$.

As an example, let $q = 3$ and $N_1 = N_2 = N_3 = N$, say. Let

$$\begin{aligned} y_\alpha &= a_1 x_\alpha^{(1)} + a_2 x_\alpha^{(2)} + a_3 x_\alpha^{(3)}, \\ z_\alpha &= b_1 x_\alpha^{(1)} + b_2 x_\alpha^{(2)} + b_3 x_\alpha^{(3)}, \end{aligned} \tag{2}$$

where $\sum_{g=1}^3 a_g = 0$, $\sum_{g=1}^3 b_g = 0$ and (a_1, a_2, a_3) and (b_1, b_2, b_3) are linearly independent. (In practice the indexing of the observations in each sample would be done randomly.) Then the hypothesis (1) is equivalent to the hypothesis

$$\mathcal{E} y_\alpha = \sum_{g=1}^3 a_g \mu^{(g)} = 0, \qquad \mathcal{E} z_\alpha = \sum_{g=1}^3 b_g \mu^{(g)} = 0. \tag{3}$$

The covariance matrix of $(y_\alpha' \ \ z_\alpha')$ is

$$\begin{pmatrix} a_1^2 \Sigma_1 + a_2^2 \Sigma_2 + a_3^2 \Sigma_3 & a_1 b_1 \Sigma_1 + a_2 b_2 \Sigma_2 + a_3 b_3 \Sigma_3 \\ a_1 b_1 \Sigma_1 + a_2 b_2 \Sigma_2 + a_3 b_3 \Sigma_3 & b_1^2 \Sigma_1 + b_2^2 \Sigma_2 + b_3^2 \Sigma_3 \end{pmatrix}. \tag{4}$$

The hypothesis (3) can be tested by a T^2-statistic

$$T^2 = N(\bar{y}' \ \bar{z}') S^{-1} \begin{pmatrix} \bar{y} \\ \bar{z} \end{pmatrix}, \tag{5}$$

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[2] Dr. Charles V. Riche called this problem to my attention.